Phase transitions and interface conditions for two phase flows

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In this paper we will consider a mathematical model of liquid-vapour flows including phase transition which was proposed by Korteweg already in 1901 [2] and which is known as the Navier-Stokes-Korteweg model. It is an extension of the compressible Navier-Stokes equation and given by the following system.

\[
\begin{align*}
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho v) & = 0 \\
\frac{\partial}{\partial t}(\rho v) + \nabla \cdot (\rho vv^t + p(\rho) I) & = \mu \Delta v + \lambda \rho \nabla \Delta \rho.
\end{align*}
\] (1)

This is a one fluid model where \( \rho, v, p(\rho) \) and \( \mu \) denote the density, velocity, pressure and the viscosity of the fluid/vapour respectively. Compared to the original Navier–Stokes equation the system (1) contains the term \( \gamma \varepsilon^2 \rho \nabla \Delta \rho \) (in which \( \lambda \) is replaced by \( \gamma \varepsilon^2 \)) which is supposed to model capillarity effects close to phase transitions. The pressure \( p(\rho) \) as a function of the density \( \rho \) is defined as

\[
p(\rho) = \rho^2 \psi'(\rho)
\] (2)

where \( \psi \) is a smooth function of \( \rho \) such that \( \rho \psi(\rho) \) is the total free energy density and of the form of a double well potential (up to a linear function).

The values \( \alpha_1 \) and \( \alpha_2 \) are defined by the extrema of \( p \). The conservation of energy is neglected in (1). Different phases of the fluid are defined by the size of \( \rho \). If \( \rho \leq \alpha_1 \) we are in the vapour phase and if \( \rho \geq \alpha_2 \) we are in the liquid phase. The equation (2) is known as the van der Waals equation of state.

In this contribution we will study the behaviour of the pressure across the interface. Since a rigorous theory about this question is not available and difficult, we will concentrate on the static version of (1), i.e.

\[
\nabla p(\rho) = \gamma \varepsilon^2 \rho \nabla \Delta \rho.
\] (3)

In particular we will study the behaviour of the pressure in the limit if \( \varepsilon \to 0 \). First we will repeat some recent results [1], [4], which show that the difference of the pressures \( [p] \) on both sides of the interface is of order \( \varepsilon \):

\[
[p] = ck_m \varepsilon + o(\varepsilon),
\] (4)

where \( k_m \) is the mean curvature of the interface. This seems to contradict the classical result of Landau and Lifschitz [3], which says that the difference of the pressures on both sides of the interface is proportional to the mean curvature of the interface:

\[
[p] = c_1 k_m.
\] (5)
Now let us consider the low Mach number limit of (1) and assume, that we have an asymptotic expansion of all quantities with respect to the Mach number $M$, in particular

$$ p(x,t) = p(\rho(x,t)) = p_0(x,t) + M p_1(x,t) + M^2 p_2(x,t) + O(M^3). $$

The non-dimensionalization form of (1) for small Mach number $M$ is given by:

$$ \partial_t (\rho v) + \nabla \cdot (\rho v v^t) + \frac{1}{M^2} \rho I = \frac{1}{Re} \Delta v + \frac{\lambda_2}{M^2} \rho \nabla \Delta \rho. $$

Then we will show that the scaling/capillarity quantity $\lambda$ can be related to the Mach number under certain conditions such that we get the expected jump relation for the difference of the pressures $p_2$ on both sides of the interface. Similarly in a second approach we can obtain the expected pressure relation across the interface by a modified definition of the pressure on the basis of a special scaling of the free energy density compared to the gradient term. Instead of the energy

$$ J_\varepsilon(\rho) := \int_\Omega W(\rho) + \varepsilon^2 \frac{1}{2} |\nabla \rho|^2 dx + \tilde{M} \rightarrow \text{Minimum} \quad (6) $$
we use the modification

$$ I_\varepsilon(\rho) := \frac{1}{\varepsilon} \int_\Omega W(\rho) + \frac{\varepsilon^2}{2} |\nabla \rho|^2 dx + \tilde{M} \rightarrow \text{Minimum} \quad (7) $$

where $W$ denotes the free energy density up to a linear function, $\frac{\varepsilon^2}{2} |\nabla \rho|^2$ penalizes the occurrence of a large interface and $\tilde{M}$ is a constant. Due to this scaling we get a modified pressure $p_\varepsilon$ with the expected jump condition.

References


