HIGH RESOLUTION WAVELET BASED CENTRAL SCHEMES
FOR MODELING NONLINEAR PROPAGATING FRONTS

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ABSTRACT. A class of wavelet-based methods, which combine adaptive grids together with second-order central and central-upwind high-resolution schemes, is introduced for accurate solution of first order hyperbolic systems of conservation laws and related equations. It is shown that the proposed class recovers stability which otherwise may be lost when the underlying central schemes are utilized over non-uniform grids. For simulations on irregular grids, both full-discretized and semi-discretized forms are derived; in particular, the effect of using certain shifts of the center of the computational allow to use standard slope limiters on non-uniform cells. Thereafter, the questions of numerical entropy production, local truncation errors, and the Total Variation Diminishing (TVD) criterion for scalar equations, are investigated.

Central schemes are sensitive for irregular grids. To cure this drawback in the present context of multiresolution, the adapted grid is locally modified around high-gradient zones by adding new neighboring points. For an adapted point of resolution scale $j$, the new points are locally added in both resolution $j$ and all successive coarser resolutions. It is shown that this simple grid modification improves stability of numerical solutions; this, however, comes at the expense that cell-centered central and central-upwind high resolution schemes, may not satisfy the TVD property. Finally, we present numerical simulations for both scalar and systems of non-linear conservation laws which confirm the simplicity and efficiency of the proposed method. These simulations demonstrate the high accuracy, the entropy production of wavelet-based algorithms which can effectively detect high gradient zones of shock waves, rarefaction regions, and contact discontinuities.

1. INTRODUCTION

Multiresolution-based studying has rapidly been developed in many branches of science; one of such powerful schemes is the wavelet theory. Wavelets act as a mathematical microscope and can detect abrupt variations. Development of this theory is simultaneously done by scientists, mathematicians and engineers [50]. Wavelets can detect different local features of data separated locally in different resolutions. Wavelets can efficiently distinguish overall smooth variation of a solution from locally high transient ones. This multiresolution feature has interested many researchers in the field of partial differential equations (PDEs) [50]. One approach is to study problems in accordance with their solution variations; i.e., using different accuracies in different computational domains. In this approach, more grid points are concentrated around high-gradient zones to detect high variations. Such simulations lead to adaptive solvers. In this case, only the important physics of problems are precisely studied for a cost-effective computations.

In general, wavelet-based adaptive methods have successfully been implemented for PDE solutions containing steep moving fronts or sharp transitions in small zones [11, 12, 1, 20, 47, 2, 55, 5, 37, 56, 5, 10, 25, 26, 9, 38, 39, 18, 13, 23, 15]. In most of these approaches, wavelets are used as a tool to detect localized spatial behaviors and corresponding zones. Wavelet coefficients of considerable values concentrate automatically in the vicinity of high-gradient regions. The wavelet coefficients have a one to one correspondence with corresponding spatial grid points. Hence, by ignoring coefficients of small values and corresponding grid points, a considered grid can be adapted. In these grid-based adaptive schemes, the degrees of freedom are considered as point values in the physical space [11, 12, 26].

In this work, propagation of nonlinear fronts are simulated by central high resolution schemes with proper integration of them with wavelet-based adaptation procedures. Accuracy and effectiveness of
numerical solutions can be obtained by such incorporation. In numerical simulation of first order hyperbolic systems, several important and challenging features exist, as: lack of inherent (natural) dissipation [7], forming of artificial (numerical) dissipation and dispersion [3, 8]. Inherent dissipation in a system improves both adaptation procedure and numerical stability. The numerical dispersion leads to developing wiggles in front and behind of propagating waves, due to distortion of different phases of propagating waves (fronts). The numerical dissipation has a tendency to flat discontinuities in numerical solutions [3, 8]. Discontinuous solutions are commonly formed in non-linear first-order hyperbolic systems, where controlling both the numerical dissipation and dispersion are challenging. Despite of hyperbolic systems, the inherent dissipation exists in elliptic and parabolic PDEs [19]. These systems are not so sensitive to small perturbations in their numerical solutions. The perturbations or errors dissipate during numerical simulations. Based on this feature, wavelet-based adaptive schemes have successfully been employed in modeling of the elliptic [5, 55] and parabolic problems [11, 12, 1, 37, 26].

Considering hyperbolic problems, however, handling the artificial effects -dissipation and dispersion, needs special treatment. Significant non-physical oscillations would strongly form around the (nearly) discontinuous solutions: the Gibb’s phenomenon. Due to such oscillations, numerical instability could occur. These spurious oscillations could propagate throughout the computational domain. Therefore, both the proper adaptation procedure and stability of numerical solutions can fail.

To guarantee solution stability of high-resolution schemes on non-uniform grids, some conditions must be satisfied. Such assurance can be obtained by using the normalized variable and space formulation (NVSF) criterion [14]. This condition is successfully incorporated with wavelet-based adaptive methods for simulation of hyperbolic systems [47, 2]. In the NVSF criterion, identifying of propagating directions are necessary. This identification needs itself complex procedures, especially for 2-D and 3-D problems [47]. To remedy this disadvantage, in this work, central and central-upwind high resolution schemes are considered (like the Kurganov and Tadmor method) [32, 30, 29]. Central/central-upwind high resolution schemes on non-staggered grids offer the following benefits: having a simple and straightforward concept; being easy to implement; having less numerical dissipation than ones on the staggered cells, like the Nessyahu and Tadmor (NT) method; offering both semi-discrete and fully-discrete forms; being a Riemann-free solver; no requiring to staggered grid points as needed in the NT scheme [40]; having comparable second and higher order accuracies with other expensive techniques. Central/central-upwind methods, however, are sensitive for cell irregularity; our investigations show that numerical instabilities appear rapidly in adaptive solutions. This is because, these methods do not originally satisfy the NVSF condition. To guarantee the numerical stability, two features should be studied: 1) performance of slope/flux limiters on non-uniform grids; 2) effects of grid density variation on numerical solutions and adaptation procedures. Most of slope limiters have been developed for working properly on uniform grids. Using of such limiters on irregular grids leads to unstable solutions. Abrupt changing of grid densities can also lead to instability, due to ill-posedness feature of irregular sampled data. To prevent this kind of instability, density variation of adapted grids should be checked. For achieving this purpose, adapted grids are locally modified in the vicinity of high-gradient zones to achieve local semi-uniform grid points (then an ill-posed problem is nearly replaced with a well-posed one). This modification is done in context of the multiresolution analysis by a post-processing stage. Considering an adapted grid, for each point of level resolution \( j_0 \), some extra new neighbor points are locally added. The new points are inserted in both the same resolution level \( j_0 \) and all successive coarser resolution levels, i.e.: \( j \in \{ j_0 - 1, j_0 - 2, \ldots, J_{\text{min}} \} \). It will be shown that even though grid modification improves numerical stability, there is no guarantee to have long-term stability. This is due to performance of slope/flux limiters on non-uniform cells. For non-uniform cells, one approach is to re-design limiters [4, 59]. In this study, it will be shown how to use limiters without their definition modification. This will be done by re-locating cell centers only in some cells acting as transmitting cells. Investigations show that the proposed method leads to stable results comparable with those of NVSF-based schemes.

In nonlinear conservation laws, discontinuous solutions develop typically. Thereby, common error estimation concepts, mainly based on the Taylor expansion, can not be used. For this reason, concept of the local truncation error is used to assess convergence of solutions to weak ones as a practical approach [28, 27]. Related formulations for such error estimation would be provided for 1-D and 2-D non-uniform cells. This kind of error has direct relationship with the Lip’-norm theory introduced by Tadmor [52, 53] for nonlinear 1-D scalar conservation laws with convex flux. The numerical results
confirm that the local truncation error can also be used for convergence study of 1-D systems or 2-D conservation laws, even with non-uniform grids.

Uniqueness of numerical solutions is checked by the concept of numerical entropy production. Theoretically, the numerical entropy production is zero in smooth regions while less than zero around shocks and discontinuities [43, 44]. This helps to study quality of numerical results especially for ones without exact solutions. All calculations will directly be done on non-uniform grids.

Concepts of the local truncation error and numerical entropy production have been used for both grid and method adaptation [31, 45, 17]. Different concepts could lead to different adapted results, especially, some concepts may not capture some phenomena. Hence, proper choosing of a adaptation approach would be crucial. In this study, wavelets are used for grid/method adaptation. Adaptation performance of this theory would be compared with above-mentioned two concepts.

Hyperbolic systems with non-convex fluxes would also be studied. These systems can explain important phenomena, such as: Euler equations of gas dynamic with a non-convex flux, polymer system used for simulation of polymer flooding processes in enhanced oil recovery and mechanical wave equations with non-convex fluxes. It will be shown that even though numerical solutions converge to weak form solutions (controlled by the local truncation error), they may not be physical (real) ones due to existence of complex waves in these problems [31]. In this work, wavelets are used to both grid and method adaptations for capturing properly physical solutions.

The Dubuc-Deslauriers (D-D) interpolating wavelets [11, 12, 36] are used for grid adaptation. This family has simple and straightforward algorithms with physical meaning. All calculations can then be done in the physical domain. The D-D wavelets use minimal spatial support for data reconstruction (approximation), and this is important, since: larger inter-distance in two sampled data is, smaller correlation between them exists [11, 12, 26].

Semi-discretized PDEs in spatial domain are solved in time by an explicit TVD integration method, such as the second order TVD Runge-Kutta scheme. As all spatio-temporal calculations are done in the physical domain, the method is simple and conceptually straightforward [26].

This paper is composed of ten parts. Section 2: explaining the main concept of multiresolution-based grid adaptation by interpolating wavelets; Section 3: presenting post-processing of adapted grids; Section 4: explaining the main concept of central or central-upwind scheme with cell-centered cell points; Section 5: evaluating of full-discrete and semi-discrete forms of conservation laws on non-uniform cells with shifted cell-centered points; section 6: estimation of the numerical entropy production; section 7: evaluating of local truncation errors on 1-D and 2-D non-uniform grids; section 8: explanation of the TVD conditions on non-uniform grids; section 9: representing some 1-D and 2-D numerical examples. The conclusion is presented at the end of the paper.

2. Multiresolution Analysis and Adaptation of 1-D Grids

2.1. Multiresolution Representation of 1-D Grids. A dyadic grid on spatial interval $[0,1]$ is assumed as follows [12]:

$$V_j = \left\{ x_{j,k} \in [0,1]: x_{j,k} = \frac{k}{2^j} \text{ for all } j \in \mathbb{Z}, \text{ and } k \in \{0,1,\ldots,2^j\} \right\},$$

where $j$ and $k$ are the resolution level (corresponding to spatial scale $1/2^j$) and spatial position ($k/2^j$), respectively. This definition of dyadic grid points $V_j$ in Eq. (2.1) leads to the condition $x_{j-1,k} = 2x_{j,k}$ and the multiresolution representation core: i.e., $V_j \subset V_{j+1}$. The points belongs to $V_{j+1} \setminus V_j$ is denoted by $W_j$, and it can be expressed as:

$$W_j = \left\{ x_{j+1,2k+1} \in (0,1): x_{j+1,2k+1} = \frac{2k+1}{2^{j+1}} \text{ for all } j \in \mathbb{Z}, \text{ and } k \in \{0,1,\ldots,2^j - 1\} \right\}.$$

So it can intuitively be concluded that: $V_j \oplus W_j = V_{j+1}$. This means the detail subspace $W_j$ with the approximation subspace $V_j$ can create (span) the next finer approximation subspace $V_{j+1}$ with more details. By repeating this decomposition procedure on $V_{J_{max}}$, it is obvious that:

$$V_{J_{max}} = V_{J_{min}} \oplus \sum_{i=0}^{N_d} W_{J_{min}+i}, \quad N_d = J_{max} - J_{min} - 1,$$
where, $J_{\text{max}}$, $J_{\text{min}}$ and $N_d$ denote the finest resolution, the coarsest resolution and number of decomposing levels, respectively [36].

A continuous function $f(x)$, defined on $V_{J_{\text{max}}}$, is assumed (i.e., $x \in V_{J_{\text{max}}}$). Regarding the multiresolution representation, the function can be decomposed as [12, 36]:

$$f(x) = \sum_{l=0}^{2^j-1} c_{J_{\text{min}},l} \phi_{J_{\text{min}},l}(x) + \sum_{j=J_{\text{min}}}^{J_{\text{max}}-1} \sum_{n=0}^{2^j-1} d_{j,n} \psi_{j,n}(x)$$

$$= P f_{J_{\text{min}}} + \sum_{j=J_{\text{min}}}^{J_{\text{max}}-1} Q f_j,$$

where $\phi(x)$ and $\psi(x)$ are scaling and wavelet functions, respectively; sets $\{\phi_{j,k}\}$ and $\{\psi_{j,l}\}$ denote dilated and shifted versions of $\phi(x)$ and $\psi(x)$, respectively. Coefficients $c_{j,l}$ and $d_{j,k}$ are respectively approximation and detail coefficients with resolution $j$. The operators $P f_j$ and $Q f_j$ show the approximation and detail information of $f(x)$, defined on grid points $V_j$ and $W_j$, respectively. The approximation on successive finer resolution $j+1$, can then be obtained as: $P f_{j+1} = P f_j + Q f_j$, [36].

In this study, the interpolating D-D wavelet of order $2M-1$ with support $\text{Supp}(\phi) = [-2M+1, 2M-1]$ is used. It can be obtained by auto-correlation of the Daubechies scaling function of order $M$ (having $M$ vanishing moments) [36]. By using this family, the transform coefficients in Eq. (2.2) can then be evaluated in the physical space with physical meanings. The approximation coefficients $c_{J_{\text{min}},l}$ are equal to sampled values of $f(x)$ at points $x_{J_{\text{min}},l} \in V_{J_{\text{min}}}$, and the detail coefficients $d_{j,n}$ are differences between $f(x_{j+1,2n+1})$ and $P f_{j+1}(x_{j+1,2n+1})$, or $d_{j,n} = f(x_{j+1,2n+1}) - P f_{j+1}(x_{j+1,2n+1})$ (for the D-D wavelets, $P f_{j+1}(x_{j+1,2n+1})$ can be obtained by the local Lagrange interpolation on points $V_j$); for details see [11, 12, 36].

In the interpolating wavelet theory, transform coefficients and grid points have one-to-one correspondence. This feature leads to a simple 1-D grid adaptation algorithm. For $f(x) \in V_{J_{\text{max}}}$, a pre-defined threshold $\epsilon$ is assumed, then, in each level of resolution $j \in \{J_{\text{min}}, J_{\text{min}}+1, \ldots, J_{\text{max}} - 1\}$, points $x_{j+1,2n+1} \in W_j$ are omitted from original calculating grid points, if corresponding detail coefficients, $d_{j,n}$, satisfy the condition $d_{j,n} < \epsilon$. This means that the function at the point $x_{j+1,2n+1}$ is smooth enough, so that its contribution in the approximation, $d_{j,n} \psi_{j,n}(x)$, can be neglected (see Eq. (2.2)). Donoho [16] showed that such truncation error is in accordance with threshold values. Finally it should be mentioned that the predefined threshold can be either level-dependent or not. One popular level-dependent threshold is $\epsilon_j = \epsilon_0/2^{J_{\text{max}}-j-1}$ for $j \in \{J_{\text{min}}, J_{\text{min}}+1, \ldots, J_{\text{max}} - 1\}$. This means a smaller threshold is assumed for finer resolution. To use a constant threshold value for all resolution levels, it is better to normalize the detail coefficient of resolution $j$ as: $d_{j,n} = d_{j,n}/f_{\text{ref}}^{eff}$. In this study constant pre-defined threshold with normalized factor $f_{\text{ref}}^{eff} = \max\{f(x_j)\}$ is used.

The presented adaptation procedure was shown to be efficient in the resolution of scalar functions. For resolution of functions in vector system, the previous procedure is modified to reflect the solutions' behaviors of all equations. Namely, the resultant adapted grid is simply superposition of all adapted grids; which each adapted grid corresponds to a functions of the vector system.

### 2.2. Multiresolution representation of 2-D grids

Consider a uniform grid of spatial locations $(x, y) \in [0, 1] \times [0, 1]$; a set of points belonging to subspace $V_j$ can be defined as [47]:

$$V_j = \{(x_{j,k}, y_{j,l}) : x_{j,k} = k2^{-j}, y_{j,l} = l2^{-j}\} \quad \text{for all} \quad j, k, l \in \mathbb{Z},$$

where $j$ denotes resolution level and $J_{\text{min}} \leq j \leq J_{\text{max}}$; coefficients $k$ & $l$ measure spatial locations. Figure 1 illustrates a schematic representation of points belonging to $V_j$ ($\ast \in V_j$) and $V_{j+1}$ ($\bullet \cup \circ \in V_{j+1}$) for case $j = 2$.

Same as the 1-D case, for 2-D information a detail subspace $W_j$ belonging to subspace $V_{j+1} \setminus V_j$ can be defined. These points are shown in Figure 1 with hollow circles; i.e.: $\circ \in W_j$. In this detail sub-space, three point types $s_1$, $s_2$ and $s_3$ can be distinguished, as:

- (1) points in even-numbered points in the $x$ direction and odd-numbered in the $y$ direction; i.e., set $s_1 = \{(x_{j+1,2k}, y_{j+1,2l+1})\}$,
Figure 1. Spatial locations of points belonging to subspaces $V_2$ and $V_3$, where $\bullet \in V_2$ and $\circ \cup \circ \in V_3$, for $(x, y) \in [0, 1] \times [0, 1]$.

(2) points in odd-numbered points in the $x$ direction and even-numbered in the $y$ direction; i.e., set $s_2 = \{(x_{j+1,2k+1}, y_{j+1,2l+1})\}$,

(3) odd-numbered grid points in both directions: $s_3 = \{(x_{j+1,2k+1}, y_{j+1,2l+1})\}$.

Based on subspaces $V_j$ and $W_j$, a 2-D multiresolution analysis can be obtained as the 1-D case. For point sets $\{(x_{j,k}, y_{j,l}) \in s_i : i \in \{1, 2, 3\}\}$, four wavelet coefficients $\{d_{j,(k,l)}^{i} : i \in \{1, 2, 3, 4\}\}$ can be defined as:

(1) For points $s_1$, wavelet coefficients are $d_{j,(k,l)}^{1}$; they measure local variations in vertical $(y)$ direction,

(2) For points $s_2$, wavelet coefficients are $d_{j,(k,l)}^{2}$; they measure local variations in horizontal $(x)$ direction,

(3) For points $s_3$, two sets of wavelet coefficients can be defined as $d_{j,(k,l)}^{3}$ and $d_{j,(k,l)}^{4}$; they measure respectively local variations in $x$ and $y$ directions.

These coefficients can be evaluated by the 1-D wavelet transform algorithm [47]. For 2-D grid adaptation, by considering a threshold value, grid points can then be adapted as the $1-D$ case [47].

3. Post processing the adapted grid points

Once adapted grids are obtained via the aforementioned wavelet-based procedures, grids are modified considering resolution level of each point; this is done to guarantee gradual density variation of grid points.

3.1. 1-D grid modification. The procedure for modification of 1-D adapted grids can be summarized as:

(1) Setting the level resolution ($j$) equal to the finest resolution, i.e.: $j = J_{\text{max}} - 1$,

(2) Considering points belonging to the detail space of resolution $j$; i.e., points: $\{x_{j+1,2k+1} = (2k + 1)/2^{j+1} \in W_j\}$,

(3) Existence controlling of $N_s$ neighbor-points for each side of the point $x_{j,k} \in W_j$ at the same level of resolution, i.e., points: $\{x_{j+1,2(k+i)+1} : i \in \{-N_s, -N_s + 1, \cdots , N_s\}, \ i \neq 0\}$,

(4) Existence controlling of $N_s$ neighbor-points for each side of the point $x_{j,k} \in W_j$ at the successive coarser resolution (subspace $W_{j-1}$). This step is only done for levels $j > J_{\text{min}}$.

(5) Adding the extra points from steps 3 and 4 to the adapted grid (updating the modified adapted grid),

(6) If $j > J_{\text{min}}$, set $j = j - 1$ and then following steps 2 through 6,

(7) If $j = J_{\text{min}}$, consider only steps 2 and 3. These points are added to the updated adapted grid points, as the final stage.

The already-mentioned post processing procedure is illustrated in Fig. 2; there, distribution of adapted grid points is shown in different levels of resolution. Solid points correspond to assumed
Figure 2. Modification of an adapted grid; solid points and hollow circles correspond to adapted and extra added points, respectively. For grid modification stage, we assume: $N_s = N_c = 1$.

Figure 3. Adding procedure in the same resolution for point types $s_1$, $s_2$ or $s_3$, where $\bullet \in V_2$, $\circ \in V_3$, $\circ \in W_2$ and bright gray solid points are the added points belonging to $W_2$.

(Original) adapted points and the hollow ones associate with points obtained after the post-processing procedure. This modification leads to locally semi-uniform distributions.

3.2. 2-D grid modification. The 2-D grid modification will be done by controlling and adding new points in the same and coarser resolution levels, as the 1-D grid case. For this purpose, for different points $s_1$, $s_2$ or $s_3$ (Fig. 1), different adding procedures will be considered.

3.2.1. Adding extra new points by the multiresolution concept.

Adding in the same resolution. Depending on point type ($s_1$, $s_2$ or $s_3$), different inserting procedures will be considered; different neighbor points will be added for each point $s_1$, $s_2$ or $s_3$, see Fig. 3. There, for an adapted point belonging to $W_2$, new neighbor points of $W_2$ are locally added. In this figure, bright gray solid points are the new extra points added around each point $s_1$, $s_2$ or $s_3$. In this figure, only one row or column of the nearest points is considered for modification stage. In general, more surrounding points of $W_2$ can be considered for each point $s_1$, $s_2$ or $s_3$.

Adding in the successive coarser resolution. In this case, for a point belonging to $W_j$ some new surrounding points including in $W_{j-1}$ are added. To guarantee symmetry and gradual concentration of modified adapted points, it may also be necessary to consider some extra points from sub-space $V_{j-1}$. For different points $s_1$, $s_2$ and $s_3$, different neighbor points will be considered. Such grid checking/adding
Figure 4. Adding procedure in the successive coarser resolution for point types $s_1$, $s_2$ or $s_3$, where $\bullet \in V_2$, $\bullet \cup \circ \in V_3$, $\circ \in W_2$ and bright gray solid points are the added points belonging to $V_2$.

procedures are shown in Fig. 4. In each illustration, points denoted by $w$ belong to sub-space $W_{j-1} = W_{2-1}$ and points with name $v$ belongs to the sub-space $V_{j-1} = V_{2-1}$. In these figures, only one row or column of the nearest distance is considered.

3.3. Post-processing 2-D adapted grids. The modification algorithm for 2-D grids is generally similar to the 1-D case; it can be reviewed as follows:

1. Set the level resolution $j$ (where $J_{\text{min}} \leq j \leq J_{\text{max}} - 1$) equal to the finest resolution level; i.e., $j = J_{\text{max}} - 1$ (with spatial sampling steps $dx = dy = 1/2^{J_{\text{max}}-1}$).
2. Consider the set of points corresponding to detail sub-space of resolution $j$; i.e., points $\{(x_{j,k}, y_{j,l}) \in W_j\}$.
3. Control $N_x$ neighbor-rows or columns of surrounding points for each side of the point $(x_{j,k}, y_{j,l})$; this control is done at the same level of resolution.
4. Control $N_x$ neighbor-rows or columns of points for each side of the point $(x_{j,k}, y_{j,l})$; this is done at the successive coarser resolution (subspace $W_{j-1}$).
5. Regarding step 3, add the surrounding points in subspace $W_j$.
6. Considering step 4, add the surrounding points in subspace $W_{j-1}$; in this stage, it may also be needed to add some new points in the sub-space $V_{j-1}$ (as explained before).
7. If $j > J_{\text{min}}$, set $j = j - 1$ and go back to step 2, else go to step 8.
8. If $j = J_{\text{min}}$, consider only steps 2, 3 and 5.

4. Central and upwind-central high resolution schemes of spatially second order accuracy

Here, the semi-discretized form of a scalar first-order hyperbolic system, $u_t + F(u)_x = 0$, will be provided for the KT method [32] and two other improvements of this scheme (to have less numerical dissipation) [30, 29]. By following the concept of the Reconstruction/Evolution/Projection (REP) procedure [32] (see Figure 5) for cell-centered non-uniform cells, it is easy to show that for $j^{th}$ cell, the semi-discrete form is:

\[
\frac{du_j}{dt} + \frac{F_{j+1/2}^* - F_{j-1/2}^*}{\Delta x_j} = 0, \quad F_{j+1/2}^* := F\left(u_{j+1/2}^*\right),
\]

where $u_j$ denotes cell center solution (estimated state values) on $j^{th}$ cell; $\Delta x_j := x_{j+1/2} - x_{j-1/2}$ is the $j^{th}$ cell length; and $F(u_{j+1/2}^*)$ shows a proper combination of left and right reconstructed state values and corresponding fluxes at cell edges $x_{j+1/2}$. The left and right reconstructed state values are shown respectively by $u_{j+1/2}^L$ and $u_{j+1/2}^R$. These values can be evaluated as $u_{j+1/2}^L := u_i + (u_{x,j})_{j} (x_{j+1/2} - x_j)$ and $u_{j+1/2}^R := u_{j+1} - (u_{x,j})_{j+1} (x_{j+1} - x_{j+1/2})$ for spatially second order methods, where $(u_{x,j})_{j}$ denotes a limited slope at point $x_j$. 
The two improvements of the central KT scheme are central-upwind methods, noting here by $M1$ and $M2$. They use two different maximum local propagation speeds for right and left directions at cell edges $x_{j+1/2}$. The speeds are shown by $a_{R,j+1/2}$ and $a_{L,j+1/2}$ for right and left directions, respectively. Such speed distinguishing leads to a narrower non-smooth zone around cell edge $x_{j+1/2}$, and therefore less dissipative schemes. The scheme $M2$ is an improvement of the $M1$ method using a narrower non-smooth zone; for more details see [30, 29].

Expressed in terms of the left and right states $F_{i+1/2}^{L/R} := F(u_{i+1/2}^{L/R})$, the reconstructed fluxes for the KT, $M1$ and $M2$ schemes are:

1. The central KT scheme:

   \[ F_{i+1/2}^* := [F_{i+1/2}^R + F_{i+1/2}^L] - a_{i+1/2} \left[ u_{i+1/2}^R - u_{i+1/2}^L \right]. \]

2. For the central-upwind method $M1$:

   \[ F_{i+1/2}^* := \frac{a_{i+1/2}^R u_{i+1/2}^R - a_{i+1/2}^L u_{i+1/2}^L}{a_{i+1/2}^R u_{i+1/2}^R + a_{i+1/2}^L u_{i+1/2}^L} + a_{i+1/2}^L a_{i+1/2}^R \left\{ \frac{u_{i+1/2}^R - u_{i+1/2}^L}{a_{i+1/2}^R - a_{i+1/2}^L} \right\}. \]

3. For the central-upwind method $M2$:

   \[ F_{i+1/2}^* := \frac{a_{i+1/2}^R u_{i+1/2}^R - a_{i+1/2}^L u_{i+1/2}^L}{a_{i+1/2}^R u_{i+1/2}^R + a_{i+1/2}^L u_{i+1/2}^L} + a_{i+1/2}^L a_{i+1/2}^R \left\{ \frac{u_{i+1/2}^R - u_{i+1/2}^L}{a_{i+1/2}^R - a_{i+1/2}^L} \right\}. \]

5. **Central high-resolution schemes on non-centered non-uniform 1-D cells**

By using the cell-centered non-uniform cells, the TVD stability condition does not satisfy without altering definition of slope/flux limiters. To preserve the TVD condition without limiter definition modification, it will be shown that by shifting cell centers in some cells, the TVD condition can be reached. These cells act as transmitting cells connecting two surrounding uniform cells of different lengths. By such cell-center shifting, a new source of the truncation error will be introduced. Effects of this new error will be discussed.

Considering the REP, the full-discrete and semi-discrete forms of the scalar hyperbolic equation $u_t + F(u)_x = 0$ will be provided on non-centered non-uniform cells. For generality, it is assumed that cell centers $x_j$ are not located at cell centers; so, the left and right cell edge positions are: $x_{j-1/2} := x_j + (1 - p_j) \Delta x_j$ and $x_{j+1/2} := x_j + p_j \Delta x_j$, where $0 < p_j < 1$ (for $p_j = 1/2$ the cell center $x_j$ is the middle point of $j^{th}$ cell).
5.1. The reconstruction and evolution stages.

(1) The spatio-temporal volume $\Delta x_{j+1/2} \times \Delta t$ is considered, where $\Delta x_{j+1/2} := x_{j+1/2}^n - x_{j+1/2}^{n-1}$, and $\Delta t := t^{n+1} - t^n$. By averaging on this volume and considering the midpoint rule in time, the solution $w_{j+1/2}^{n+1} := w(x_{j+1/2}, t^n)$ can be obtained as:

\[
w_{j+1/2}^{n+1} = \frac{1}{\Delta x_{j+1/2}} \int_{x_{j+1/2}^{n-1}}^{x_{j+1/2}^n} u(x, t^n) \, dx
\]

\[
= \frac{1}{\Delta x_{j+1/2}} \int_{x_{j+1/2}^{n-1}}^{x_{j+1/2}^n} u(x, t^n) \, dx - \frac{1}{\Delta x_{j+1/2}} \int_{x_{j+1/2}^{n-1}}^{x_{j+1/2}^n} \left[ F(u_{j+1/2}^n) - F(u_{j+1/2}^{n-1}) \right] \, dt
\]

\[
(5.1)
\]

(2) The spatio-temporal volume $\Delta x_j \times \Delta t$ is considered, where $\Delta x_j := x_{j+1/2} - x_{j-1/2}$. By averaging on this volume and considering the midpoint rule in time, evolved solution $w_{j+1}^{n+1} := w(x_j, t^{n+1})$ is:

\[
w_{j+1}^{n+1} = \frac{1}{\Delta x_j} \int_{x_{j-1/2}^{n-1}}^{x_{j+1/2}^n} u(x, t^n) \, dx
\]

\[
= \frac{1}{\Delta x_j} \int_{x_{j-1/2}^{n-1}}^{x_{j+1/2}^n} u(x, t^n) \, dx - \frac{1}{\Delta x_j} \int_{x_{j-1/2}^{n-1}}^{x_{j+1/2}^n} \left[ F(u_{j+1/2}^n) - F(u_{j-1/2}^n) \right] \, dt
\]

\[
(5.2)
\]

(3) Regarding volume $\Delta x_{j-1/2} \times \Delta t$, where $\Delta x_{j-1/2} := x_{j-1/2}^n - x_{j+1/2}^{n-1}$, evolved solution $w_{j-1/2}^{n+1} := w(x_{j-1/2}, t^{n+1})$ can be obtained with the similar procedure explained in (5.1).

Considering $w_{j+1/2}^{n+1}$, $w_{j+1}^{n+1}$, and $w_{j-1/2}^{n+1}$ definitions in Eqs. (5.1) and (5.2), a piecewise-linear approximation on the staggered grid in the evolution stage is (see Figure (5)):

\[
\hat{w}(x, t^{n+1}) := \sum \left\{ w_{j+1/2}^{n+1} + (u_x)_{j+1/2}^{n+1} (x - x_{j+1/2}) 1_{[x_{j+1/2}, x_{j+1/2}+\Delta x_j]} + w_{j+1}^{n+1} 1_{[x_{j+1/2}, x_{j+1/2}+\Delta x_j]} \right\},
\]

where $u_x$ denote limited derivatives and $1_{[a,b]}$ shows a unit function on spatial interval $[a, b]$.

5.2. The projection stage. The fully discrete second-order central scheme can be obtained by averaging $\hat{w}(x, t^{n+1})$ on interval $[x_{j-1/2}, x_{j+1/2}]$, as:

\[
w_{j}^{n+1} = \frac{1}{\Delta x_j} \int_{x_{j-1/2}}^{x_{j+1/2}} \hat{w}(x, t^{n+1}) \, dx
\]

\[
= \frac{1}{\Delta x_j} \left\{ \frac{1}{2} \left[ \Delta t \left( F_{j+1/2}^{n+1} - F_{j-1/2}^{n} \right) + \Delta x_j (2u_x)_{j+1/2} + 2u_j \right) \right. \\
+ 2\Delta x j \left( x_{j-1} - x_{j-1/2} \right) (u_x)_{j-1/2} - \Delta x_j (x_{j+1} - x_{j+1/2}) (u_x)_{j+1/2} - u_{j-1} + u_{j+1} \right) \\
- \Delta t^2 \left( a^n_{j-1/2} x_{j-1/2} + (u_x)_{j-1/2}^n - 2(u_x)_{j+1/2} - (u_x)_{j+1/2}^n \right) + \Delta t^2 \left( a^n_{j+1/2} x_{j+1/2} + (u_x)_{j+1/2}^n - 2(u_x)_{j+1/2} - (u_x)_{j+1/2}^n \right) \right\}
\]

\[
(5.3)
\]
5.3. The semi-discrete form. Regarding the full discrete form (5.3), for small enough $\Delta t$ values, the following approximations can be assumed: 1) ignoring of terms corresponding to $\Delta t$ central/central-upwind schemes.

The entropy function $\psi_j$ can be controlled by small-enough spatial sampling steps $\Delta x_j$. This can be done by a proper grid adaptation procedure around high-gradient zones, 

Since $u_{j+1/2} := u_j + p_j \Delta x_j (u_x)_j^n$ and $u_{j-1/2} := u_j - (1 - p_j) \Delta x_j (u_x)_j^n$, (5.4) can be rewritten as:

$$\frac{\Delta u_j}{\Delta t} + F^*_j + 1/2 - F^*_j - 1/2 \approx Q,$$

where $F^*_j + 1/2 := F^*_{j+1/2} + F^*_{j+1/2} - a_{j+1/2} \left( u^R_{j+1/2} - u^F_{j+1/2} \right)$ and $Q := -\left( \frac{1}{2} - p_j \right) (u_x)_j^n \frac{\Delta x_j}{\Delta t}$.

We conclude this section with the following remarks regarding ((5.5)).

1. The term $Q$ acts as a new source of the truncation error,
2. Effects of $\Delta t$ in $Q$ can be controlled by small-enough spatial sampling steps $\Delta x_j$. This can be done by a proper grid adaptation procedure around high-gradient zones,
3. If $x_j$ is in the middle of the $j^{th}$ cell, i.e. $p = 1/2$, the truncation error $Q$ vanishes: $Q = 0$,
4. Around discontinuities, since limited slopes $(u_x)_j^n$ are nearly zero, the error $Q$ approaches to zero,
5. Average solution on $j^{th}$ cell is $(u^*_{j})_{ave} = u^*_j + Q \times \Delta t$.

6. Numerical entropy production of central/central-upwind schemes

Calculation of numerical entropy production helps one to control quality/uniqueness of numerical results especially for ones without exact solutions. This is because, theoretically, the numerical entropy production is zero in smooth regions while less than zero around shocks and discontinuities.

Puppo [43, 44] showed how to estimate the numerical entropy production for staggered central high resolution schemes. After these works, Puppo et al. [45] improved the previous works for non-staggered central/central-upwind schemes.

The entropy function $\eta(u)$ with flux $\psi(u)$ satisfies the relationship:

$$\eta_t + \psi_x(u) \leq 0, \quad \eta := \eta(x).$$

Following works [43, 44], by integrating this relationship in spatio-temporal volume $[x_j - \Delta x_j, x_j + \Delta x_j] \times [\ell^n, \ell^{n+1}]$, and discretizing the inequality by considering the REP concept (used in the KT, M1 or M2 method), we have:

$$\eta^{n+1}_j - \left( \eta^n_j - \frac{1}{\Delta x_j} \int_{\ell^n}^{\ell^{n+1}} \left[ \psi^*_j + 1/2 - \psi^*_j - 1/2 \right] \right) \leq 0, \quad \eta^n_j := \eta(x_j, \ell^n).$$

Based on this inequality, the density of numerical entropy production at $x_j, S^n_j$, can be defined as:

$$S^n_j = \frac{1}{\Delta x_j} \left\{ \eta^{n+1}_j - \left( \eta^n_j - \frac{1}{\Delta x_j} \int_{\ell^n}^{\ell^{n+1}} \left[ \psi^*_j + 1/2 - \psi^*_j - 1/2 \right] \right) \right\}, \quad S^n_j := S(x_j, \ell^n).$$
where $v_{j+1/2}^- := v_{j+1/2}^+ \left( \left( u_{j+1/2}^L \right)^n \left( u_{j+1/2}^R \right)^n \right)$.

In Eqs. (6.1) and (6.2), the terms in parentheses estimate the evolved values of the entropy at the next time step (this can be done with the KT, M1 or M2 method). Theoretically, as mentioned before, the parameter $S^0$ is zero in smooth regions, however, in numerical solutions, it may be slightly less or more than zero [45]. This kind of entropy definition cannot efficiently detect some phenomena like contact discontinuities even by increasing number of resolution levels (see the Sod and the Lax problems in the Euler gas dynamic system). To remedy this, other approaches like the entropy viscosity scheme can be recommended [21].

7. Local truncation error on non-uniform cells

7.1. One dimensional systems. The aim of this section is to estimate local truncation errors in hyperbolic systems of conservation laws with the governing equation: $u_t + F(u)_x = 0$, $u(x, t) = u_0(x)$.

It is easy to show that solution of this equation is also a weak solution satisfying the relationship:

$$E(u, \phi) = - \int_{t=0}^{\infty} \int_X \left( u(x, t) \phi(x, t) + F(u(x, t)) \phi_x(x, t) \right) dxdt$$

(7.1)

where $\phi(x, t)$ is a test function that $\phi(x, t) \in C^0_{0}(X \times [0, \infty])$. One effective and practical way to measure convergence of a numerical solution $u(x, t)$ is to check how much it fails to satisfy (7.1); this can be measured by evaluating $E(u, \phi)$ [28, 27]. For a convex scalar hyperbolic system $(dF(u)/du) > \alpha > 0$, the function $E(u, \phi)$ measures point-wisely real errors in weak $\text{Lip}^-$ norm; this norm is developed by Tadmor to measure errors in simulation of the nonlinear scalar conservation laws [52, 53]. In nonlinear conservation laws, discontinuous solutions develop typically; in these cases standard methods of error estimation are not valid. Such approaches consider the Taylor expansion which is based on the smoothness assumption.

The final point is the relationship of this local truncation error with the weak $\text{Lip}^-$ norm theory studied for convex scalar one dimensional hyperbolic systems. Numerical results confirm that it is also an effective tool for systems of one dimensional PDEs, 2-D problems, and even non-convex systems.

To be sure that $E(u, \phi)$ measures errors of $u(x, t)$, some constraints should be met by the test function $\phi(x, t)$: 1) the space spanned by the test functions should have a higher order accuracy; so that the order of error $E(u, \phi)$ is affected by solutions $u(x, t)$; 2) the test function have continuous derivatives; 3) to estimate locally the error $E(u, \phi)$, the test function has to be compact support in spatio-temporal domains.

We now consider the truncation error $E^n_j := E(u^n, \phi)$ for the piecewise constant approximate solution $u^n(x, t)$

$$u^n(x, t) = \sum_{j,n} u^n_{j} 1_{C_j \times T^n}(x, t), \quad C_j \times T^n := [x_{j-1/2}, x_{j+1/2}] \times \left[ t^{n-1/2}, t^{n+1/2} \right],$$

The compact test function $\phi(x, t)$ is assumed to be $\phi^n_j(x) := B_j(x) B^n(t)$, where $B_j(x)$ and $B^n(t)$ are the quadratic B-splines with center points $x = x_j$ and $t = t^n$. In this case, supports of $B_j(x)$ and $B^n(t)$ belong to $x \in [x_{j-3/2}, x_{j+3/2}]$ and $t \in [t^{n-3/2}, t^{n+3/2}]$, respectively.

To obtain higher-order B-splines, the recurrence feature between higher-order and lower-order B-splines: recall that the B-spline of order $k$ with centered position $j$ can be obtained as

$$B_{j,k} = B_{j,k-1} w_{j,k} + B_{j+1,k} (1 - w_{j+1,k}), \quad w_{j,k}(x) := \begin{cases} \frac{x-x_j}{x_{j+k+1}-x_j}, & \text{if } x_j \neq x_{j+k+1}, \\ 0, & \text{otherwise}. \end{cases}$$

(7.2)

B-splines should also satisfy the partition of unity condition $\sum_j B_{j,k}(x) \equiv 1$. For second order spline (7.2) leads to:

$$B_{j,2} = B_{j,1} w_{j,2} + B_{j+1,2} (1 - w_{j+1,2}).$$
Let us consider three successive cells, with length ratios, \( a = \Delta x_{j-1}/\Delta x_j \) and \( b = \Delta x_{j+1}/\Delta x_j \), where \( \Delta x_j := x_{j+1/2} - x_{j-1/2} \) denotes \( j^{th} \) cell length with cell middle point \( x_j \). It is easy to show that \( B_j(x) \) is:

\[
B_j(x) = \begin{cases} 
\frac{(2x-x_j+3\Delta x)2}{4(1+a)\Delta x^2}, & \text{if } x_j-3/2 \leq x \leq x_j-1/2, \\
\frac{(2x-x_j+3\Delta x)2}{4(1+a+b)\Delta x^2} - 4(a-b)(x-x_j)^2, & \text{if } x_j-1/2 \leq x \leq x_j+1/2, \\
\frac{(2x-x_j+3\Delta x)2}{4(1+a+b)\Delta x^2} - 4(a-b)(x-x_j)^2 - 4(a-b)(x-x_j+\Delta x)^2, & \text{if } x_j+1/2 \leq x \leq x_j+3/2, \\
0, & \text{otherwise,}
\end{cases}
\]  

(7.3) where: \( \Delta x := \Delta x_j; x_j = (x_{j-1/2}+x_{j+1/2})/2; x_j-3/2 := x_j - \Delta x(1/2+a); x_j-1/2 := x_j - \Delta x/2; x_j+1/2 := x_j + \Delta x/2; x_j+3/2 := x_j + \Delta x(1/2+b) \).

It is easy to check that on uniform grids, where \( a = b = 1 \), the above-mentioned B-spline definition leads to the B-spline on uniform grids. Since in time-domain, in this study, a constant time-step is used, the definition of \( B^n(t) \) is the quadratic B-spline on uniform grids with step \( \Delta t \), as:

\[
B^n(x) = \begin{cases} 
\frac{(t-t^n+nx/2)^2}{2\Delta t^2}, & \text{if } t_{n+1/2} \leq t \leq t_{n+1} - \frac{1}{2}, \\
\frac{3}{4} - \frac{(t-t^n+nx/2)^2}{2\Delta t^2}, & \text{if } t_{n+1} - \frac{1}{2} \leq t \leq t_{n+1}, \\
\frac{(t-t^n+nx/2)^2}{2\Delta t^2}, & \text{if } t_{n+1} \leq t \leq t_{n+1} + \frac{1}{2}, \\
0, & \text{otherwise,}
\end{cases}
\]  

(7.4) here: \( t_{n+1/2} := (t_n + \Delta t/2) \) and \( t_{n+1} := (t_n + \Delta t) \).

For spatially non-uniform grid points, it is straightforward to show that the local truncation error can be expressed as:

\[
E^n_j = \Delta x U^n_j + \Delta t F^n_j,
\]

where \( U^n_j \) is expressed in terms of the time differences \( \Delta u^n_a := 1/2 (u^n_{a+1} - u^n_a) \),

\[
U^n_j = \frac{a^2}{3(a+1)} \Delta u^n_{j-1} + \frac{(a3b+2b+1)}{3(a+1)(b+1)} \Delta u^n_j + \frac{b^2}{3(b+1)} \Delta u^n_{j+1},
\]

and \( F^n_j \) is expressed in terms of the time averages \( \mu F^n_a := 1/6 (F^n_{a-1} + 4F^n_a + F^n_{a+1}) \),

\[
F^n_j = -\frac{(b-a)}{(a+1)(b+1)} \mu F^n_{j-1} + \frac{a}{a+1} \mu F^n_{j+1} + \frac{b}{b+1} \mu F^n_{j+1}.
\]

7.2. Two-dimensional problems. Let us consider a two dimensional scalar hyperbolic PDE as:

\[
u_x + F(u)_x + G(u)_y = 0, \quad u_0(x, y) := u(x, y, t = 0).
\]

As in the 1-D case, the weak solution of (7.6) satisfies

\[
E(u, \phi) = -\iint_T u(x, y, t) \phi_t(x, y, t) + F(u(x, y, t)) \phi_x(x, y, t) + G(u(x, y, t)) \phi_y(x, y, t) \, dx \, dy \, dt \\
+ \int_{\Omega} u(x, y, t) \phi_t(x, y, t) \, dx \, dy = 0.
\]

We quantify the truncation error \( E^n_{i,j,k} := E(u^\Delta, \phi) \) for the piecewise constant approximate solution, \( u^\Delta(x, y, t) \),

\[
u^\Delta(x, y, t) := \sum_{j,k,n} u^n_{j,k} C_{j,k} \times T^n := \{x_{j-1/2}, x_{j+1/2}\} \times \{y_{k-1/2}, y_{k+1/2}\} \times \{t_{n-1/2}, t_{n+1/2}\}.
\]

The test function \( \phi(x, y, t) \) can also be defined as: \( \phi^n_{i,j,k}(x, y, t) := B_j(x)B_k(y)B^n(t) \). Where all functions \( B_j(x) \), \( B_k(y) \) (from (7.3)), and \( B^n(t) \) (from (7.4)) denote quadratic B-splines. Let us assume the successive cell length ratios in \( x \) and \( y \) directions to be: \( a = \Delta x_{j-1}/\Delta x_j \), \( b = \Delta x_{j+1}/\Delta x_j \} \) and \( c = \Delta y_{k-1}/\Delta y_k \), \( d = \Delta y_{k+1}/\Delta y_k \).
where: $\Delta x = \Delta x_j := x_{j+1/2} - x_{j-1/2}$ and $\Delta y = \Delta y_k := y_{k+1/2} - y_{k-1/2}$. By these assumptions, the local truncation error $E^m_{i,k}$ can be represented as:

\begin{equation}
E^m_{i,k} = \{ \Delta x \Delta y \mathcal{U}_{i,k}^m + \Delta t \Delta y F^m_{i,j} + \Delta t \Delta x G^m_{i,k} \},
\end{equation}

where $\mathcal{U}_{i,k}^m$ is expressed in terms of the time differences $\Delta u_{i,j}^n := 1/2 \left( u_{i,j+1}^{n+1} - u_{i,j}^{n-1} \right)$

\begin{equation}
\mathcal{U}_{i,k}^m = \frac{a^2 d^3}{9(a+1)b(b+1)} \Delta u_{i,j-1,k}^{n-1} + \frac{a^2(a(3b+2)+2b+1)u_{i,j-1,k}^{n-1}}{9(a+1)^2(b+1)} \Delta u_{i,j-1,k}^{n-1} + \frac{b^2(a(3b+2)+2b+1)}{9(a+1)^2(b+1)^2} \Delta u_{i,j+1,k+1}^{n-1} + \frac{a^2(a(3b+2)+2b+1)}{9(a+1)^2(b+1)^2} \Delta u_{i,j-1,k}^{n-1} + \frac{b^2(a(3b+2)+2b+1)}{9(a+1)^2(b+1)^2} \Delta u_{i,j+1,k+1}^{n-1},
\end{equation}

and $F^m_{i,j}$ and $G^m_{i,k}$ are expressed in terms of the time averages $\mu Z_{i,j}^{n-1} + 2Z_{i,j}^n + Z_{i,j}^{n+1}$.

\begin{align*}
F^m_{i,j} &= - \frac{c^3}{3(a+1)^2(3a+b+1)} \mu F_{i-1,j}^{n-1} + \frac{(a-b)c^3}{3(a+1)^2(3a+b+1)} \mu F_{i+1,j}^{n-1} + \frac{bc^3}{3(a+1)^2(3a+b+1)} \mu F_{i,j}^{n-1} - \frac{a(2b+a(3b+2)+1)}{3(a+1)^2(b+1)} \mu F_{i,j}^{n-1} - \frac{ad^3}{3(a+1)^2(b+1)} \mu F_{i,j}^{n-1} + \frac{(a-b)(2b+a(3b+2)+1)}{3(a+1)^2(b+1)^2} \mu F_{i,j}^{n-1} + \frac{d^3}{3(a+1)^2(b+1)^2} \mu F_{i,j}^{n-1} + \frac{a^2(b+1)}{3(a+1)^2(b+1)^2} \mu F_{i,j}^{n-1} + \frac{b^2}{3(a+1)^2(b+1)^2} \mu F_{i,j}^{n-1} + \frac{d^3}{3(a+1)^2(b+1)^2} \mu F_{i,j}^{n-1},
\end{align*}

\begin{align*}
G^m_{i,k} &= - \frac{c^3}{3(a+1)^2(3a+b+1)} \mu G_{i,j-1,k}^{n-1} + \frac{(a-b)c^3}{3(a+1)^2(3a+b+1)} \mu G_{i,j+1,k}^{n-1} + \frac{bc^3}{3(a+1)^2(3a+b+1)} \mu G_{i,j}^{n-1} - \frac{a(2b+a(3b+2)+1)}{3(a+1)^2(b+1)} \mu G_{i,j}^{n-1} - \frac{ad^3}{3(a+1)^2(b+1)} \mu G_{i,j}^{n-1} + \frac{(a-b)(2b+a(3b+2)+1)}{3(a+1)^2(b+1)^2} \mu G_{i,j}^{n-1} + \frac{d^3}{3(a+1)^2(b+1)^2} \mu G_{i,j}^{n-1} + \frac{a^2(b+1)}{3(a+1)^2(b+1)^2} \mu G_{i,j}^{n-1} + \frac{b^2}{3(a+1)^2(b+1)^2} \mu G_{i,j}^{n-1} + \frac{d^3}{3(a+1)^2(b+1)^2} \mu G_{i,j}^{n-1}.
\end{align*}

8. The TVD Property of Semi-Discrete Schemes

8.1. Global and local TVD conditions. To control the development of spurious oscillations in numerical simulation of hyperbolic systems, it is necessary to show that a monotone (non-increasing or non-decreasing) profile remains monotone during time evolution. To achieve high-order of accuracy, a relaxed monotonicity condition — so called Total Variation Diminishing (TVD) is sought [22, 34]. We consider a general class of non-linear semi-discrete scalar scheme which takes the incremental form

\[ \frac{du_i}{dt} = C^+_{i+1/2}(u_{i+1} - u_i) - C^-_{i-1/2}(u_i - u_{i-1}), \quad u_i := u_i(t). \]

A general (global) TVD condition derived in [22] is expressed in terms of the global positivity condition, e.g.,[24, 41, 46]:

\begin{equation}
C^+_{i+1/2} \geq 0, \quad C^-_{i-1/2} \geq 0, \quad C^+_{i+1/2} + C^-_{i+1/2} \leq 1.
\end{equation}

Local TVD conditions were studied in [51] where it was shown that for the TVD property to hold, it suffice to verify the positivity condition (8.1) in the extreme cells and their surrounding grid points. In the present context of (4.1), these local TVD conditions amount to:

1. In extreme points, for both uniform and non-uniform cells, the numerical fluxes $F^*_{i\pm1/2}$ satisfy (see [51], Lemma 2.1):
   (a) at maximum values $u_i : F^*_{i+1/2} \geq F^*_{i-1/2}$;
   (b) at minimum values $u_i : F^*_{i+1/2} \leq F^*_{i-1/2}$.
(2) Osher [42] showed that in order to satisfy the positivity conditions, and therefore the TVD property, we should have:

\[ 0 \leq \frac{\Delta x}{\Delta x - u_i}(u_{ix}) \leq 1 \quad \text{and} \quad 0 \leq \frac{\Delta x}{\Delta x + u_i}(u_{ix}) \leq 1, \]

(3) For neighbor points of an extreme \( u_i \) (with estimated derivative \( u_{ix} \) = 0; this is based on condition 2), for uniform grids with width \( \Delta x \), the neighbor derivatives meet conditions (see [51], example 2.4):

\[
\frac{1}{2} \left| \frac{\Delta x_i}{\Delta x - u_i}(u_{ix})_{i-1} \right| \leq 1 \quad \text{and} \quad \frac{1}{2} \left| \frac{\Delta x_i}{\Delta x + u_i}(u_{ix})_{i+1} \right| \leq 1.
\]

In these conditions, we have: \( u_i := u(x_i) \); \( \Delta x_i := x_{i+1/2} - x_{i-1/2} \) denotes \( i^{th} \) cell width; in case of uniform grids: \( \Delta x = \Delta x_i \); \( F^{j \pm 1/2}_x \) represent reconstructed fluxes at right and left cell boundaries \( x_{i+1/2}; \Delta x, u_i := \pm (u_{ix} \pm u_i); (u_{ix})_{i \pm 1} \) are the estimated (limited) first derivatives at points \( x_{i \pm 1} \).

The conditions 1-3 have physical meaning; the first condition states that an extreme solution with maximum value should decrease in time (i.e., \( u_i^{n+1} \leq u_i^n \)), while an extreme solution with minimum value should increase through time (i.e., \( u_i^{n+1} \geq u_i^n \)). From the second condition, it is clear that at extreme points we must have: \( (u_{ix})_i = 0 \). Due to the third condition, the reconstructed values in cell edges should satisfy the monotonicity feature: magnitude of the reconstructed values should be less than immediately neighbor cell center values. This means, for example, if \( u_{i-1}^n \leq u_i^n \leq u_{i+1}^n \), then \( u_{i-1}^n \leq u_{i-1}^{n+1} \leq u_i^n \leq u_{i+1}^{n+1} \leq u_{i+1}^n \).

The third local TVD condition is for uniform cells; in case of non-uniform cells, due to the E-condition at extreme values \( u_i \), with estimated derivative \( (u_{ix})_i = 0 \), we have:

\[
\text{Sign}(u_{i+1} - u_i) = \text{Sign}\left(\left\{ \left( u_{i+1} - \frac{\Delta x_i}{\Delta x - u_i}(u_{ix})_i \right) - (u_i) \right\}\right),
\]

\[
\text{Sign}(u_i - u_{i-1}) = \text{Sign}\left(\left\{ (u_i) - \left( u_{i-1} + \frac{\Delta x_i}{\Delta x + u_i}(u_{ix})_{i-1} \right) \right\}\right).
\]

According to these relationships, TVD preserving derivatives should satisfy:

\[
\frac{1}{2} \left| \frac{\Delta x_i}{\Delta x - u_i}(u_{ix})_{i-1} \right| \leq 1 \quad \text{and} \quad \frac{1}{2} \left| \frac{\Delta x_i}{\Delta x + u_i}(u_{ix})_{i+1} \right| \leq 1.
\]

These two relationships confirm also as before necessity of the monotone reconstruction in cell edges for non-uniform cells.

8.2. Irregularity effects on slope limiters. The central/central-upwind schemes (e.g., the KT method) is originally developed for uniform cells. Most of them satisfy the TVD and monotonicity preserving conditions. On irregular cells, these methods do not completely meet these conditions and therefore do not remain necessarily stable. In the following, effects of cell irregularities will be studied by considering the monotonicity preserving necessity and the local TVD conditions.

It will be shown that only the grid modification (for gradual variation of grids) is not generally enough to guarantee TVD results.

8.2.1. Slope limiters on uniform grids. Most of slope limiters meet some necessary conditions, as: 1) the TVD; 2) preserving linear approximation; and 3) symmetric feature. These features are explained, in brief, as follows.

The TVD property. Considering physical meanings of the local TVD conditions 1-3, the upper limit of a TVD limiter for \( i^{th} \) cell is (for more details, one can see [4]):

\[
(8.2) \quad u_i^\prime = \text{min}\left\{ \frac{2 (\Delta_+ u_i)}{\Delta x}, \frac{2 (\Delta_- u_i)}{\Delta x} \right\},
\]

where \( u_i \) denotes cell center solution on the \( i^{th} \) cell and operator \( u_i^\prime \) is upper limit of a discrete approximation of \( \text{grad}(u_i) \) in a way that solutions remain TVD. To have TVD solutions, estimated derivative \( (u_{ix})_i := du(x_i, t)/dx \) should be limited by a slope limiter \( \phi_i := \phi_i(R_i) \) as \( \phi_i(u_{ix})_i \). The function \( \phi_i \) is a slope limiter at \( x_i \), where \( 0 \leq \phi_i \leq 1 \). Parameter \( R_i \) measures smoothness of \( \text{rel} \) of successive gradients around point \( x_i \); for uniform grids its definition is: \( R_i = \frac{\Delta x_i}{\Delta x_i u_i} = \frac{u_{ix}^2}{\Delta x_i u_i} \).

For all TVD slope limiters, it is necessary that: \( \phi_i(u_{ix})_i \leq u_i^\prime \) (this upper TVD limiter will be derived on non-uniform cells).
Linear approximation preserving. For a linear function with slope $s$ on a uniform grid, it is clear: $R_i = \Delta_+ u_i / \Delta_- u_i = \{ s(\Delta_+ x_i) \} / \{ s(\Delta_- x_i) \} = 1$. In this regard, the linear preserving condition is: $\phi(1) = 1$. This constraint can also be obtained by considering the MINMOD limiter definition: the linear preserving feature is conserved if the forward and backward derivatives are the same, i.e., $R_i = 1$.

The symmetric feature. The condition is: $\phi(R_i) = \phi(1/R_i)$. This condition assure that slope limiter effects are the same for forward and backward propagating solutions. To more clarify this feature, a new parameter $f$ is defined as: $f = (\Delta_- u_i) / (\Delta_0 u_i)$, where $\Delta_0 u_i := u_{i+1} - u_{i-1}$. For monotone solutions, $u_i$ is always between $u_{i-1}$ and $u_{i+1}$, so $f$ belongs always to the range $[0, 1]$. The parameters $R$ and $f$ have then a relationship with each other, as: $R = (1-f)/f$. The symmetric property is now clear, since: $\phi(f) = \phi(R) = \phi((1-f)/f)$ and $\phi(1/R) = \phi(f/(1-f)) = \phi(1-f)$, then $\phi(f) = \phi(1-f)$ for $0 \leq f \leq 1$.

8.3. Slope limiters on cell-centered non-uniform grids. A grid with non-uniform cells is considered. Widths of successive cells can be related to each other by coefficients: $a = \Delta x_{i-1}/\Delta x_i$ and $b = \Delta x_{i+1}/\Delta x_i$. Possible jump in solutions between points $x_{j+1}$ and $x_{j-1}$ is denoted by $\Delta_0 u_i$, where $\Delta_0 u_i := u_{i+1} - u_{i-1}$. The forward $(D_+ u_i)$, backward $(D_- u_i)$ and centered $(D_0 u_i)$ differences can then be written as: $D_+ u_i := \Delta_{x_i}^+ u_i; D_0 u_i := \Delta_{x_i}^0 u_i$. Considering definition of $f$, these derivatives can be rewritten as: $D_+ u_i = (2(1-f)\Delta u_i)/((1+b)\Delta x_i); D_- u_i = (2f\Delta u_i)/((1+a)\Delta x_i); \text{ and } D_0 u_i = ((2\Delta_0 u_i)/((2+a+b)\Delta x_i$).

The TVD property. This can be obtained by the local TVD conditions 1-3 (see [4]). It has the same definition as the uniform case (Eq. (8.2)):

$$u'_i = \min \left\{ \frac{2\Delta_+ u_i}{\Delta x_i}, \frac{2\Delta_- u_i}{\Delta x_i} \right\}.$$  

Linear approximation preserving. For non-uniform grids, the linear preserving condition for a slope limiter is [4]:

$$\phi_0 D_i(f = f_p) = \frac{2\Delta_0 u_i}{(a + b + 2)\Delta x_i},$$

where $f_p = \frac{a + 1}{a + b + 2}$. The linear preserving feature is conserved if the backward and forward derivatives are equal to each other: $D_+ u_i = D_- u_i$. By considering the MINMOD definition, this happens for case $f = f_p$, or values of $\phi_0 D_i(f_p)$ are values of the MINMOD limiter at the $f_p$.

Symmetric condition. This feature is not fulfilled, in general.

8.4. Performance of the generalized MINMOD limiter on non-uniform grids. The generalized MINMOD (GMINMOD), can be rewritten as:

$$\phi_0 D_i \text{ GMINMOD} = \text{GMINMOD} \{ \theta D_- u_i, D_0 u_i, \theta D_+ u_i \}, \quad 1 \leq \theta \leq 2.$$  

In case of non-uniform grids, different definitions of $D_0 u_i$ can be considered. One is based on the above-mentioned definition: $D_0 u_i = \Delta_0 u_i / (\Delta_+ x_i + \Delta_- x_i)$, and the other one obtains by the first order least square based estimated slope, as [4]:

$$D_0 u_i = D_{lsq} u_i := \frac{\Delta_+ x_i^2}{\Delta_- x_i^2 + \Delta_+ x_i^2} D_+ u_i + \frac{\Delta_- x_i^2}{\Delta_- x_i^2 + \Delta_+ x_i^2} D_- u_i.$$  

On uniform grids, both definitions lead to the first order central difference equation: $(u_{i+1} - u_{i-1})/(2\Delta x_i)$. Performance of these two GMINMOD limiters will be illustrated on non-uniform grids.

Regarding $f$ and $\Delta_0 u_j$ definitions, functions $D_{lsq} u_i$ (Eq. (8.5)) and $u'_i$ (Eq. (8.3)) can be rewritten as:

$$D_0 u_i = D_{lsq} u_i \frac{(1+b)(1-f) + (1+a)}{(1+b)^2 + (1+a)^2} \left\{ \frac{f\Delta_0 u_i}{\Delta x_i} \right\},$$

$$u'_i = \min \{ f, 1-f \} \frac{2\Delta_0 u_i}{\Delta x_i}.$$
Performance of the GMINMOD limiters on uniform and non-uniform grids are shown in Figure 6. Figure 6(a) is for uniform case and Figures 6(b,c) are for non-uniform cases. In all figures, center of each cell, \( x_i \), is in the middle of \( i^{th} \) cell, i.e.: \( x_i = (x_{i+1/2} + x_{i-1/2})/2 \). A gradual variation of cell lengths is considered for irregular grids. In Figure 6(b) length of cells are: \( \Delta x_{i-1} = 2dx \), \( \Delta x_i = dx \), and \( \Delta x_{i+1} = 0.5dx \); and in Figure 6(c) the lengths are: \( \Delta x_{i-1} = 2dx \), \( \Delta x_i = 1.5dx \), and \( \Delta x_{i+1} = dx \). The results confirm that: 1) on gradually varying grids, limiters may not completely remain in the TVD region; 2) the symmetry condition may not satisfy; 3) more gradual variation of grids is, more stability exists; 4) the long-term numerical stability cannot guarantee; and 5) the GMINMOD limiter by the direct-definition of central differencing, \( D_0 u_i = (\Delta_0 u_i)/(\Delta_- x_i + \Delta_+ x_i) \), leads to more symmetric behaviors, so this definition will be considered in this work.

It will be shown that to have a TVD solution (defined on non-uniform grids) without modifying limiter definitions (the GMINMOD limiter, here), the cell middle point, \( x_i \), should be shifted slightly; and this is only necessary for transmitting cells (a cell between two surrounding uniform cells with different cell lengths). In the following, at first, it will be shown how to choose properly cell centers/edges by the adaptive wavelet transform. It will then mathematically be proved that why such spatial configurations lead to stable and TVD solutions without modifying limiter definitions.

8.5. Choosing of cell centers and edges by the wavelet-based adaptation algorithm. In this work, at first, cell center positions, \( x_j \), are evaluated by the adaptive wavelet transform, and then, cell edges are simply assumed to be the middle point of them, as: \( x_{j+1/2} = (x_j + x_{j+1})/2 \).

The interpolating wavelet theory in Sec. 2 uses the pyramid algorithm. In this formulation, distance between detail coefficients in the resolution \( j \) is twice those in the resolution \( j + 1 \). Consider an adapted grid where for every adapted points of resolution level \( j + 1 \), there exist always two surrounding adapted points of resolution level \( j \). For such adapted grids, inter-distances of successive points increase or decrease gradually by the dyadic pattern. Let us consider an adapted cell-centers as \{ \cdots, 2d, 2d, d, d, \cdots \}. This means, for grid points \{ \cdots, x_{j-2}, x_{j-1}, x_j, x_{j+1}, x_{j+2}, \cdots \}, we have: \( x_{j-1} - x_{j-2} = x_j - x_{j-1} = 2dx \) and \( x_{j+1} - x_j = x_{j+2} - x_{j+1} = dx \). Such configuration is illustrated in Figure 7(a). For this adapted points, it is assumed that there exists always three successive points with equal distances from each other. In this figure, cell edges are middle point of two successive cell centers, i.e.: \( x_{j+1/2} = (x_j + x_{j+1})/2 \). By this, length of created cells are: \{ \cdots, \Delta x_{j-1} = 2dx, \Delta x_j = 1.5dx, \Delta x_{j+1} = dx, \cdots \}. Note that \( j^{th} \) cell acts as a transiting cell with a shifted cell center (\( x_j \) is no longer in the middle of cell \( j \)). In all the surrounding cells, all cell centers remain in the middle of cells. In the next subsection, it will be proved that such cell configuration leads to stable and TVD results. This cell configuration can obtain by the post-processing stage, Sec. 3.
8.6. Stability and TVD conditions on wavelet-based adapted grids. In the proposed wavelet-based grid adaptation, as mentioned, cell center of transmitting cells do not remain in the middle of cells, Figure 7(a). In Figure 7, three pattern of successive cells are distinguishable: 1) \( \{j - 2, j - 1, j\} \); 2) \( \{j - 1, j, j + 1\} \); and 3) \( \{j, j + 1, j + 2\} \). These sets are shown in Figures 7 (b) to (d). In the set (2), the transmitting cell \( j \) is in the middle, while in the remaining groups, the transmitting cell \( j \) is the first or the last cell.

In the following, the TVD local conditions are checked and modified for such spatially non-centered cell centers. This will be done for the set (2) (see Figure 7(b)), and then it will be checked for the groups (1) and (3).

8.6.1. The TVD condition when transmitting cell is in the middle of surrounding cells (the set (2)). To provide the TVD condition for the cell set \( \{j - 1, j, j + 1\} \) (see Figure 7(b)), a right propagating scalar advection equation is considered as: \( u_t + a u_x = 0 \) for \( a > 0 \). For simplicity, the forward Euler discretization in time will be used. The upwind finite volume method with second order accuracy will be considered for the spatial discretization. The resulted discretized system is:

\[
\begin{align*}
\left. u_j^{n+1} = u_j^n - \frac{a \Delta t}{\Delta x_j} \left( u_{j+1}^L - u_{j-1}^L \right) \right. 
\end{align*}
\]

The symbol \( L \) represents the upwind flux and therefore the upwind-based reconstructed value of \( u_{j+1/2}^L \) is:

\[
\begin{align*}
u_{j+1/2}^L = u_j + (p \Delta x_j) S_j,
\end{align*}
\]

where \( S_j := \phi_j D_j \) is a limited slope at point \( x_j \). Let assume in the transmitting cell \( j \), the cell center \( x_j \) locates in a way that \( x_{j+1/2} - x_j = p \Delta x_j \) and \( x_j - x_{j-1/2} = (1 - p) \Delta x_j \). For the wavelet based adapted grids, it is easy to show that \( p = 1/3 \).
To show a method is TVD, it should be confirmed that: 1) a monotone increasing (or decreasing) solution remains monotone increasing (or decreasing) in time (the monotonicity preserving feature, resulted from the positivity condition [51]); 2) if \( u^n_j \) is a local maximum (or minimum), then at the next time step: \( u^{n+1}_j \leq u^n_j \) (or \( u^{n+1}_j \geq u^n_j \)) (due to the first local TVD condition).

**Controlling of the monotonicity preserving condition.** Assume a monotonic increasing solution at time step \( t = t^n \) as: \( u^n_j \leq u^n_{j+1} \). This feature should satisfy at the next time step, i.e.: \( u^{n+1}_j \leq u^n_{j+1} \leq u^{n+1}_{j+1} \).

For the monotone increasing solution \( \{ u^n_j \} \), slopes \( S_{j-1} \) and \( S_j \) are positive, so: \( u^n_{j-1/2} = u^n_{j-1} + (\Delta x_j/2)S_{j-1} \leq u^n_j \) and \( u^n_{j+1/2} = u^n_j + (p\Delta x_j)S_j \geq u^n_{j+1} \). These inequalities are obtained due to the monotone reconstruction constraint: the local TVD condition 3. Hence from Eq. (8.6), we have:

\[
(8.7) \quad u^{n+1}_j \leq u^n_j - \frac{\bar{a}\Delta t}{\Delta x_j} (u^n_j - u^n_n) = u^n_n.
\]

To complete the proof, we need to estimate a below bound for \( u^{n+1}_j \). At first, due to the local TVD constraint on reconstruction edge values (the condition 3), it is clear that: \( u^n_j - (1 - p)\Delta x_j S_j \geq u^n_{j-1} \). By considering this relationship, and conditions \( S_j \geq 0 \) & \( S_{j-1} \geq 0 \), from Eq. (8.6), we have:

\[
(8.8) \quad u^{n+1}_j \geq u^n_{j-1},
\]

where \( \lambda_j := \bar{a}\Delta t/\Delta x_j \). The relationship ((8.8)) valids for \( \lambda_j \leq 0.5 \). Eqs. (8.7) and (8.8) lead to the condition \( u^{n+1}_j \leq u^n_{j-1} \leq u^{n+1}_j \leq u^n_j \leq \cdots \). This means a monotone solution remains monotone through time.

**Controlling of the extreme conditions (the first local TVD condition).**

1. **local maximum:** let have a right propagating wave, \( \bar{a} > 0 \) and \( u^n_{j+1} \) is a local maximum on group cell (2) (Figure 7(b)). Due to the extreme condition, we have: 1) \( S_{j+1} = 0 \) (the second local TVD condition); 2) \( u^n_{j+1} \geq u^n_n \) and \( u^n_{j+1} \geq u^n_{j-1} \); and 3) \( S_j \geq 0 \) (due to the monotone reconstruction feature). Rewriting Eq. (8.6) for the maximum point, we have:

\[
(8.9) \quad u^{n+1}_j = u^n_j + \frac{\bar{a}\Delta t}{\Delta x_{j+1}} \left( u^n_{j+1} - u^n_{j+2} \right)
\]

To guarantee that \( u^{n+1}_{j+1} \leq u^n_{j+1} \), the term in the parentheses should be non-negative; hence:

\[
0 \leq S_j \leq \frac{u^n_{j+1} - u^n_j}{p\Delta x_j} \leq \frac{\Delta x_j u^n_j}{p\Delta x_j}.
\]

2. **local minimum** let the wave propagate to the left, \( \bar{a} < 0 \) and \( u^n_{j-1} \) is a local minimum. This means: \( u^n_{j-1} \leq u^n_j \), \( u^n_{j-1} \leq u^n_{j-2} \), \( S_{j-1} = 0 \) and \( S_j \geq 0 \). By the spatio-temporal discretization
The TVD condition when transmitting cell is the first or last cell (the set (1) or (3)).

(8.10)

From Eqs. (8.9) and (8.10), the TVD constraint on non-uniform grids is:

(8.11)

8.6.2. The TVD condition when transmitting cell is the first or last cell (the set (1) or (3)). In this case, it is easy to show that the TVD limiter constraint, \( u_j^0 \) is the same as cell-centered cells, see Eq. (8.3).

8.7. Controlling of the TVD condition for slope limiters on wavelet-based adapted grids.

A modified wavelet-based adapted grid is considered; the modification is done with the post-processing stage. It is assumed also there exists always at least three neighbor cell centers of equal distances from each other, as explained in Sec. 8.5 and illustrated in Figure 7(a). In this figure, we have: \( x_{j-1} - x_{j-2} = 2dx, x_j - x_{j-1} = 2dx, x_{j+1} - x_j = dx \) and \( x_{j+2} - x_{j+1} = dx \). This configuration of grid points leads to stable and TVD solutions, which will be studied later. Depending of the transmitting cell location, three cell sequences are detectable: 1) \( \{j - 2, j - 1, j\} \); 2) \( \{j - 1, j, j + 1\} \); 3) \( \{j, j + 1, j + 2\} \) (Figure 7(a)).

**Cell sequence \( \{j - 1, j, j + 1\} \): Transmitting cell \( j \) (with a shifted cell center) is the middle one.** In this case, we have \( \Delta_+ x_j = [a/2 + (1 - p)]\Delta x_j \) and \( \Delta_- x_j = [p + b/2]\Delta x_j \). So, the backward, forward and central derivatives (in the GMINMOD limiter, Eq. (8.4)) can be rewritten as: \( D_- u_j = \frac{2\Delta_0 u_j}{[a + 2(1 - p)]\Delta x_j}, D_+ u_j = \frac{2(1 - f)\Delta_0 u_j}{2p + b}\Delta x_j, \) and \( D_0 u_j = \frac{2\Delta_0 u_j}{[2 + (a + b)]\Delta x_j} \). Where \( a := \Delta x_{j-1}/\Delta x_j = 4/3 \) and \( b := \Delta x_{j+1}/\Delta x_j = 2/3 \); \( p \) measures cell-center shifting and here \( p = 1/3 \) (see Figures 7(a)-(b)).

The functions \( (\phi_j)_TVD D_j \) (Eq. (8.11)) and GMINMOD (Eq. (8.4)) are illustrated in Figure 7(b). The comparison offers: 1) the limiter remains completely in the TVD region; 2) the linear preserving feature is satisfied; 3) at the expense of the symmetric feature, the transmitting cell \( j \) acts properly for joining surrounding cells; and 4) shifting of the cell center leads to a TVD result (Figure 7(b)), while cell-centered one does not (Figure 6(c)).

**Cell sequence \( \{j - 2, j - 1, j\} \) (see Figure 7(c)).** In this case, we have: \( \Delta_- x_j = (1 + a)\Delta x_j \) and \( \Delta_+ x_j = [1 + 2b(1 - p)]\Delta x_j \). Hence: \( D_- u_j = \frac{2\Delta_0 u_j}{(1 + a)\Delta x_j}, D_+ u_j = \frac{2(1 - f)\Delta_0 u_j}{[1 + 2b(1 - p)]\Delta x_j} \) and \( D_0 u_j = \frac{2\Delta_0 u_j}{[1 + a + 2(1 - p)b]\Delta x_j} \), where \( a = 1, b = 3/4 \) and \( p = 1/3 \). Definition of the \( (\phi_{j - 1})_{TVD} D_{j - 1} \) is the same as the uniform case, Eq. (8.3)). Comparison of this limiter with the GMINMOD is shown in Figure 7(c). It is obvious, the GMINMOD limiter remains in the TVD region.

**Cell sequence \( \{j, j + 1, j + 2\} \) (see Figure 7(d)).** Here we have: \( \Delta_- x_j = \frac{(1 + 2b)\Delta x_j}{2} \) and \( \Delta_+ x_j = \frac{(1 + b)\Delta x_j}{2} \). Hence: \( D_- u_j = \frac{2\Delta_0 u_j}{(1 + 2b)\Delta x_j}, D_+ u_j = \frac{2(1 - f)\Delta_0 u_j}{(1 + b)\Delta x_j} \) and \( D_0 u_j = \frac{2\Delta_0 u_j}{(2 + 2ab + b)\Delta x_j} \); where \( a = 3/2, b = 1 \) and \( p = 1/3 \). The function \( (\phi_{j + 1})_{TVD} D_{j + 1} \) is the same as the uniform case. This function is compared with the GMINMOD limiter in Figure 7(d), where stable and TVD results are attached.

In General, it can be concluded that on irregular grid points with typical grid configuration illustrated in Figure 7(a), numerical solutions will remain TVD and thereby stable.
8.8. Constraint on cell center adaptation in the wavelet-based algorithm. As mentioned before, it is always assumed that there exist at least three neighbor cell centers of equal distance from each other in non-uniform adapted cell centers. Let assume an adapted cell with weaker cell-sequence condition: \( x_{j+1} - x_j = 0.5(x_j - x_{j-1}) \) or \( x_{j+1} - x_j = 2(x_j - x_{j-1}) \). As an example let consider: \( x_{j-1} - x_j - 2 = 8dx, x_j - x_{j-1} = 4dx, x_{j+1} - x_j = 2dx \) and \( x_{j+2} - x_{j+1} = dx \) (Figure 8(a)). For this case, if cell edges are assumed to be in the middle of cell centers, \( x_{j+1/2} = 0.5(x_j + x_{j+1}) \), then cell lengths are: \( \Delta x_{j-1} = 6dx, \Delta x_j = 3dx \) and \( \Delta x_{j+1} = 1.5dx \). Hence: \( \Delta_j x_j = [(1 - p) + ap] \Delta x_j, \Delta_j x_j = \left[ p + (1 - p)/2, D_{-uj} = \frac{f_{\Delta x_j}}{(1 + p) + \Delta x_j}, D_{+uj} = \frac{(1 - f)_{\Delta x_j}}{p(1 - p)/\Delta x_j} \right] \), and \( D_{0uj} = \frac{1}{(1 + p) + (1 - p)/2} \); where \( a = 2 \) and \( b = 0.5 \). For all cells, shifting coefficient is equal to \( p = 1/3 \) (see Figure 8(a)).

The GMINMOD (Eq. (8.4)) and \( u_i' \) (from non-uniform cases; Eq. (8.11)) are compared in Figure 8(b). It is clear that the limiter is slightly outside the TVD domain. For this reason, long term stability of numerical solutions could vanish. For cell-centered cases, where \( x_j = (x_{j+1/2} + x_{j-1/2})/2 \), the results are not also TVD, see Figure 6(b).

To have stable results, cell centers are located in a way that there exists always at least three neighbor grids (cell centers) of equal distance from each other. For guarantee this condition, in the grid modification stage (the post-processing stage), it is assumed, at least, to have: \( \Delta x = \Delta x_c = 1 \).

9. Numerical examples

The following examples are to study the effectiveness of the proposed method concerning nonlinear 1-D and 2-D first order hyperbolic systems. The main assumptions are: 1- applying the D-D interpolating wavelet of order 3; 2- using the generalized MINMOD flux/slope limiter in all problems; 4- repeating re-adaptation processes every time step; 5- using the semi-discrete form of central and central-upwind schemes; 6- integrating in time by the TVD Rung-Kutta second-order solver.

Burgers’ equation. The Burgers equation is defined as follows:

\[
\frac{\partial u}{\partial t} + \frac{1}{2} \left( \frac{\partial u^2}{\partial x} \right) = 0,
\]

where \( u \) is the conserved quantity and its flux is \( F(u) = u^2/2 \). The system is nonlinear, so that discontinuous fronts will develop during front propagations. Here it is assumed that the initial and boundary conditions are:

\[
BCs : u(x = 0, t) = u(1, t) = 0, \quad ICs : u(x, t = 0) = \sin(2\pi x) + \frac{1}{2}\sin(\pi x).
\]
For above conditions, a discontinuity starts to appear around $t \approx 0.158$. This discontinuous front will propagate to the right side after this time. Assumptions for the numerical simulations are: $\epsilon = 10^{-3}$ (threshold) and $\theta = 2$ (the flux limiter parameter), $N_c = N_s = 2$ (for post-processing stage), $J_{\text{max}} = 11$, and $J_{\text{min}} = 5$. The numerical results are illustrated in Figure 9 at times 0.158, 0.5, and 1. This figure contains numerical results, exact solutions and corresponding adapted grids in different resolutions. The results confirm that adapted points are properly concentrated around propagating fronts.

The KT central scheme and NVSF-based method [12] are compared on wavelet-based adapted grids (the NVSF-based method is basically developed for non-uniform grid points). The results are presented in Fig. 10 at time 0.158. For the NVSF formulation, two types of flux limiters are considered: the SMART and MINMOD limiters [12]. For all simulations, we have $\epsilon = 10^{-3}$.

The results offer that the proposed method is comparable with methods originally provided for non-uniform grid points.

**Euler system of equations.** For this system, the governing equation is:

$$
\frac{\partial}{\partial t} \begin{pmatrix}
\rho \\
\rho u \\
\rho E
\end{pmatrix} + \frac{\partial}{\partial x} \begin{pmatrix}
\rho u \\
\rho u^2 + P \\
u(E + P)
\end{pmatrix} = \begin{pmatrix}
0 \\
0 \\
0
\end{pmatrix},
$$

where $\rho$, $u$ and $E$ are gas density, velocity and total energy, respectively. The pressure $P$ can be obtained by: $P = (\gamma - 1) \left( E - \frac{\rho u^2}{2} \right)$, where $\gamma$ is the ratio of specific heats, and here it is assumed to be $\gamma = 1.4$.

In the following three different problems with different initial and boundary conditions will be studied. These diverse conditions lead to different bench-mark problems; they are: 1) the Sod problem [49]; 2) the Lax problem [33]; 3) Interaction of an entropy sine wave with a Mach 3 right-moving front [48].

**Sod Problem.** Corresponding initial conditions are:

$$
\begin{pmatrix}
\rho \\
u \\
P
\end{pmatrix}_{t=0} = \begin{cases}
\{0, 0, 1\}^T, & x \leq 0.5, \\
\{0.125, 0, 0.1\}^T, & x > 0.5,
\end{cases}
$$
An unbounded 1-D domain is assumed: a Riemann problem. Assumptions for numerical simulations are: $J_{\text{max}} = 11$, $N_d = 6$, $\epsilon = 10^{-3}$, $\theta = 2$, $N_e = 2$, $N_s = 1$ and $dt = 10^{-5}$. The numerical solutions and corresponding adapted grid points are illustrated in Figure 11 at time 0.2 with methods KT, $M_1$ and $M_2$. For the KT method, results are presented in Figures 11(a)-(c); these figures contains numerical results, corresponding entropy productions ($S^n_j$), adapted grids and local truncation errors ($E^n_j$). These results for the $M_1$ and $M_2$ schemes are provided respectively in Figures 11(d-f) and Figures 11(g-i).

Figure 11 provides that: 1) methods $M_1$ & $M_2$ have less numerical dissipation in comparison with the KT scheme; 2) the $M_2$ scheme leads to the smallest dissipation; 3) using a less dissipative method, more grid points concentrate automatically in different resolutions. This is confirmed by comparing $N_g$ values of these three methods during simulations, see Figures 11(b,e,h) and Figure 12. The numerical entropy production cannot detect the contact discontinuity in this example (Figures 11(a),(d) & (g)), even by less dissipative methods with fine enough resolutions. Entropies $S^n_j$ have small values in the rarefaction zone (for $0.25 < x < 0.45$), but can properly detect shock waves. The local truncation error can capture both the shock wave and contact discontinuity. The local errors $E^n_j$ have considerable values in rarefaction zones. This zone is not detected by the wavelet theory; as a result, grid points do not adapt there. Considering the wavelet-based adapted points, results of $S^n_j$, and $E^n_j$, different criteria lead to different adapted grids. In this example, it seems that wavelet-based adaptation method leads to more realistic adapted grids.

Effects of the post-processing stage are investigated by some numerical simulations in the following. At first, effect of considering the post-processing stage is studied. Two simulations with and without the post-processing step are done and results are presented in Figure 13. Figures 13(a-b) and Figures 13(c-d) include solutions with and without the post-processing step, respectively. The results indicate that post-processing adapted grids have significant effects on solution stability. The numerical instability grows rapidly in absence of the post-processing step.

The post-processing stage contains both grid modification in the same resolution and successive coarser resolution, see 3. To study effects of them, two types of modifications are considered: full and partial grid modification (by a post-processing). The modifications are: 1) partial post processing: for a grid point having resolution $j$, new points are only added at the corresponding resolution level; here we assume: $N_e = 2$ & $N_s = 0$; 2) full post processing: both resolution level $j$ and $j+1$ are controlled; we choose: $N_s = 2$ & $N_e = 1$. The former, the semi-modification, is frequently used in wavelet-based adaptation procedures. Numerical results are presented in Figure 14. Figure 14(a) and Figure 14(b) correspond to methods using the full and partial post-processing stages, respectively. This figure shows that long-term stability can be obtained in case of having full-modification, 14(a).
Figure 11. Numerical results, corresponding truncation errors, entropy productions, and adapted points in different resolutions for the Sod problem with the KT, $M_1$ & $M_2$ schemes at $t = 0.2$; a-c) the KT scheme; d-f) the $M_1$ method; g-i) the $M_2$ scheme.

Figure 12. Number of adapted grid points $N_g$ during simulations. In the finest resolution, number of grid points is $2^{11} + 1$.

Lax Problem. The initial conditions are:

$$\begin{pmatrix} \rho \\ u \\ P \end{pmatrix}_{t=0} = \begin{cases} \{0.445, 0.69887, 3.5277\}, & x \leq 0.5, \\
\{0.5, 0, 0.571\}, & x > 0.5, \end{cases}$$
and the problem is a Riemann problem. For simulations, it is assumed: $\epsilon = 10^{-3}$, $\theta = 2$, and $dt = 10^{-5}$.

For the three methods KT, $M_1$ and $M_2$, solutions $\rho$, corresponding entropies $S^n$, adapted grids, and truncation errors $E^n$ are presented in Figure 15 at time 0.16. Same as the Sod problem, the results offer that: 1) less dissipative methods mobilize more adapted grid points of fine resolutions; 2) numerical entropy production of these methods cannot detect the contact discontinuity; 3) the local truncation errors $E^n$ can detect the contact discontinuity zones; 4) $E^n_j$ can also detect rarefaction zones; 5) the wavelet transform can properly capture all phenomena: shock waves, rarefaction and contact discontinuity zones.

**Interaction of an entropy sine wave with a Mach 3 right-moving front.** This challenging problem was developed to reveal high order scheme capabilities by Shu and Osher [48]. Here, it is assumed the ratio of specific heats is $\gamma = 1.4$. The Riemann initial condition is [48, 35]:

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**Figure 13.** Post-processing effects on stability of solutions; a-b) with the post-processing stage; c-d) without the post-processing step. In figures (a) & (c), solid lines and hollow shapes are exact and numerical solutions, respectively.

**Figure 14.** Full and semi post processing effects; a) full post processing, $N_s = N_c = 2$; b) semi-post processing, $N_s = 2, N_c = 0$. Solid lines and hollow shapes are exact and numerical solutions, respectively.
The considered computational domain is: $\Omega \in (-5, 5) \times (0, T)$; Assumed parameters are: $\epsilon = \epsilon_0 = 5 \times 10^{-3}$, $J_{\text{max}} = 11$, $J_{\text{min}} = 5$ (or $N_d = 6$), $N_c = 1$, $N_s = 2$, $\theta = 2$, $dt = 0.00025$.

The numerical entropy production, numerical and exact solutions are illustrated in Figure 16 at $t = 1.8$. There, the solid lines and hollow shapes are the reference [35] and numerical solutions, respectively. Regarding numerical entropy productions, it is clear that both the $M1$ and $M2$ methods lead to less numerical dissipations in comparison to the KT scheme. The $M2$ scheme leads to the least dissipative results, since magnitude of entropy $S$ is larger than both the KT and $M1$ methods. Distribution of adapted points at different resolution levels for these three schemes are shown in Figure 17 at $t = 1.8$. The methods $M1$ and $M2$ lead to more adapted points of high resolutions. In Figure 18 local truncation errors for these three methods are presented, which confirm numerical convergence.

**2-D Euler equation of gas dynamics for ideal gases.** The governing equation for the 2-D system is:

$$ u_t + \mathbf{F}_x + \mathbf{G}_y = 0, $$

where state values are $u = [\rho, \rho u, \rho v, E]^T$; the flux vectors in the $x$ and $y$ directions are $\mathbf{F} = \{\rho u, \rho u^2 + P, \rho uv, u(E + P)\}^T$ and $\mathbf{G} = \{\rho v, \rho uv, \rho v^2 + P, v(E + P)\}^T$, respectively.
Figure 16. Numerical results, and entropy productions for the right propagating front with the KT, M1 & M2 schemes at $t = 1.8$. In these figures, solid black lines are the reference solutions, hollow shapes are numerical ones, and the gray solid lines are numerical entropy productions.

Figure 17. Distribution of adapted grid points in different levels of resolution for the KT, M1 & M2 schemes at $t = 1.8$.

In this example, radially symmetric initial conditions are assumed with respect to the origin. There exists initial higher density and higher pressure inside a circle with radius $r = 0.4$; corresponding values are: $\rho_{in} = 1, \rho_{out} = 0.1$ & $P_{in} = 1, P_{out} = 0.1$. Other initial values are: $u_{in} = u_{out} = v_{in} = v_{out} = 0$ [31, 54]. The computation domain belongs to $\Omega \in [-1.5,1.5] \times [-1.5,1.5]$. To control the symmetric of solutions and corresponding adapted grids in simulations, total of the computing domain $\Omega$ is considered.
The numerical results are presented in Figure 19 at $t = 0.4$. Figures 19(a) & (b) are from the KT scheme and Figures 19(c) & (d) belong to the M1 method. The results offer that: 1) all of the solutions and corresponding adapted grids are symmetric; 2) due to numerical dissipation, the KT solver leads to slightly different result from the M1 one. To clarify the numerical dissipation effects, cut of solutions are compared along $y = 0$, Figure 20. This figure confirms that the KT scheme ends to more dissipative results.

The local truncation errors $E^n_j$ for the two schemes (KT and M1) and corresponding adapted grids are presented in Figure 21 at $t = 0.2$. It is clear that the errors are properly concentrated in high-gradient zones detected properly by the wavelet transform.

A non-convex example: the polymer system. The governing equation of the polymer system is:

\[
\begin{align*}
\frac{\partial}{\partial t} \begin{pmatrix} s \\ b \end{pmatrix} + \frac{\partial}{\partial x} \begin{pmatrix} f(s, c) \\ cf(s, c) \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix},
\end{align*}
\]

where $s$ denotes water saturation; parameter $c$ is the polymer concentration; function $f := f(s, c)$ presents the fractional flow function of water; parameter $b$ is function of $s$ and $c$ where $b := b(s, c)$. Functions $b$ and $f$ are assumed to be: $b(s, c) = sc + a(c)$ and $f(s, c) = \frac{\epsilon}{s + (0.5 + c)(1-s)^2}$, where $a(c)$ denotes the adsorption function and in this example, it is: $a(c) = \frac{c}{5(1+c)}$.

The eigenvalues of the polymer system ((9.1)) are: $\lambda_1 = f_s(s, c)$ and $\lambda_2 = f(s, c)(s + a'(c))^{-1}$.

For numerical simulations, we assume: $J_{\text{max}} = 11$, $N_d = 6$, $\epsilon = 10^{-4}$, $dt = 0.00025$, and $N_s = N_c = 1$ (for grid modification).

In numerical simulations two types of $\theta$ are assumed: constant and adaptive. In adaptive case, in this work, it is assumed $\theta$ depends linearly on spatial positions of cell centers $x_j$, as:

\[
\theta(x_j) = 1 + \frac{\left( (\Delta x_j + \Delta x_{j+1})/2 \right) - \Delta x_{\text{min}}}{\Delta x_{\text{max}} - \Delta x_{\text{min}}}, \quad \theta_j := \theta(x_j),
\]

where $\Delta x_j := x_j - x_{j-1}$, $\Delta x_{\text{min}} := \min \{ \Delta x_j \}$, and $\Delta x_{\text{max}} := \max \{ \Delta x_j \}$. So, around high-gradient solutions $\theta \rightarrow 1$ and in smooth regions $\theta \rightarrow 2$. To study resolution effect, this example is also re-simulated for a fine resolution with resolution number: $J_{\text{max}} = 13$. In this case, number of decomposition levels is $N_d = 8$ and the time step is chosen in such a way that the CFL number does not change.

Numerical results and corresponding local truncation errors for the parameter $s$ are illustrated in Figure 22. The results provide that: 1) for case $\theta = 2$, the numerical solution does not converge to real
one, even though it is a converged weak solution (Figure 22(b)); 2) by using an adaptive scheme with adaptive $\theta$, the result is nearly in accordance with the real solution, Figure 22(c); by increasing $J_{max}$ values (or using finer resolutions), the numerical solution approaches to the reference one, Figure 22(e); 4) all of the results are the converged weak solutions due to errors $E_j^n$; 5) using more higher resolution level $J_{max}$, smaller local truncation errors are.
2-D scalar conservation laws with non-convex fluxes. The assumed conservation law is:

\[ u_t + \{\sin(u)\}_x + \{\cos(u)\}_y = 0, \]

where \( u := u(x, y, t) \) and the initial condition is:

\[
  u(x, y, t = 0) = \begin{cases} 
  \frac{14\pi}{4}, & x^2 + y^2 < 1, \\
  \frac{\pi}{4}, & x^2 + y^2 \geq 1. 
\end{cases}
\]

For numerical simulations, we have: \( J_{\text{max}} = 8, J_{\text{min}} = 5 \) (or \( N_d = 3 \)), \( \epsilon = 10^{-4}, dt = 0.5 \times 10^{-3} \), and \( N_x = N_c = 1 \) (for modification of adapted grid). For modeling, two different choices of \( \theta \) are assumed: 1) the constant one with value \( \theta = 2 \); 2) the adaptive implementation of \( \theta \). The latter is also based on the 1-D linear interpolation of \( \theta \) on adapted grid points, as:
Figure 23. Adaptive solutions of 2-D non-convex conservation law system at \( t = 1 \);
a, b) obtained with \( \theta \)-adaptive KT scheme; c, d) based on the KT scheme with \( \theta = 2 \).

\[
\theta(Z_j) = 1 + \frac{(\Delta Z_{\text{ave}})_j - \Delta Z_{\text{min}}}{\Delta Z_{\text{max}} - \Delta Z_{\text{min}}},
\]

where \((\Delta Z_{\text{ave}})_j := \{(\Delta Z_j + \Delta Z_{j+1})/2\}, \Delta Z_j := Z_j - Z_{j-1}, Z_j \in \{x_j, y_j\}, \Delta Z_{\text{max}} = 1/2^j_{\text{max}}, \text{ and } \Delta Z_{\text{min}} = 1/2^j_{\text{min}}. \) For each direction, \( \theta(Z_j) \) is calculated independently.

The numerical results and corresponding adapted grid points are shown in Figure 23 at \( t = 1 \). Figures (a, b) correspond to the \( \theta \)-adaptive results and figures (c, d) are from the constant \( \theta \). Again the \( \theta \)-adaptive solver converges to proper and physical results [31].

10. Conclusion

In this study, a wavelet-based adaptation procedure is properly integrated with central/central-upwind high resolution schemes for simulation of first order hyperbolic PDEs. It is shown that central high-resolution schemes become unstable on non-uniform cells even those have gradual grid density variations. This is because, the NVSF criterion is not satisfied by central schemes. Since their slope/flux limiters do not remain TVD on irregular cell-centered cells. Two key ideas are followed to remedy the instability problem: 1) replacing local irregular grids with abrupt changing with grids having gradual variations (replacing an ill-posed problem with a nearly well-posed one); 2) studying performance of limiters on irregular cells. The grid modification stage is done in framework of multiresolution analysis. The TVD conditions are reviewed and provided for non-uniform cells. It is shown that on cell-centered cells, limiter definitions should be modified. Another approach is using of non-cell-centered cells. In this case common limiters can be used without modification. The TVD conditions are derived for both cell-centered and non-cell-centered cells. Based on these conditions, proper configuration of adapted
cells and corresponding cell-centers are derived. It is shown that cell-center shifting is necessary only in some cells acting as transmitting cells. They connect surrounding uniform cells with different cell lengths.

The local truncation errors for 1-D and 2-D problems are provided on non-uniform cells for convergence studying. Also concept of the numerical entropy production is investigated for uniqueness insurance. These two concepts have also been used as criteria for grid adaptation. In this regard, performance of these concepts are compared with the wavelet-based algorithm. It is numerically shown that: 1) the numerical entropy production can not detect some phenomena such as contact discontinuities; 2) this concept also have enough values in rarefaction domains. This can lead to unnecessary concentration of grid points in such regions (this can also be seen in [17, 45]); 3) it seems performance of the local truncation errors as detector is better than the numerical entropy production; 4) wavelets can properly detect all of the shock waves, rarefaction regions, and contact discontinuities.

Non-linear hyperbolic systems with non-convex fluxes are studied. It is shown by local truncation errors that both physical and non-physical solutions converge to weak solutions. Both method (flux/slope limiters) and grid adaptations are used for capturing physical solutions.

**References**


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