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Adaptive filters for piecewise smooth spectral data†

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We introduce a new class of exponentially accurate filters for processing piecewise smooth spectral data. Our study is based on careful error decompositions, focusing on a rather precise balance between physical space localization and the usual moments condition. Exponential convergence is recovered by optimizing the order of the filter as an *adaptive* function of both the projection order and the distance to the nearest discontinuity. Combined with the automated edge detection methods, e.g. Gelb & Tadmor (2002, *Math. Model. Numer. Anal.*, **36**, 155–175), adaptive filters provide a robust, computationally efficient, black box procedure for the exponentially accurate reconstruction of a piecewise smooth function from its spectral information.

Keywords: Fourier series; filters; localization; piecewise smooth; spectral projection.

1. Introduction

The Fourier projection of a 2π-periodic function, $S_N f(\cdot)$, enjoys the well-known spectral convergence rate, i.e. the convergence rate is as rapid as the *global* smoothness of $f(\cdot)$ permits. Specifically, if $f(\cdot)$ has s bounded derivatives, then $|S_N f(x) - f(x)| \leqslant \text{Const.}\|f\|_{C^s} N^{1-s}$. This interplay between global smoothness and spectral convergence is reflected in the dual Fourier space through the rapidly decaying Fourier coefficients $|\widehat{f}(k)| \leqslant 2\pi \|f\|_{C^s}|k|^{-s}$. On the other hand, spectral projections of piecewise smooth functions suffer from the well-known Gibbs' phenomena, where the uniform convergence of $S_N f(x)$ is lost in the neighbourhood of discontinuities and the *global* convergence rate of $S_N f(x)$ deteriorates to first order. These related phenomena are manifestations of unacceptably slowly decaying Fourier coefficients.

Two interchangeable processes are available for recovering the rapid convergence in the piecewise smooth case. These are mollification, carried out in the physical space, and filtering, carried out in the Fourier space. Filtering accelerates convergence when premultiplying the Fourier coefficients $\widehat{f}(k)$ by a rapidly decreasing function $\sigma(\cdot)$, resulting in modified coefficients, $\widehat{f}(k)\sigma(|k|/N)$, with a greatly accelerated decay rate as $|k| \uparrow N$. This accelerated decay in the dual space corresponds to a smoothly localized mollification in the physical space. In Tadmor & Tanner (2002) we showed how to parameterize an

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optimal mollifier in order to gain the exponential convergence for piecewise analytic f values. The key ingredient in our approach was *adaptivity*, where the optimal mollifier is adapted to the maximal region of local smoothness. Here we continue the same line of thought by introducing *adaptive filters*, which allow the same optimal recovery of piecewise smooth functions from their Fourier coefficients. In particular, piecewise analytic functions are recovered with exponential accuracy. A brief overview follows.

We consider a family of general filters $\sigma(\cdot)$ which are characterized by two main properties. First, we seek the rapid decay of $\sigma_k := \sigma(|k|/N)$ which is tied to a regular, compactly supported multiplier $\sigma \in C_0^q[-1,1]$. Being compactly supported, such filters are restricted to N-Fourier expansions,

$$S_N^\sigma f(x) := \sum_{|k|\leqslant N} \sigma\left(\frac{|k|}{N}\right) \widehat{f}(k)\, \mathrm{e}^{\mathrm{i}kx}. \tag{1.1}$$

The operation of such filters in Fourier space corresponds to mollification in physical space, expressed in terms of the associated mollifier, $\Phi^\sigma(y) := 1/2\pi \sum_{|k|\leqslant N} \sigma(|k|/N)\, \mathrm{e}^{\mathrm{i}ky}$,

$$S_N^\sigma f(x) \equiv f * \Phi^\sigma(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \Phi^\sigma(y) f(x-y)\,\mathrm{d}y, \quad \Phi^\sigma(y) := \frac{1}{2\pi}\sum_{|k|\leqslant N} \sigma\left(\frac{|k|}{N}\right) \mathrm{e}^{\mathrm{i}ky}. \tag{1.2}$$

Second, such filters are required to satisfy the usual moments condition, e.g. (Majda *et al.*, 1978; Vandeven, 1991)

$$\int_{-\pi}^{\pi} y^n \Phi^\sigma(y) = \delta_{n0}, \quad n = 0, 1, \ldots, p-1 < q. \tag{1.3}$$

The first requirement of C_0^q-smoothness is responsible for localization—the essential part of the associated mollifier, Φ^σ, is supported near the origin, see (2.5) below. The second property drives the accuracy of the filter by annihilating an increasing number of its moments.

The rich subject of filters includes the classical filters of finite-order accuracy where finite $p \leqslant q$ dictate a fixed convergence of polynomial order, $\mathcal{O}(N^{-p})$, see Vandeven (1991). By letting $q \uparrow \infty$, one obtains a $C_0^\infty[-1,1]$-filter, i.e. an infinitely differentiable compactly supported filter σ, which respects (1.3) for increasing orders p. Majda *et al.* (1978) employed such filters to postprocess piecewise solutions with propagating singularities and achieve spectral convergence in the sense of having a convergence rate faster than any fixed order. Vandeven (1991) constructs spectrally accurate filters by relating the order of the filter, (q, p), to the increasing order of the projection, $q = q(N)$, $p = p(N)$. An alternative approach for spectral accuracy employs highly oscillatory mollifiers which are activated in physical space. Gottlieb & Tadmor (1985) constructed such (properly dilated) mollifiers of the form $\Phi(y) = \rho(y) D_p(y)$, where $D_p(\cdot)$ stands for the usual Dirichlet kernel of degree $p \sim \sqrt{N}$ and ρ is a standard $C_0^\infty[-1,1]$ cut-off function normalized so that $\rho(0) = 1$.

The different filters and mollifiers advocated in these works enable us to reconstruct the underlying piecewise smooth data from its given spectral content. Spectral accuracy is achieved in smooth regions as long as they are bounded away from discontinuities, but the error deteriorates in the neighbourhood of such discontinuities due to spurious oscillations. The latter difficulty was addressed by Gottlieb, Shu and collaborators, by invoking Gegenbauer expansions which are driven by a judicious choice of a localizer $(1-y^2)^\lambda$ which is appended to the Dirichlet kernel $D_p(y)$, see Gottlieb & Shu (1998) and the references therein. Their approach allows for high resolution *uniformly up to* the discontinuities, but its precise (p, λ)-parameterization as a function of N has a rather sensitive f-dependence which impacts the overall robustness of the Gegenbauer reconstruction, e.g. Boyd (2005). In Tadmor & Tanner (2002) we have introduced an alternative approach where the accuracy is *adapted* according to the maximal region of local smoothness. Specifically, we have shown how the Gottlieb–Tadmor mollifiers are optimized when their order is chosen *adaptively* as $p \sim Nd(x)$. Here $d(x)$ is the distance from the location x to its nearest discontinuity, $d(x) = \text{distance}(x, \text{sinsupp} f(\cdot))$; the distance function $d(x)$ could be recovered from the Fourier coefficients by edge detection, e.g. Gelb & Tadmor (2000, 2002). The resulting adaptive mollifiers lead to exponentially accurate, numerically robust mollifiers of order $\exp(-\alpha(\kappa N d(x))^{1/\alpha})$ with $\alpha > 1$ dictated by the detailed C_0^∞-regularity of ρ; specifically, $\alpha > 1$ reflects the Gevrey regularity of ρ (for Gevrey regularity and the similar class of ultramodulation spaces, we refer to, e.g. John, 1982 and Pilipovic & Teofanov, 2002, respectively). The key ingredient in our adaptive approach is giving up the *exact* moments condition; instead, it is satisfied modulo exponentially small errors, by replacing the exact (1.3) with the requirement

$$\sigma^{(n)}(0) = \delta_{n0}, \quad n = 0, 1, \ldots, p - 1 < q. \tag{1.4}$$

The precise relation between (1.4) and (1.3) is quantified in Theorem 2.2 below. We note that it is rather simple to construct admissible filters satisfying the last requirement for an *arbitrary* p; a prototype example is given by the $C_0^\infty[-1, 1]$-filters

$$\sigma_p(\xi) = \begin{cases} \exp\left(\frac{\xi^p}{\xi^2 - 1}\right), & |\xi| < 1, \\ 0, & |\xi| \geqslant 1. \end{cases} \tag{1.5}$$

The purpose of this paper is to construct a new class of exponentially accurate *adaptive filters*. As before, the key issue is the parameterization of their order, p. Here we develop the rigorous study for the optimal parameterization for such filters. We advocate adaptive filters in the sense that their order, $p = p(N, d(x))$, depends on both the order of the projection, N, and the distance function $d(x)$. Summarized in Theorem 2.1 below, our main result states that the optimal adaptive filter is determined to be of order $p(N) \sim (Nd(x))^{1/\alpha}$ with $\alpha > 1$ reflecting the Gevrey regularity of σ. While achieving exponential accuracy away from discontinuities, the new filters are adapted so as to prevent spurious oscillations *throughout* the computational domain, including discontinuous neighbourhoods. We mention here the adaptive filters introduced by Boyd (1995, 1996). Boyd's procedure was based on the acceleration summability by the so-called Euler-lag averaging; the acceleration was limited, however, since the resulting piecewise constant filters of order $p \sim Nd(x)$ were consistently larger than the optimal order and they exhibit slower convergence than the non-adaptive order of Majda *et al.* (1978).

Our current discussion on adaptive filters follows a similar approach for the adaptive Gottlieb–Tadmor mollifiers constructed in Tadmor & Tanner (2002), $\rho(y)D_p(y)$, where the precise Gevrey regularity of ρ allows us to obtain tight error bounds which in turn reveal the optimal adaptive parameterization, $p = p(N, d(x))$. New tight error bounds are outlined in Section 2 and are confirmed by numerical simulations in Section 3.

2. Adaptive-order filters

In this section we show how the regularity and moments properties of the filter σ are translated into precise statements of localization and accuracy of the associated mollifier $\Phi^\sigma(x)$. We begin by decomposing the filtering error $f(\cdot) - S_N^\sigma f(\cdot) = f - f * \Phi^\sigma$ into the two terms

$$\begin{aligned} f(x) - f * \Phi^\sigma(x) &= \int_{-\pi}^{\pi} \Phi^\sigma(y)[f(x) - f(x-y)][1 - \chi(y)]\, \mathrm{d}y \\ &\quad + \int_{-\pi}^{\pi} \Phi^\sigma(y)[f(x) - f(x-y)]\chi(y)\, \mathrm{d}y =: \mathcal{I}_1 + \mathcal{I}_2. \end{aligned} \tag{2.1}$$

Here $\chi(\cdot) = \chi_x(\cdot)$ is an auxiliary cut-off function adapted to the smoothness region of f. To this end, we let $d(x)$ denote the distance between x and its nearest discontinuity so that the y-function $f(x) - f(x-y)$ remains smooth for the largest symmetric interval, $|y| \leqslant d(x)$. We then set $\chi(y) \equiv \chi_x(y) := \rho(y/d(x))$, where ρ is a standard C_0^∞ cut-off function,

$$\chi(y) \equiv \chi_x(y) := \rho\left(\frac{y}{d(x)}\right), \quad \rho(y) \equiv \begin{cases} 1, & |y| \leqslant 1/2, \\ 0, & |y| \geqslant 1. \end{cases}$$

We observe that the dilated cut-off function $\chi_x(y) = \rho(y/d(x))$ enforces the support of the first integrand on the right-hand side of (2.1) to be bounded $d(x)/2$-away from x, while the second term is supported in the $d(x)$-neighbourhood of x. To simplify matters, we assume that ρ is adapted to the same C_0^∞-regularity of σ.

We turn to estimate the first error term on the right-hand side of (2.1) which measures the essential localization of the mollifier. To this end, we use the following aliasing formula, expressing our N-degree mollifier $\Phi^\sigma(y)$ in terms of the equally sampled inverse Fourier transform, $\varphi^\sigma(y) := \int \sigma(\xi)\,\mathrm{e}^{\mathrm{i}y\xi}\,\mathrm{d}\xi$([1]),

$$\Phi^\sigma(y) \equiv \frac{N}{2\pi}\sum_{n=-\infty}^{\infty}\varphi^\sigma((N(y+2\pi n)), \quad \Phi^\sigma(y) = \frac{1}{2\pi}\sum_{|k|\leqslant N}\sigma\left(\frac{|k|}{N}\right)\mathrm{e}^{\mathrm{i}ky}. \tag{2.2}$$

The usual Fourier decay rate estimates then yield

$$\begin{aligned} |\Phi^\sigma(y)| &\leqslant \frac{N}{2\pi}\sum_{n=-\infty}^{\infty}|\varphi^\sigma(N(y+2\pi n))| \leqslant N^{1-p}\|\sigma\|_{C^p}\sum_{n=-\infty}^{\infty}|y+2\pi n|^{-p} \\ &< \text{Const.} N\|\sigma\|_{C^p}(N|y|)^{-p} \quad \forall p,\ \ 0<|y|<\pi. \end{aligned} \tag{2.3}$$

We observe that the first integrand on the right-hand side of (2.1) is supported across the possible discontinuities of $f(x) - f(x-\cdot)$. Lack of smoothness excludes the possibility of high-oscillatory cancellations. Instead, we now seek a tight upper bound on the decay of the associated mollifier, Φ^σ for $|y| \geqslant d(x)/2$. To this we need to quantify the C^∞-regularity of our filter σ. We focus on σ values which have *Gevrey regularity* of order α, denoted G_α below; in our case, $\sigma = \sigma_p$ in (1.5) belong to G_2, namely, there exist constants, $M = M_\sigma$ and $\eta = \eta_\sigma > 0$ (independent of p) such that

$$\|\sigma_p\|_{C^p} \leqslant M_\sigma (p!)^\alpha \eta_\sigma^{-p}, \quad \alpha = 2, \ \ \sigma_p(\xi) = \begin{cases} \exp\left(\frac{\xi^p}{\xi^2-1}\right), & |\xi| < 1, \\ 0, & |\xi| \geqslant 1. \end{cases} \tag{2.4}$$

Details are outlined in Lemma 2.1 below. Incorporating the above growth rate into the localization estimate, (2.3) yields

$$|\Phi^\sigma(y)| \leqslant \text{Const.} M_\sigma (p!)^2 \left(\frac{1}{\eta_\sigma N|y|}\right)^p, \quad 0 < |y| < \pi,$$

[1]The result follows, e.g. by sampling the Fourier transform $\sigma(\xi) = 1/2\pi\int\varphi^\sigma(y)\,\mathrm{e}^{-\mathrm{i}y\xi}\,\mathrm{d}y$,

$$\begin{aligned} \sigma\left(\frac{|k|}{N}\right) &= \frac{N}{2\pi}\int\varphi^\sigma(Ny)\,\mathrm{e}^{-\mathrm{i}Ny\frac{|k|}{N}}\,\mathrm{d}y \\ &= \frac{N}{2\pi}\sum_{n=-\infty}^{\infty}\int_{-\pi}^{\pi}\varphi^\sigma(N(y+2\pi n))\,\mathrm{e}^{-\mathrm{i}|k|y}\,\mathrm{d}y = \frac{N}{2\pi}\int_{-\pi}^{\pi}\left(\sum_{n=-\infty}^{\infty}\varphi^\sigma(N(y+2\pi n))\right)\mathrm{e}^{-\mathrm{i}|k|y}\,\mathrm{d}y, \end{aligned}$$

and comparing with the discrete inverse Fourier transform, $\sigma(|k|/N) = \int\Phi^\sigma(y)\,\mathrm{e}^{-\mathrm{i}|k|y}\,\mathrm{d}y$.

which is minimized at $p = p_{\min} := (\eta|y|)^{1/2}$. This shows that with this choice of adaptive p, the mollifier associated with our σ-filter, Φ^σ, is essentially localized in the neighbourhood of x as it admits an exponential decay

$$|\Phi^{\sigma_p}(y)| \leqslant \text{Const.}_\sigma (1 + N|y|)\, \mathrm{e}^{-(\eta_\sigma N|y|)^{1/2}}. \tag{2.5}$$

Here and below, η is a positive constant which may differ among the different estimates. In particular, since $[1 - \chi_x(y)]$ and hence the first integrand on the right-hand side of (2.1) is supported at $|y| \geqslant d(x)/2$, the exponential bound follows

$$|\mathcal{I}_1| \leqslant \text{Const.}_{\sigma,f} (1 + Nd(x))\, \mathrm{e}^{-(\eta_\sigma Nd(x))^{1/2}}. \tag{2.6}$$

We turn to the second error term, $\mathcal{I}_2 = \int \Phi^\sigma(y)[f(x) - f(x-y)]\chi_x(y)\,\mathrm{d}y$. Traditionally, such a term is bounded above by $(d(x))^p \|f\|_{C^p[x-d(x),x+d(x)]}/p!$ through Taylor-expanding $f(x-y)$ about $y = 0$ and by invoking the moments condition (1.3). This bound is useful for a vanishing neighbourhood, $d(x) \ll 1$, while suffering by increasing the contribution of the first term on the right-hand side of (2.1), as reflected through its upper bound (2.6). We therefore let $d(x)$ be as large as possible so that we cannot argue by localization. Instead, this portion of the error decreases due to *cancellation* of oscillations by increasing the order p of Φ^{σ_p}. To this end, we write

$$\mathcal{I}_2 = \int_{-\pi}^{\pi} \Phi^{\sigma_p}(y) g(y)\,\mathrm{d}y \equiv \sum_{|k|\leqslant N} \sigma\left(\frac{|k|}{N}\right)\widehat{g}(k), \quad g(y) = g_x(y) := [f(x) - f(x-y)]\chi_x(y), \tag{2.7}$$

and we turn to estimate the Fourier coefficients on the right-hand side. By our assumption, $f(x) - f(x-y)$ remains analytic for $|y| \leqslant d(x)$ and hence $g_x(y) = [f(x) - f(x-y)]\chi(y)$ is C^∞. We quantify the C^∞-regularity of $\chi_x(\cdot)$ in terms of the same Gevrey regularity of order $\alpha = 2$ that σ has, so that $\|\chi_x\|_{C^p} \leqslant M(p!)^2(\eta_\rho d(x))^{-p}$. Thanks to the analyticity of $f(x) - f(x-y)$, it follows that if $\rho(\cdot)$ and hence $\chi_x(\cdot)$ belongs to Gevrey class G_α, so does $g_x(y) = [f(x) - f(x-y)]\chi_x(y)$, and hence

$$\|g_x(y)\|_{C^p} \leqslant M\frac{(p!)^\alpha}{(d(x)\eta)^p}, \quad |y| < d(x).$$

The constants $M = M_{\rho,\eta}$ and $\eta = \eta_{\rho,f}$ capture the detailed Gevrey and analyticity properties of $\rho(y)$ and $f(x-y)$ for $|y| < d(x)$; the order p is arbitrary. The Fourier coefficients $\widehat{g}(k)$ in (2.7) do not exceed

$$|\widehat{g}(k)| \leqslant \text{Const.}\|g_x(y)\|_{C^p}|k|^{-p} \leqslant \text{Const.}M\frac{(p!)^2}{(\eta|k|d(x))^p}, \quad \eta = \eta_{\rho,f}.$$

For $\sigma(|k|/N)$, we distinguish between the low modes $|k| \leqslant N/2$ and the high modes $N/2 < |k| \leqslant N$, setting

$$\mathcal{I}_{21} := \sum_{|k|\leqslant N/2}\left[\sigma_p\left(\frac{|k|}{N}\right) - 1\right]\widehat{g}(k)$$

$$\mathcal{I}_{22} := \sum_{N/2<|k|\leqslant N}\left[\sigma_p\left(\frac{|k|}{N}\right) - 1\right]\widehat{g}(k).$$

Since $g(y) = [f(\cdot) - f(\cdot - y)]\chi(y)$ vanishes at $y = 0$, we have $\sum \widehat{g}(k) = g(0) = 0$ and hence $\mathcal{I}_2 = \mathcal{I}_{21} + \mathcal{I}_{22} + \mathcal{I}_{23}$, where $\mathcal{I}_{23} := -\sum_{|k|>N}\widehat{g}(k)$. For the first term, we use a Taylor expansion around the origin: the accuracy assumption (1.4) yields

$$\left|\sigma_p\left(\frac{|k|}{N}\right)-1\right| \leqslant \frac{1}{p!}\|\sigma\|_{C^p([-\frac{1}{2},\frac{1}{2}])}\left(\frac{|k|}{N}\right)^p, \quad |k| \leqslant \frac{N}{2}.$$

Restricted to the interval $[-1/2, 1/2]$, σ retains an analytic bound of lemma 2.1 below, $\|\sigma\|_{C^p} \leqslant \text{Const.}\, p!\eta_\sigma^{-p}$, and hence $\mathcal{I}_{21}$ does not exceed

$$|\mathcal{I}_{21}| := \left|\sum_{|k|\leqslant N/2}\left[\sigma_p\left(\frac{|k|}{N}\right)-1\right]\widehat{g}(k)\right| \leqslant \frac{\text{Const.}}{p!}\|\sigma\|_{C^p([-\frac{1}{2},\frac{1}{2}])}\sum_{|k|\leqslant N/2}\left(\frac{|k|}{N}\right)^p\frac{(p!)^2}{(\eta_{\rho,f}|k|d(x))^p}$$
$$\leqslant \text{Const.}(p!)^2\frac{1}{(\eta N d(x))^p}, \quad \eta = \eta_\sigma\eta_{\rho,f}. \tag{2.8}$$

For the high modes, $\widehat{g}_x(k)$ is sufficiently small so that the simple bound of $|\sigma_p(|k|/N)| \leqslant 1$ will do for $\mathcal{I}_{22}$,

$$|\mathcal{I}_{22}| := \left|\sum_{N/2<|k|\leqslant N}\left[\sigma_p\left(\frac{|k|}{N}\right)-1\right]\widehat{g}(k)\right| \leqslant \left|\sum_{N/2<|k|\leqslant N}\frac{(p!)^2}{(\eta_\sigma|k|d(x))^p}\right|$$
$$\leqslant \text{Const.}(p!)^2\frac{1}{(\eta_\sigma N d(x)/2)^p}. \tag{2.9}$$

Similarly, by taking into account that $g_x(\cdot)$ is supported on an interval of length $2d(x)$ we find that $\mathcal{I}_{23}$ does not exceed

$$|\mathcal{I}_{23}| := \left|\sum_{|k|>N}\widehat{g}(k)\right| \leqslant \text{Const.}d(x)(p!)^2\sum_{|k|>N}\frac{1}{(\eta_\sigma|k|d(x))^p} \leqslant \text{Const.}\frac{N}{p}d(x)(p!)^2\frac{1}{(\eta_\sigma N d(x))^p}. \tag{2.10}$$

We combine the last three bounds to conclude

$$|\mathcal{I}_2| \leqslant \text{Const.}(1+Nd(x))(p!)^2\frac{1}{(\eta N d(x))^p}, \quad \eta = \min(\eta_\sigma\eta_{\rho,f}, \eta_\sigma/2, \eta_{\rho,f}),$$

which is minimized at the same value as before, $p = p_{min} := (\eta N d(x))^{1/2}$, so that

$$|\mathcal{I}_2| \leqslant \text{Const.}(1+Nd(x))\,\mathrm{e}^{-(\eta N d(x))^{1/2}}. \tag{2.11}$$

Finally, we recall that the assumed regularity of ρ is in fact dictated by that of σ and hence the various bounds, $\eta = \eta_{\sigma,f}$. We summarize by stating the following theorem.

THEOREM 2.1 Given the Fourier projection $S_N f$ of a piecewise analytic function $f(\cdot)$, we consider a $C_0^\infty[-1,1]$-filter $\sigma(\xi)$,

$$S_N^\sigma f(x) = \sum_{|k|\leqslant N}\sigma\left(\frac{|k|}{N}\right)\widehat{f}_k\,\mathrm{e}^{\mathrm{i}kx}.$$

Assume that σ has G_α-regularity and that it is accurate of order p in the sense of satisfying the moments condition

$$\sigma^{(n)}(0) = \delta_{n0}, \quad n = 0, 1, \ldots, p-1. \tag{2.12}$$

We set the adaptive order $p(x) := (\eta N d(x))^{1/\alpha}$, depending on the distance function $d(x) = \mathrm{dist}(x, \mathrm{sinsupp}\, f(\cdot))$. The resulting *adaptive* filter, $S_N^\sigma f$, recovers the point values $f(x)$ with the following exponential accuracy

$$|f(x) - S_N^\sigma f(x)| \leqslant \mathrm{Const.}(1 + Nd(x))\, \mathrm{e}^{-\alpha(\eta N d(x))^{1/\alpha}}. \tag{2.13}$$

The constant $\eta = \eta_{\sigma,f}$ is dictated by the specific Gevrey and piecewise-analyticity properties of σ and f.

We close this section with the promised statements on the exponential error bound (2.4).

LEMMA 2.1 Consider the p-order filter $\sigma_p(\xi) = \left\{\begin{smallmatrix} \exp(\xi^p/(\xi^2-1)), & |\xi|<1, \\ 0, & |\xi|\geqslant 1 \end{smallmatrix}\right\}$. Then there exists a constant η such that

$$\|\sigma_p\|_{C^p} \leqslant \mathrm{Const.}(p!)^2 \eta^{-p}, \tag{2.14}$$

$$\|\sigma_p\|_{C^p\left(\left[-\frac{1}{2},\frac{1}{2}\right]\right)} \leqslant \mathrm{Const.}\, p!\eta^{-p}. \tag{2.15}$$

Proof. To verify (2.14) we first note that $\sigma_p^{(s)}$ is a collection of polynomial terms which premultiply the exponential in the variable $\xi^p/(\xi^2-1)$. Each derivative of σ_p doubles the number of such terms; thus, by successive application of Leibniz's rule, $\sigma_p^{(s)}$ consists of 2^s α-terms, each of which is of the form

$$C_\alpha \prod_{\alpha_j} \left(\frac{\xi^p}{\xi^2-1}\right)^{(\alpha_j)} \exp\left(\frac{\xi^p}{\xi^2-1}\right), \quad |\alpha| = \sum_j \alpha_j = s. \tag{2.16}$$

Here the C_α values are constant integers with $|C_\alpha| \leqslant \eta_1^{-s}$, for some fixed $\eta_1 > 0$. We consider the prototype term $T_{p,s} := \left(\frac{\xi^p}{\xi^2-1}\right)^{(s)} \exp\left(\frac{\xi^p}{\xi^2-1}\right)$, corresponding to $\alpha = (0, 0, \ldots, s)$,

$$\begin{aligned} T_{p,s} &= \left(\frac{\xi^p}{\xi^2-1}\right)^{(s)} \exp\left(\frac{\xi^p}{\xi^2-1}\right) \\ &= \sum_{k=0}^{s} \binom{s}{k} \frac{p!}{(p-s+k)!} \xi^{p-s+k} \frac{k!}{(\xi^2-1)^{k+1}} \exp\left(\frac{\xi^p}{\xi^2-1}\right) + \text{ lower-order terms }. \end{aligned} \tag{2.17}$$

Here by 'lower-order terms' we refer to the singular behaviour of $(\xi^2-1)^{-j}$, $j \leqslant k$ near $\xi = \pm 1$, which is weaker than the leading term $(\xi^2-1)^{-k+1}$. To control the amplitude of $T_{p,s}$ we let $a(\xi) := (\xi^2 - 1)$ and note that the expression $|a(\xi)|^{-k} \exp(\alpha a(\xi) + \beta/a(\xi))$ is maximized at $\xi = \xi_{\max}$ so that $a(\xi_{\max}) \sim -\beta/k$, yielding

$$|T_{p,s}| \leqslant \mathrm{Const.} \sum_{k=0}^{s} \binom{s}{k} \frac{p!}{(p-s+k)!} k! k^k\, \mathrm{e}^{-k} < \mathrm{Const.}\, p! \sum_{k=0}^{s} \binom{s}{k} k! \leqslant \mathrm{Const.} 2^s p! s!.$$

The other 2^s terms in (2.16) admit similar bounds and the resulting $T_{p,p}$ bound yields (2.14) with $\eta = 4\eta_1$. To prove (2.15), we restrict our attention to a subinterval which is bounded away from ± 1, so that the ξ-dependent terms in (2.17) remain uniformly bounded, $(\xi^2-1)^{-j} \exp(\xi^p/(\xi^2-1)) \leqslant \eta_2^{-k}$,

$j \leqslant k+1$, and we are left with the desired upper bound

$$|T_{p,p}| \leqslant \text{Const.} \sum_{k=0}^{p} \binom{p}{k} \frac{p!}{k!} k! \eta_2^{-k} < \text{Const.} 2^p p! (\eta_2)^{-p},$$

and (2.15) follows with $\eta = 4\eta_1\eta_2$. □

The intricate part in the construction of such highly accurate filters or mollifiers is the further requirement for their localization (in physical space) or smoothness (in Fourier space). One cannot increase the order p arbitrarily without steepening Φ^σ or, equivalently, without losing smoothness of σ. The solution taken here was to satisfy the moments condition *approximately*, modulo exponentially negligible errors while retaining the desired smoothness properties. We note that our optimal adaptive filter is essentially localized in the physical space in the sense that the associated mollifier Φ^σ is exponentially small for $|y| \gg 1/N$, (2.5), see Fig. 1. In contrast, the adaptive mollifiers constructed in Tadmor & Tanner (2002), $\rho(y/d(x))D_p(y/d(x))/d(x)$, were compactly supported in physical space (adapted to the smoothness neighbourhood of x) and only essentially localized in the dual Fourier space. The precise result is quantified in the following theorem.

THEOREM 2.2 Consider the even filter σ with Gevrey regularity G_α satisfying the p-order accuracy condition (1.4), with $p \sim N^{1/\alpha}$. Then the associated mollifier, Φ^σ, satisfies the moments condition (1.3) modulo an exponentially negligible error,

$$\int_{y=-\pi}^{\pi} y^n \Phi^\sigma(y)\,\mathrm{d}y = \delta_{n0} + \text{Const.}\,\mathrm{e}^{-(\eta N)^{1/\alpha}}, \quad n \leqslant \text{Const.} N^{1/\alpha}.$$

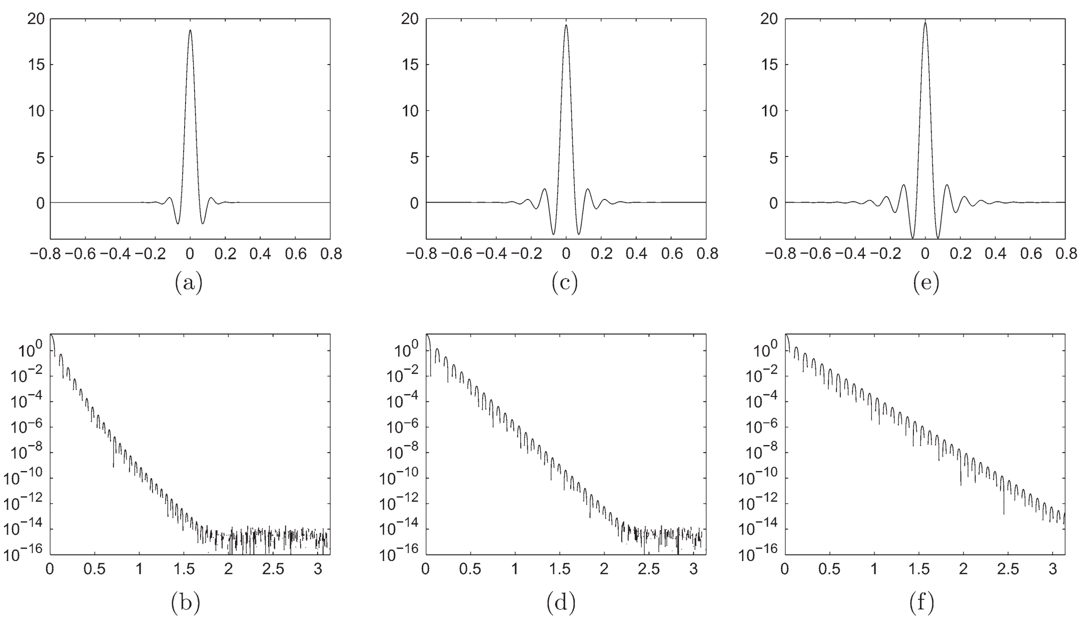

FIG. 1. The mollifier (top) and its semi-log plot (bottom) with the mollifier defined from the filter (3.1) used in the numerical experiments, with $N = 128$ and filter orders $p = 4$, 8 and 12 in (a,b), (c,d) and (e,f), respectively.

Proof. For proof, we appeal to (2.2)

$$\int_{y=-\pi}^{\pi} y^n \Phi^\sigma(y)\,\mathrm{d}y = \frac{N}{2\pi}\int_{y=-\pi}^{\pi} y^n \varphi^\sigma(Ny)\,\mathrm{d}y + \frac{N}{2\pi}\sum_{n\neq 0}\int_{y=-\pi}^{\pi} y^n\varphi^\sigma(N(y+2\pi n))\,\mathrm{d}y$$
$$=: \mathcal{I}_1 + \mathcal{I}_2.$$

For the first term on the right-hand side, we have

$$\mathcal{I}_1 = \frac{N}{2\pi}\int_{y=-\infty}^{\infty} y^n\varphi^\sigma(Ny)\,\mathrm{d}y - \frac{N}{2\pi}\int_{|y|\geqslant\pi} y^n\varphi^\sigma(Ny)\,\mathrm{d}y =: \mathcal{I}_{11} + \mathcal{I}_{12}.$$

We now have $\mathcal{I}_{11} = (-iN)^n\sigma^{(n)}(0) = \delta_{n0}$ by (1.4), where the usual decay rate $|\varphi^\sigma(y)| \leqslant \text{Const.}\|\sigma\|_{C^s}$ $|y|^{-s}$ yields

$$|\mathcal{I}_{12}| \leqslant \text{Const.}\frac{N^{1-s}}{2\pi}\|\sigma\|_{C^s}\int_{\pi}^{\infty} y^{n-s}\,\mathrm{d}y \leqslant \text{Const.}(\eta N)^{1-s}(s!)^\alpha, \quad n \leqslant s-2.$$

The remainder amounts to a similarly exponentially small term

$$|\mathcal{I}_2| \leqslant \text{Const.}\frac{N^{1-s}}{2\pi}\|\sigma\|_{C^s}\int_{y=-\pi}^{\pi} |y|^n\frac{1}{(2\pi-|y|)^s}\,\mathrm{d}y \leqslant \text{Const.}(\eta N)^{1-s}(s!)^\alpha, \quad n \leqslant s,$$

which is minimized at $s \sim (\eta N)^{1/\alpha}$ and the lemma follows. □

We note in passing that the last theorem could be used as a starting point for an alternative proof of the main result stated in Theorem 2.1.

3. Numerical experiments

For the following examples we utilize the filter

$$\sigma_p(\xi) = \begin{cases} \exp\left(\frac{c_p\xi^p}{\xi^2-1}\right), & |\xi| < 1, \\ 0, & |\xi| \geqslant 1, \end{cases} \tag{3.1}$$

which has Gevrey regularity of order $\alpha = 2$. Its advocated order is then optimized at the adaptive order, $p = p(x) = \sqrt{\kappa N d(x)}$. For a given filter, the free constant c_p should be selected to enhance the immediate localization of $\Phi^\sigma(\cdot)$ by minimizing $\|\sigma\|_{C^1}$. The value of such an optimal c_p does not permit a closed-form expression; an approximate condition used in the numerical examples below is $\sigma^{(2)}(1/2) = 0$, resulting in

$$c_p := 2^p\frac{3}{4}\frac{9p^2+3p+14}{9p^2+12p+4}.$$

To allow a direct comparison between our adaptive filters and the adaptive mollifiers advocated in Tadmor & Tanner (2002), we concern ourselves with the two prototypes of piecewise analytic functions, $f_1(x)$ and $f_2(x)$, given below.

$$f_1(x) = \begin{cases} \sin(x/2), & x \in [0,\pi), \\ -\sin(x/2), & x \in [\pi, 2\pi), \end{cases} \tag{3.2}$$

$$f_2(x) = \begin{cases} (2\mathrm{e}^{2x} - 1 - \mathrm{e}^\pi)/(\mathrm{e}^\pi - 1), & x \in [0, \pi/2), \\ -\sin(2x/3 - \pi/3), & x \in [\pi/2, 2\pi) \end{cases}. \tag{3.3}$$

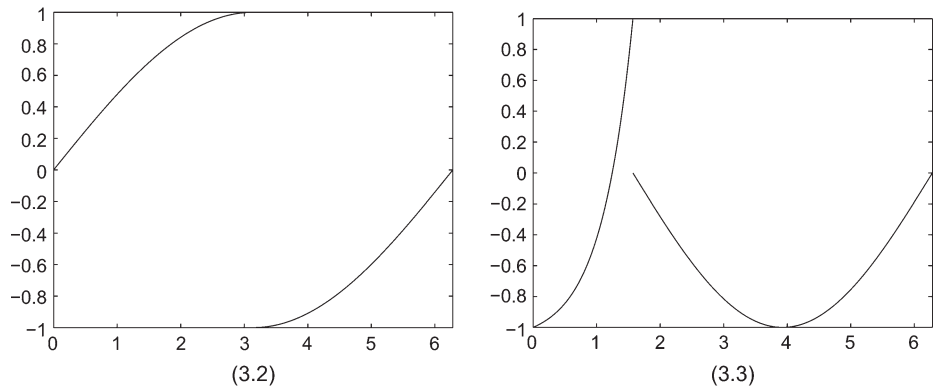

FIG. 2. (a) the graph of the function (3.2) and (b) the graph of function (3.3)

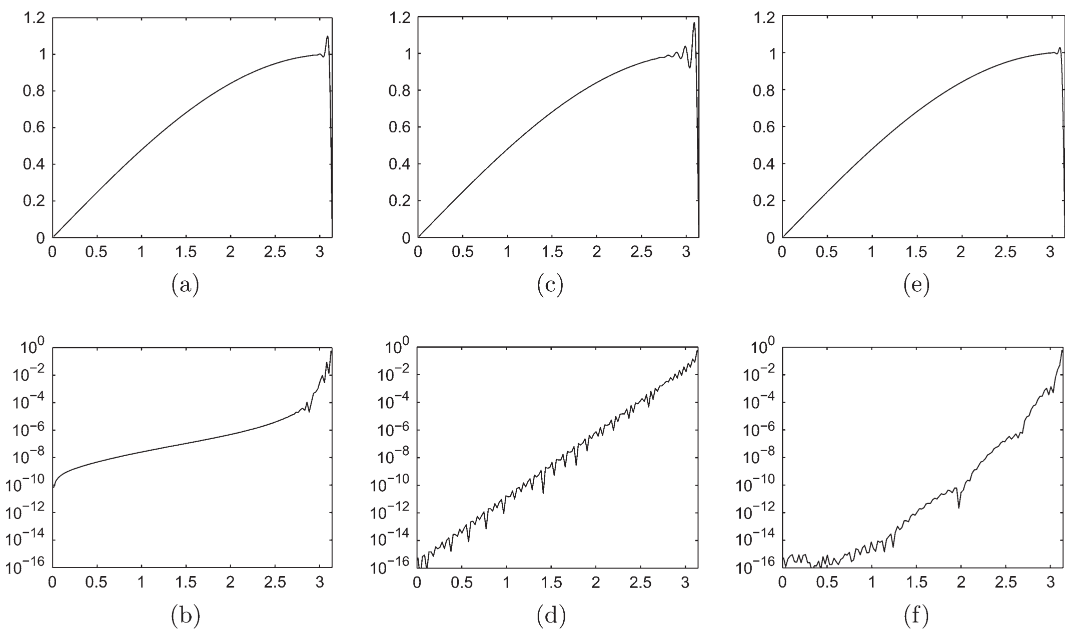

FIG. 3. Recovery of $f_1(x)$ (top) and the approximation error (bottom) from their $N = 128$-mode spectral projections. The filter (3.1) was of orders $N^{1/4}$ in (a,b), $N^{1/2}$ in (c,d) and $\max\left(2, \frac{1}{2}(Nd(x))^{1/2}\right)$ in (e,f).

The first function, $f_1(\cdot)$, shown in figure 2(a), possesses a mild regularity constant and a single discontinuity at $x = \pi$; consequently, $d(x) = |x - \pi|$ for $x \in [0, 2\pi]$. The second function, $f_2(\cdot)$, shown in figure 2(b), was constructed as a more challenging test problem with a large gradient to the left of the discontinuity at $x = \pi/2$. Moreover, lacking periodicity $f_2(\cdot)$ feels three discontinuities per period;

$$d(x) = \min(|x|, |x - \pi/2|, |x - 2\pi|), \quad x \in [0, 2\pi].$$

For both functions, the exact Fourier coefficients, $\{\widehat{f}(k)\}_{k \leqslant N}$, are given and then filtered to recover the intermediate point values $\frac{\pi}{N}(\nu - \frac{1}{2})$ for $\nu = 1, 2, \ldots 2N$. Graphs (a–d) in figures 3, 4 and 5 use

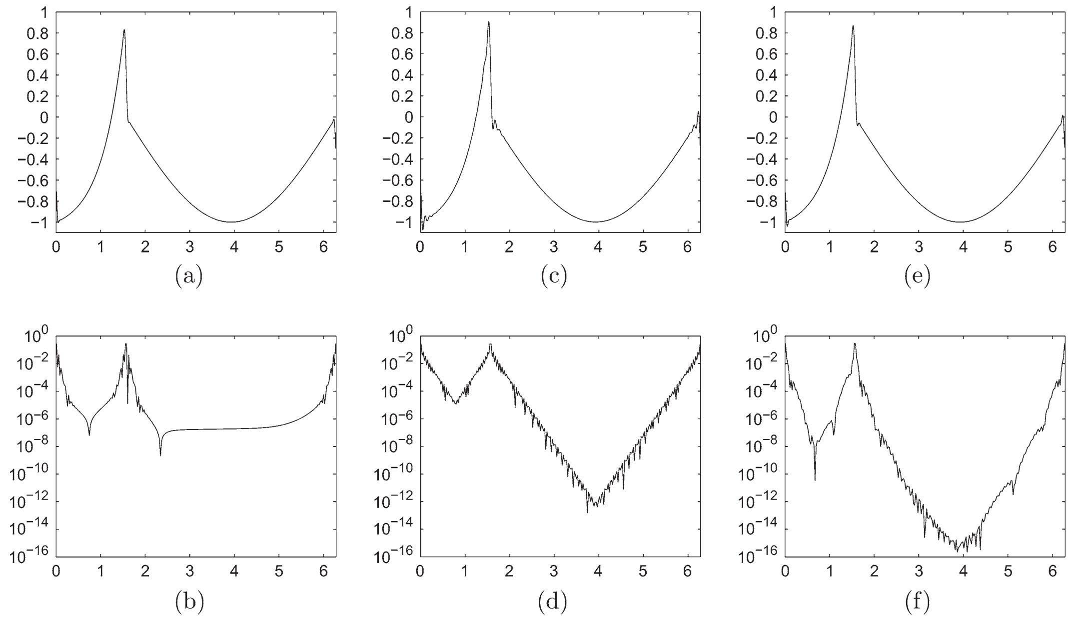

FIG. 4. Recovery of $f_2(x)$ (top) and the approximation error (bottom) from its $N = 128$-mode spectral projections. The filter (3.1) was of orders $N^{1/4}$ in (a,b), $N^{1/2}$ in (c,d) and $\max(2, \frac{1}{2}(Nd(x))^{1/2})$ in (e,f).

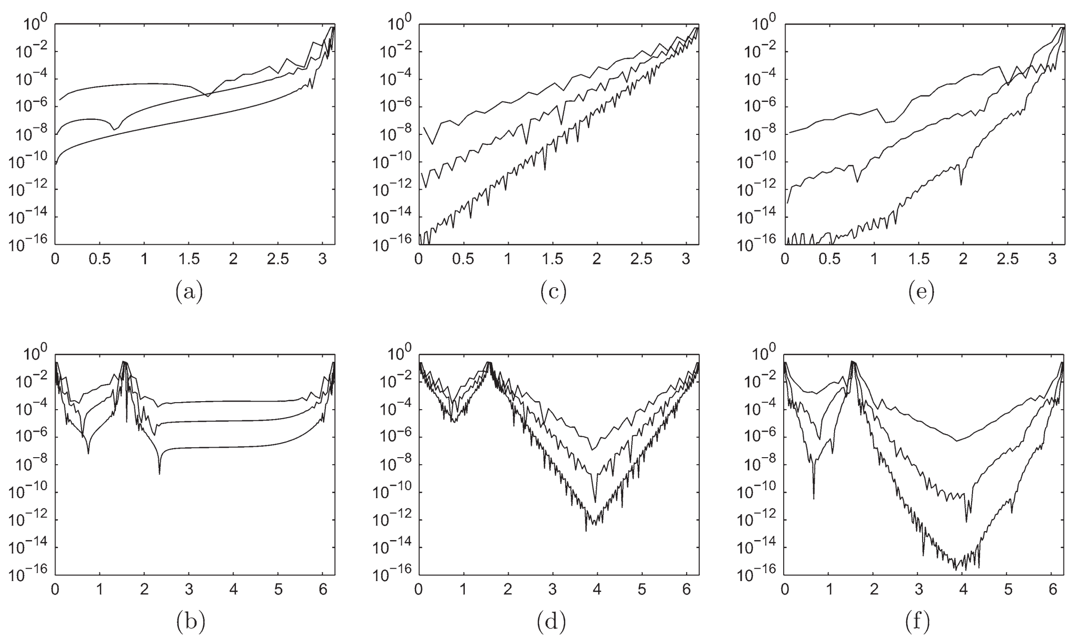

FIG. 5. Error plots for the recovery of $f_1(x)$ (top) and $f_2(x)$ (bottom) from their $N = 32$-, 64- and 128-mode spectral projections. The filter (3.1) was of orders $N^{1/4}$ in (a,b), $N^{1/2}$ in (c,d) and $\max(2, \frac{1}{2}(Nd(x))^{1/2})$ in (e,f).

fixed-order filters, verifying the well-known fact that higher-order filters give superior convergence away from discontinuities and lower-order filters near discontinuities. Graphs (e–f) in figures 3, 4 and 5 illustrate the superior convergence for the adaptive filter described in Theorem 2.1, computed with adaptive order $p = p(x) = \max(2, \frac{1}{2}(Nd(x))^{1/2})$. We note in passing that the same filter order is used for both $f_1(\cdot)$ and $f_2(\cdot)$, ignoring the different analyticity properties of f_1 and f_2 (reflected by different analyticity constants η_f) and achieving exponential accuracy in both instances. Results of the adaptive filter are contrasted with the spectrally accurate filter of Vandeven (1991) where the order, $p = N^\gamma$, remains uniform throughout the computational domain.

4. Summary

The analysis presented here quantitatively resolves the classical methodology that for improved accuracy low-order filters should be used near discontinuities and high-order filters away from discontinuities. The optimal adaptive filters presented here retain the traditional robustness associated with low-order filtering, yet achieve a significant increase in accuracy with minimal increase to computational cost. Combined with the automated edge detection methods (Gelb & Tadmor, 2000) adaptive-order filtering is a *black box* procedure for the exponentially accurate reconstruction of a piecewise smooth function from its spectral information.

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