Homework 1 (due Feb. 11 at midnight)

Note: the percentages are with respect to the total for this homework.

1. [5%] Compute the polynomial interpolant \( Q_2(x) \) of degree \( \leq 2 \) that satisfies \( Q_2(0) = 0, Q_2(1) = 1, Q_2(2) = 0 \).

2. [5%] Find the polynomial interpolant \( Q_3(x) \) of degree \( \leq 3 \) that agrees with \( \sqrt{x} \) at 0, 1, 3, 4. Compare the approximation \( Q_3(2) \) with \( \sqrt{2} \approx 1.414216 \).

3. [10%] Suppose that we are interpolating a function on \([0, 1]\) by using only two points \( x_0, x_1 \). Directly and explicitly show (that is, without resorting to the proven properties of Chebyshev points) how to choose \( x_0 \) and \( x_1 \) so that the term \((x - x_0)(x - x_1)\) in the error estimate is minimized.

4. [5%] Let \( V_n \) be the Vandermonde matrix for the points \( x_0, \ldots, x_n \). Show that \( \det V_1 = x_1 - x_0 \), \( \det V_2 = (x_1 - x_0)(x_2 - x_1)(x_2 - x_0) \).

5. [5%] Provide an upper bound for the error when approximating \( f(x) = \cosh(x) \) by polynomial interpolation at the points

\[
x_i = -1 + \frac{i}{2}, \quad i = 0, \ldots, 4.
\]

6. [10%] Show that the Lagrange polynomials satisfy, for any \( n \geq 1 \),

\[
\sum_{i=0}^{n} \ell_i^n(x) = 1 \quad \forall x.
\]

7. [25%] Approximate the function \( f(x) = (1 + 25x^2)^{-1} \) in \( x \in [-1, 1] \) through polynomial interpolation using:

- Equally spaced points in \([-1, 1]\): \( x_i = -1 + (2i)/n \), \( i = 0, 1, \ldots, n \), for \( n = 3, 9, 15, 21 \).
- Chebyshev points (with \( n = 3, 9, 15, 21 \) as before).

Plot the error function \(|f(x) - Q_n(x)|\) versus \( x \) for each value of \( n \) and comment on the difference(s) between the two approaches. Explain the root of the observed behavior.

In both cases (equally spaced and Chebyshev points) plot, using a fine mesh of \( x \) points (say, 1,000 of them) \( \max_{x \in [-1, 1]} |f(x) - Q_n(x)| \) versus \( n \). Discuss the results.

8. [15%] For a fine sample of, say, 1,000 \( x \)-points, plot (versus \( n \)) the maximum of the absolute value of the nodal polynomial

\[
\omega_{n+1}(x) = (x - x_0) \ldots (x - x_n)
\]

in \( x \in [-1, 1] \),

\[
\max_{x \in [-1, 1]} |\omega_n(x)|
\]

when using \((n+1)\) equally spaced node points \( \{x_0, x_1, \ldots, x_n\} \) and when using \((n+1)\) Chebyshev ones. Compare to the theoretical expected results and discuss.

9. [10%] Prove that the Hermite interpolant is unique.
10. [10%] Prove that the error for the Hermite interpolant \( p(x) \) of a function \( f(x) \) at node points \( \{x_0, x_1, \ldots, x_n\} \) satisfies

\[
f(x) - p(x) = \frac{1}{(2n+2)!} f^{(2n+2)}(\xi) \prod_{i=0}^{n} (x - x_i)^2
\]

for some \( \xi \), where \( f^{(j)} \) denotes the \( j \)-th derivative,

\[
f^{(j)} := \frac{d^j}{dx^j} f(x).
\]