Homework 6 (due April 11 at noon)

1. [30%] Consider the matrix

\[
A = \begin{pmatrix}
2 & 1 & 1 & 0 \\
4 & 3 & 3 & 1 \\
8 & 7 & 9 & 5 \\
6 & 7 & 9 & 8
\end{pmatrix}
\]

Using “pencil and paper”:

(a) Solve the system \( Ax = b \), for an arbitrary \( b \) vector, using Gaussian elimination.
(b) Use the multipliers from the previous item to compute the LU decomposition of \( A \). Check that the product of \( L \) times \( U \) does give \( A \) as expected.

Note: present/sketch the intermediate steps rather than only the final results.

2. [10%] Compute the \( L_1 \), \( L_2 \) and \( L_\infty \) norms of the following vector and matrix, respectively,

\[
\begin{pmatrix} 2 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}
\]

3. [30%] Consider a linear system of equations \( Ax = b \)

where, as usual, \( A \) and \( b \) are known and \( x \) is the unknown. Suppose now that \( b \) changes

\[ b \to b + \delta b \]

Show that the condition number of the problem can be bounded by the condition number of \( A \), the latter defined as

\[ t\kappa(A) = ||A||||A^{-1}|| \]

4. [30%] Residuals and ill-conditioning. Suppose that \( \tilde{x} \) is an approximant to the solution of a system \( Ax = b \).

One way to try to quantify the accuracy of \( \tilde{x} \) would be by computing the residual vector,

\[ r := b - A\tilde{x} \]

If \( \tilde{x} = x \) then the residual would be zero. Thus, we would expect \( r \) to be small if \( \tilde{x} \) were a good approximation to \( x \), and vice versa. This is true in some cases but the magnitude of \( r \) can be misleading if \( A \) is ill-conditioned.

Consider the system of equations

\[
\begin{align*}
0.780x_1 + 0.563x_2 &= 0.217 \\
0.913x_1 + 0.659x_2 &= 0.254
\end{align*}
\]

and two (very different!) approximate solutions,

\[
\begin{align*}
\tilde{x}_1 &= \begin{pmatrix} 0.341 \\ -0.087 \end{pmatrix}, & \tilde{x}_2 &= \begin{pmatrix} 0.999 \\ -1.001 \end{pmatrix}
\end{align*}
\]

First, verify that the exact solution is

\[
x = \begin{pmatrix} 1 \\ -1 \end{pmatrix}
\]

Next, compute the residuals associated with \( \tilde{x}_1 \) and \( \tilde{x}_2 \). Notice that based on them you would expect \( \tilde{x}_1 \) to be a better approximation to \( x \), while it is the other way around.