Homework 7

1. Show that the number of operations in Gaussian elimination for a generic \( n \times n \) matrix is, for large \( n \), of order \( O(n^3) \).

2. Consider the matrix

\[
A = \begin{pmatrix}
2 & 1 & 1 & 0 \\
4 & 3 & 3 & 1 \\
8 & 7 & 9 & 5 \\
6 & 7 & 9 & 8
\end{pmatrix}
\]

Using "pencil and paper" (as opposed to Problem 4):

(a) Solve the system \( Ax = b \), for an arbitrary \( b \) vector, using Gaussian elimination.

(b) Use the multipliers from the previous item to compute the LU decomposition of \( A \). Check that the product of \( L \) times \( U \) does give \( A \) as expected.

Note: present/sketch the intermediate steps rather than only the final results.

3. We have seen that Gaussian elimination yields a factorization \( A = LU \), where \( L \) has ones in the diagonal but \( U \) in general does not. Describe at a high level the factorization that results if this process is varied in the following ways:

(a) Elimination by columns from left to right, rather than by rows from top to bottom, so that \( A \) is made lower-triangular.

(b) Gaussian elimination applied after a preliminary scaling of the columns of \( A \) by a diagonal matrix \( D \). What form does a system \( Ax = b \) take under this rescaling? Is it the equations or the unknowns that are rescaled by \( D \)?

(c) Gaussian elimination carried further, so that after \( A \) (assumed non-singular) is brought to upper-triangular form, additional column operations are carried out so that this upper triangular matrix is made diagonal.

4. Write a program that solves \( Ax = b \) through Gaussian elimination, for:

(a) \( A \) the matrix of Problem 2, and \( b = (1, 7, 5, 2) \). Compare with the "pencil and paper" solution from Problem 2.

(b) \( A \) an \( n \times n \) matrix with coefficients \( a_{ij} = i^{j-1} \) and \( b_i = i \) for \( n = 7, 15, 50 \). From the numerical solution \( \tilde{x} \) compute \( A\tilde{x} \) and compare with the components of \( b \). Discuss.