Using transformation media to manipulate waves

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Collaborators

- Key contributor: Huanyang (Kenyon) Chen
- Ho Bou, Prof. WJ Wen's group: experimental work
- Jiaotung U (Prof. H. Ma), Institute of Microsystem and Information Technology Prof. X. Jiang, Fudan U (Prof. Z Jian)

Invisibility cloak for EM waves: Transformation media

$$\nabla \times \mathbf{E} + i\omega\mu\mathbf{H} = 0, \quad \nabla \times \mathbf{H} - i\omega\varepsilon\mathbf{E} = 0,$$

Maxwell's equations at fixed frequency

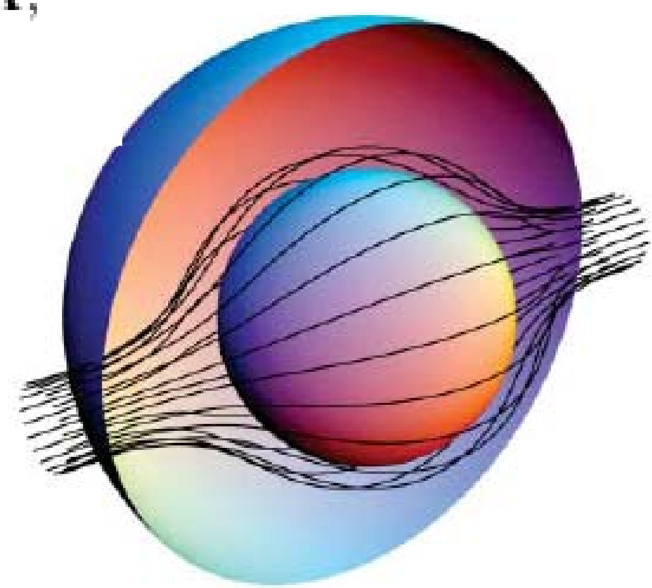
$$\mathbf{x}' = \mathbf{x}'(\mathbf{x}), \quad \mathbf{E}'(\mathbf{x}') = (\mathbf{A}^{\mathrm{T}})^{-1}\mathbf{E}(\mathbf{x}), \quad \mathbf{H}'(\mathbf{x}') = (\mathbf{A}^{\mathrm{T}})^{-1}\mathbf{H}(\mathbf{x}),$$

$$\nabla' \times \mathbf{E}' + i\omega\mu'\mathbf{H}' = 0, \quad \nabla' \times \mathbf{H}' - i\omega\varepsilon'\mathbf{E}' = 0,$$

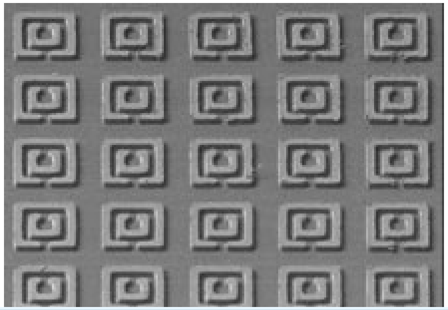
$$A_{kl} = \frac{\partial x'_k}{\partial x_l}.$$

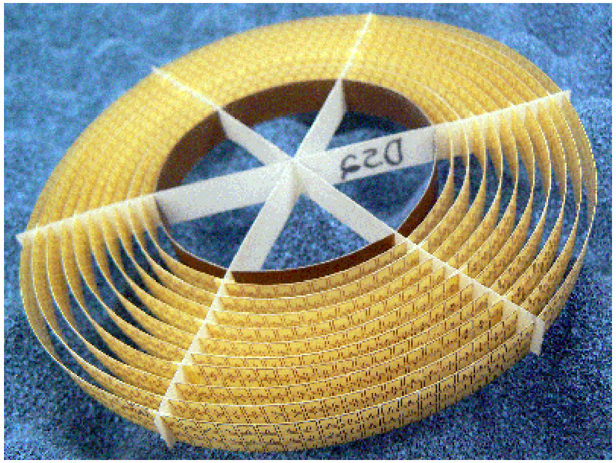
$$\mu'(\mathbf{x}') = \mathbf{A}\mu(\mathbf{x})\mathbf{A}^{\mathrm{T}}/\det\mathbf{A}, \quad \varepsilon'(\mathbf{x}') = \mathbf{A}\varepsilon(\mathbf{x})\mathbf{A}^{\mathrm{T}}/\det\mathbf{A},$$

J. B. Pendry, D. Schurig, and D. R. Smith, Science **312**, 1780 (2006).
G. W. Milton, M. Briane and J. R. Wills, New J. Phys **8**, 248 (2006).

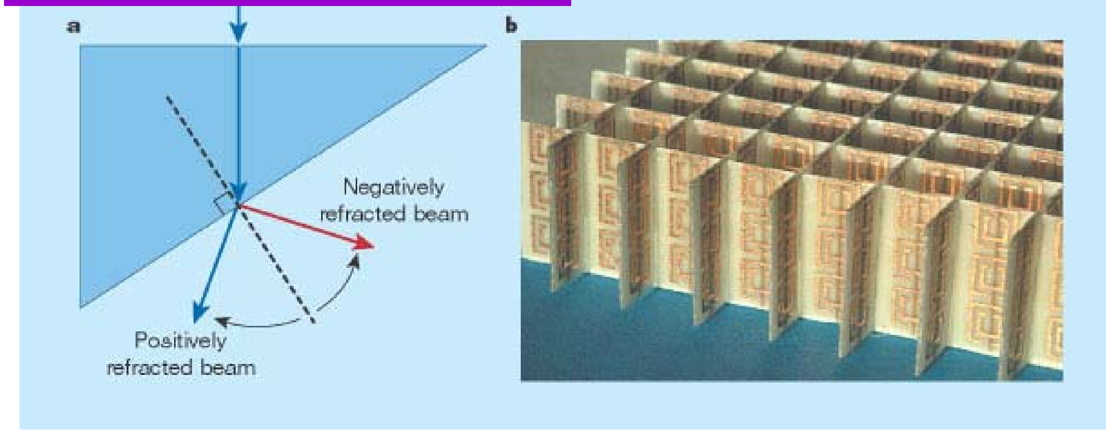


Metamaterials (a << λ)





Negative refractive index

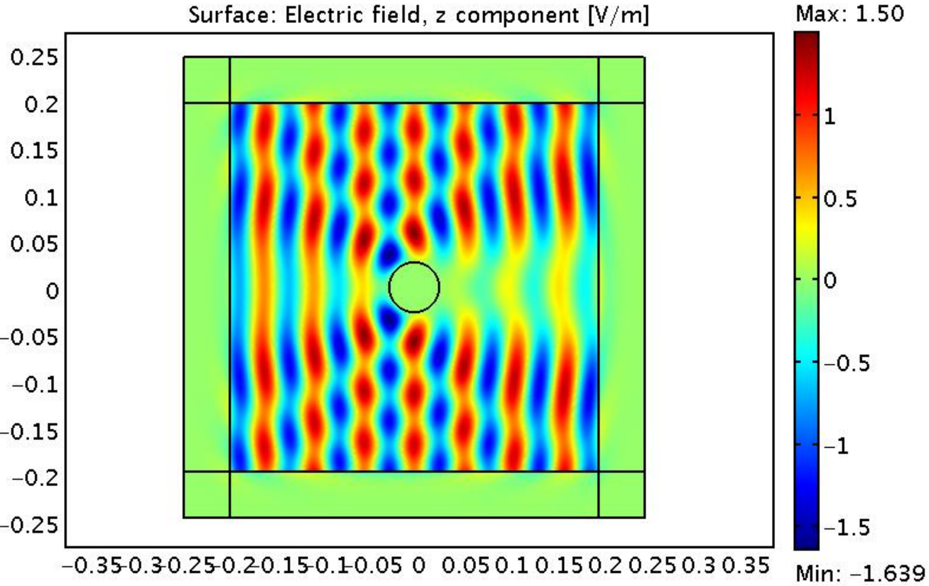


Man-made materials with small embedded artificial resonators which gives nearly any value of ε,μ

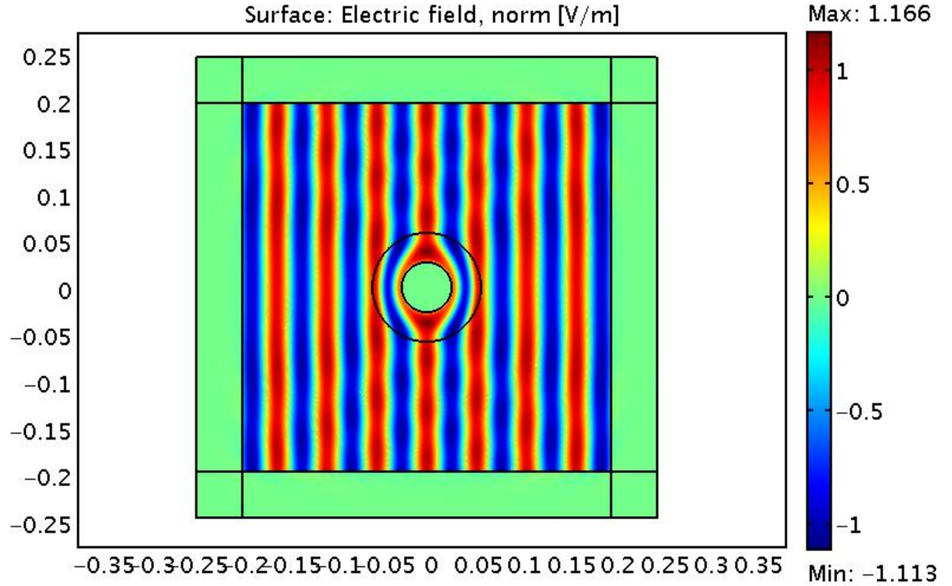
Forward vs reverse

- Photonic crystals, metamaterials
  - Define the structure first, see the effect on wave propagation later
  - “Design” requires iteration, search in a large parameter space
- Transformation media
  - Define the path of light first, obtain the material specifications directly
  - Strange material properties
  - If there were no “metamaterials”, task would be impossible

Invisibility Cloaking



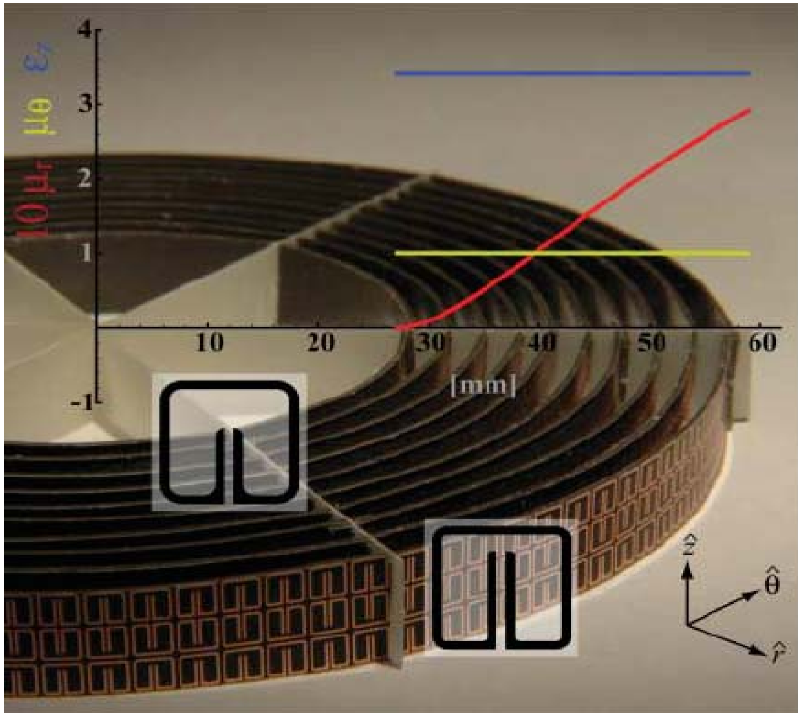
without cloak

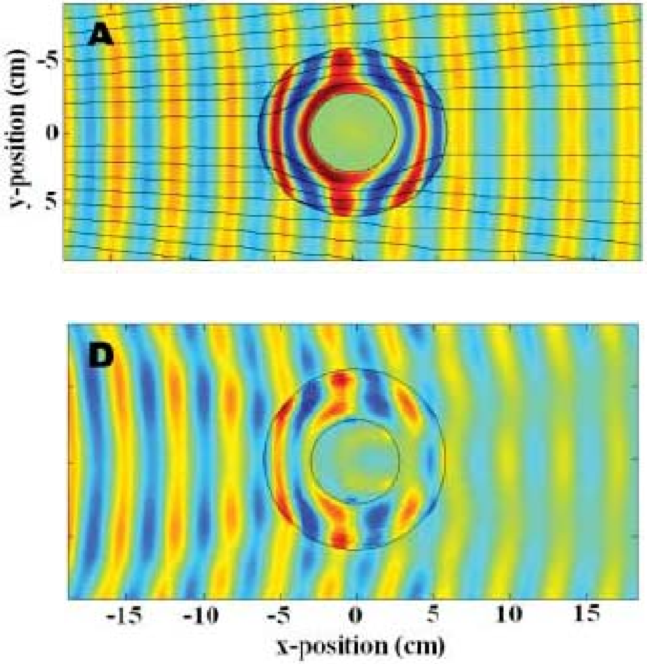


with cloak

EM Cloaking at Microwave Frequency

- Metamaterials provide the spatial profile
- Reduced material parameters used
- Proof-of-concept experiment





D Schurig, et. al., Science **314**, 977 (2006).

Some “What if”s

- What if the transformation is “angular” instead of radial?
- What if we have other kinds of wave?
- Can transformation media cloak hide everything inside?
- Can we extend the bandwidth?

- Invisibility cloak is a radial transform
- What will happen if we perform a angular transform?

An invisible cloak that rotate fields

- Rotation mapping:

$$r < a:$$

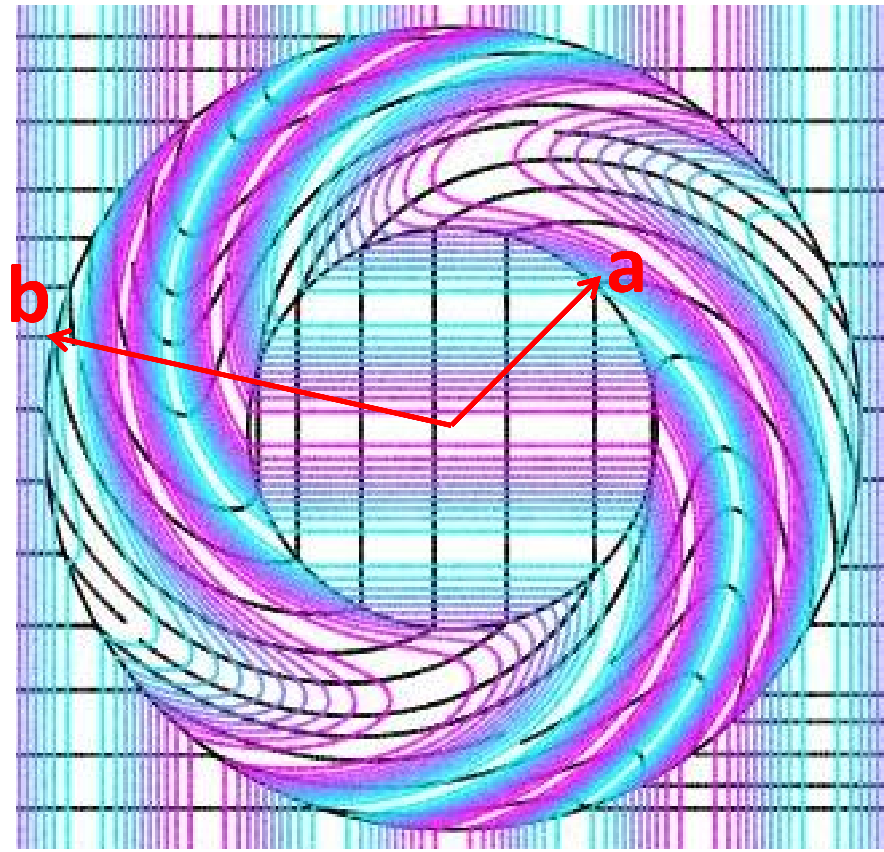
$$r' = r, \quad z' = z, \quad \theta' = \theta + \theta_0;$$

$$a < r < b:$$

$$r' = r, \quad z' = z, \quad \theta' = \theta + \theta_0 \frac{f(b) - f(r)}{f(b) - f(a)};$$

$$r > b:$$

$$r' = r, \quad z' = z, \quad \theta' = \theta.$$



Space

$$\overleftrightarrow{\varepsilon} = \overleftrightarrow{\mu} = \begin{vmatrix} 1 + 2t\cos\theta'\sin\theta' + t^2\sin^2\theta' & -t^2\cos\theta'\sin\theta' - t(\cos^2\theta' - \sin^2\theta') & 0 \\ -t^2\cos\theta'\sin\theta' - t(\cos^2\theta' - \sin^2\theta') & 1 - 2t\cos\theta'\sin\theta' + t^2\cos^2\theta' & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} 1 + 2t(x'y'/r^2) + t^2(y'^2/r^2) & -t^2(x'y'/r^2) - t(x'^2/r^2 - y'^2/r^2) & 0 \\ -t^2(x'y'/r^2) - t(x'^2/r^2 - y'^2/r^2) & 1 - 2t(x'y'/r^2) + t^2(x'^2/r^2) & 0 \\ 0 & 0 & 1 \end{vmatrix},$$

$$t = \theta_0 \frac{rf'(r)}{f(b) - f(a)}$$

Define auxiliary angle

$$\cos\tau = \frac{t}{\sqrt{t^2+4}}, \quad \sin\tau = \frac{2}{\sqrt{t^2+4}}.$$

2D Field Rotator

$$\overleftrightarrow{\varepsilon} = \begin{vmatrix} \varepsilon_{xx} & \varepsilon_{xy} \\ \varepsilon_{xy} & \varepsilon_{yy} \end{vmatrix}$$

For TE wave: (Ex,Ey, Hz)

$$\varepsilon_{xx} = \varepsilon_u \cos^2(\theta + \tau/2) + \varepsilon_v \sin^2(\theta + \tau/2)$$
$$\varepsilon_{xy} = (\varepsilon_u - \varepsilon_v)\sin(\theta + \tau/2)\cos(\theta + \tau/2)$$
$$\varepsilon_{yy} = \varepsilon_u \sin^2(\theta + \tau/2) + \varepsilon_v \cos^2(\theta + \tau/2)$$

$$\mu_z = 1$$

$$\varepsilon_u = 1 + (1/2)t^2 - (1/2)t\sqrt{t^2+4}$$
$$\varepsilon_v = 1/\varepsilon_u = 1 + (1/2)t^2 + (1/2)t\sqrt{t^2+4}$$

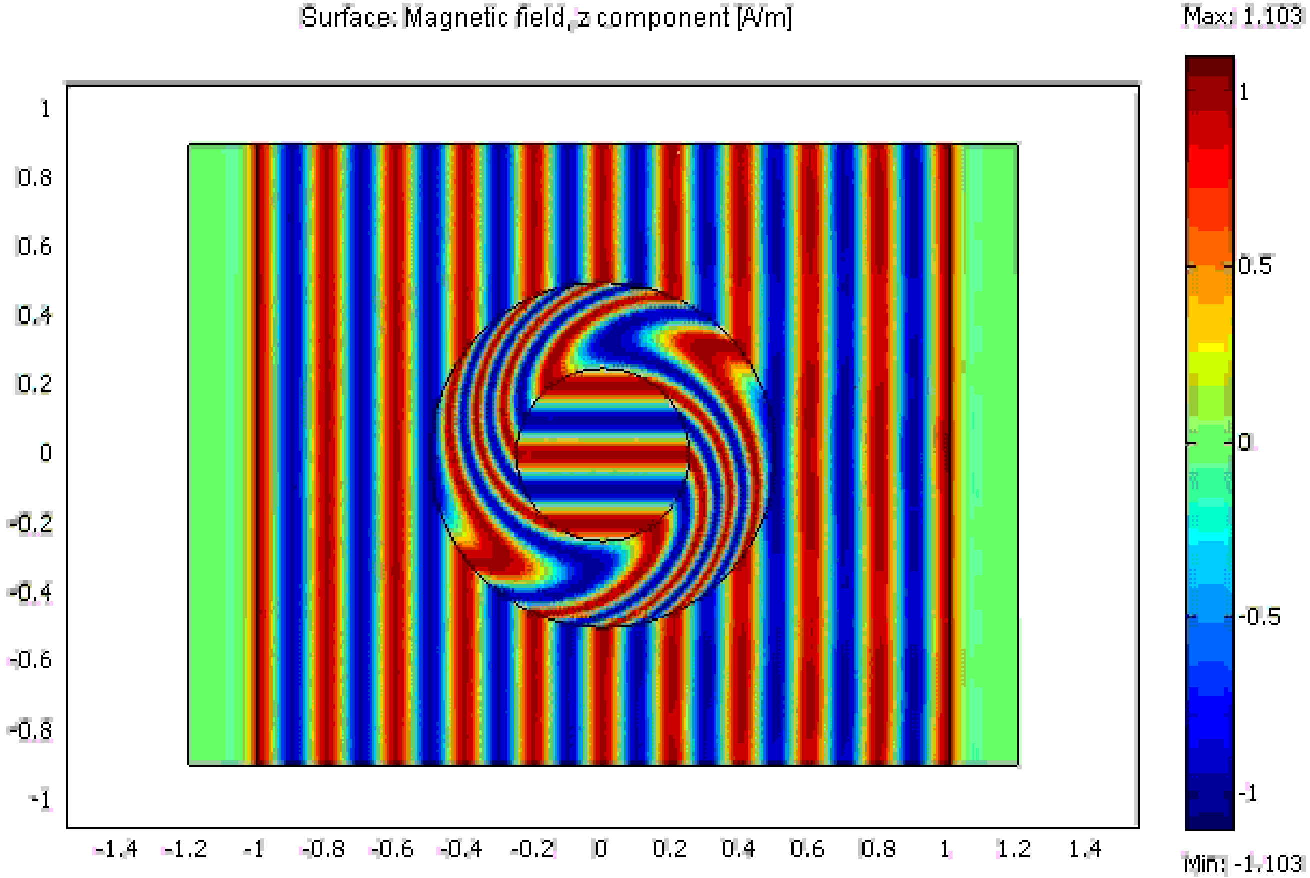
$$\cos\tau = t/\sqrt{t^2+4}$$
$$\sin\tau = 2/\sqrt{t^2+4}$$

$$If \;\; f(r) = \ln(r),\; t = \frac{\theta_0}{\ln(b/a)}$$

$\varepsilon_u \; \varepsilon_v \; \mu_z \; \tau$ are constants

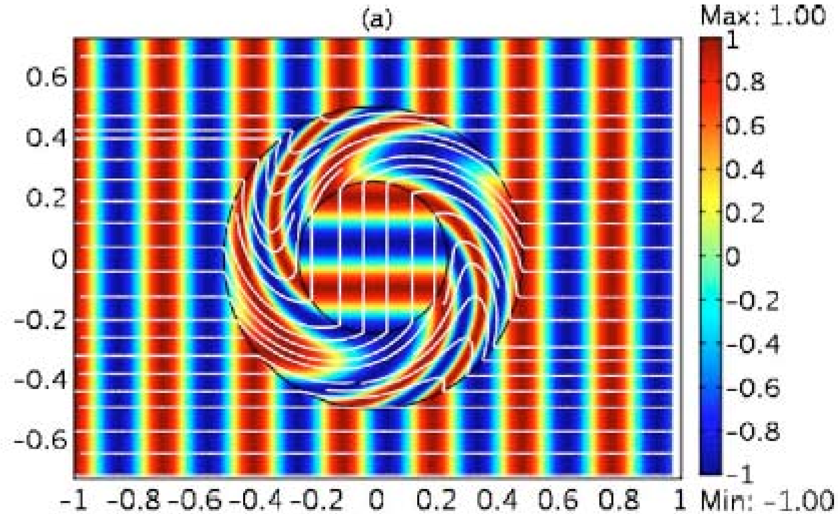
Rotation Transformation: Cloak rotates field

- observers inside/outside the rotation coating would see a rotated world with respect to each other.
- Cloak is “invisible”

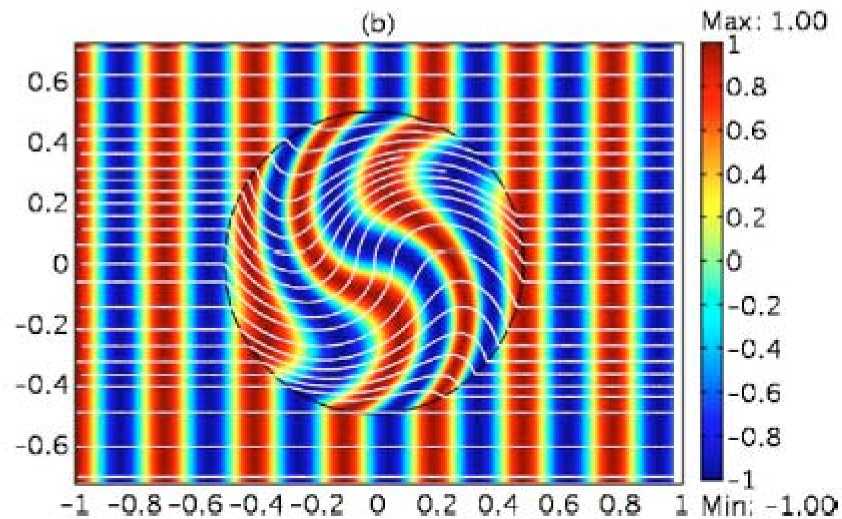


Appl. Phys. Lett. **90**, 241105 2007

Inner radius *a*

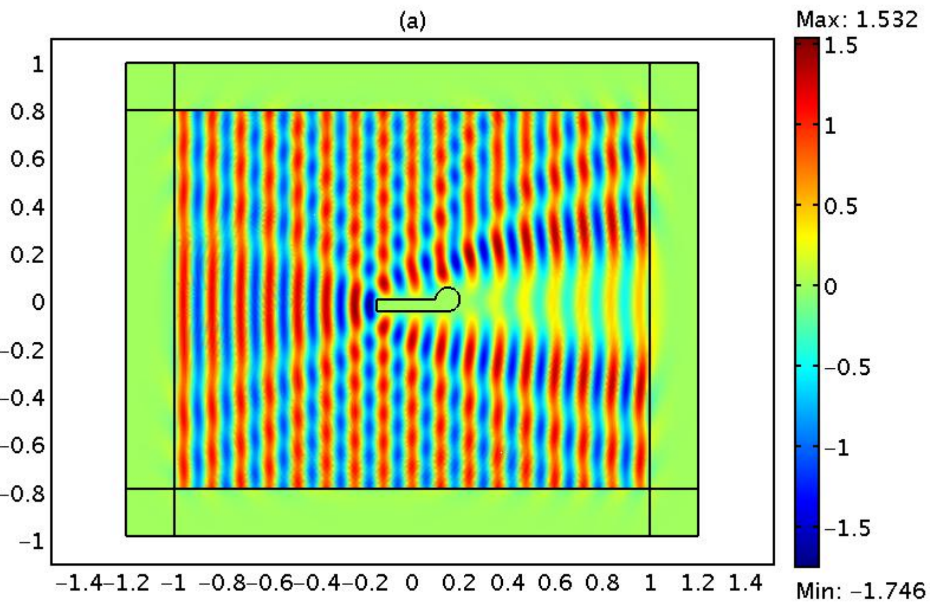


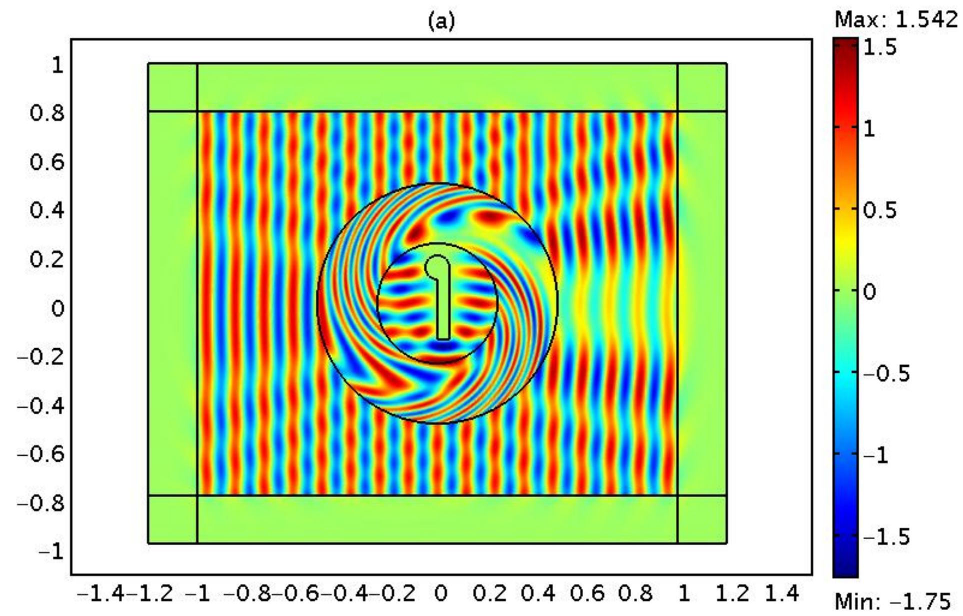
Inner radius *a*=0.



Magnetic-field distribution in the vicinity of the rotation coating. Power-flow lines in white show the flow of EM power.

Rotation Cloaking



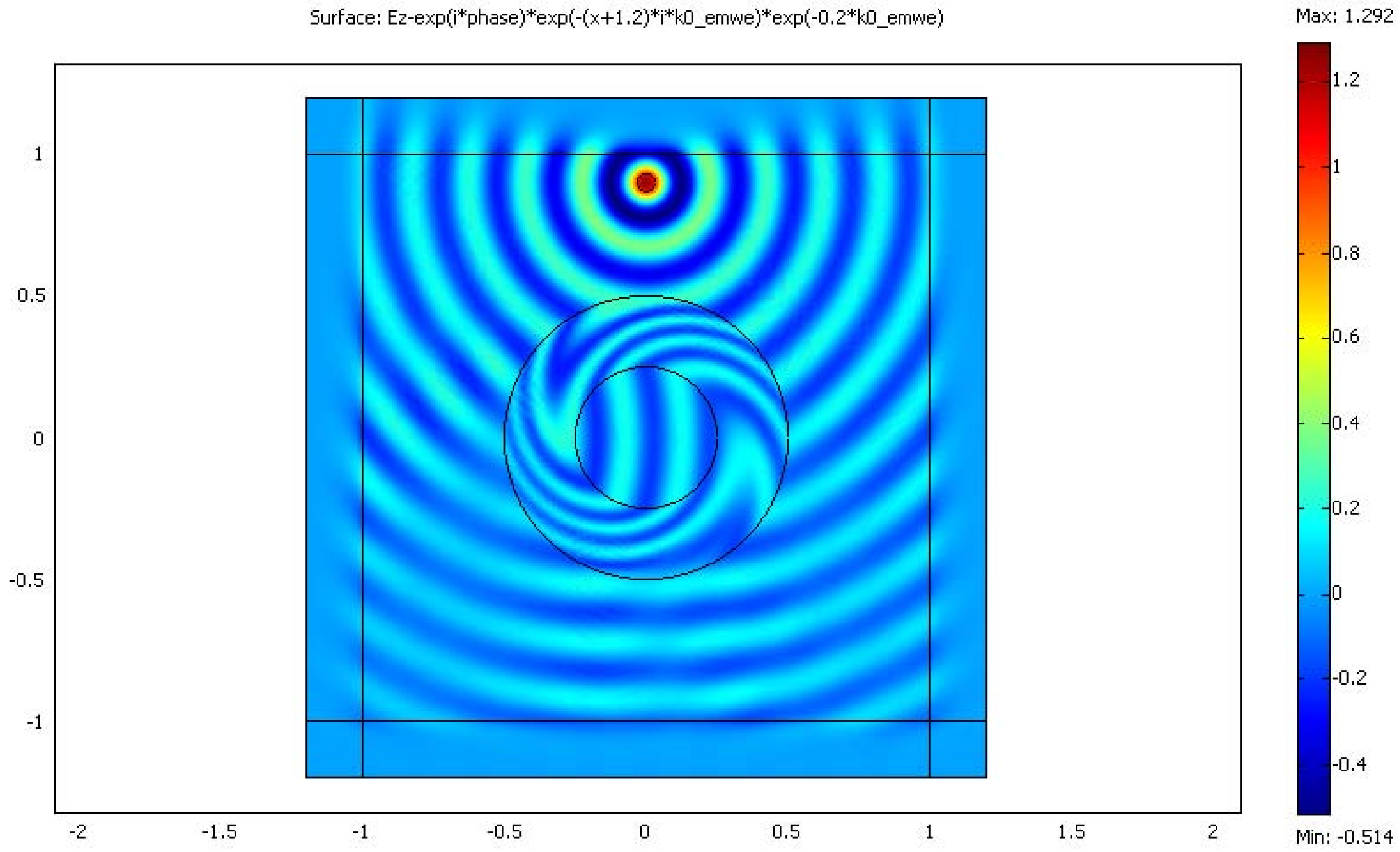


without cloak

With cloak:
Object apparently rotated to external observer

Same functionality for any kind of source

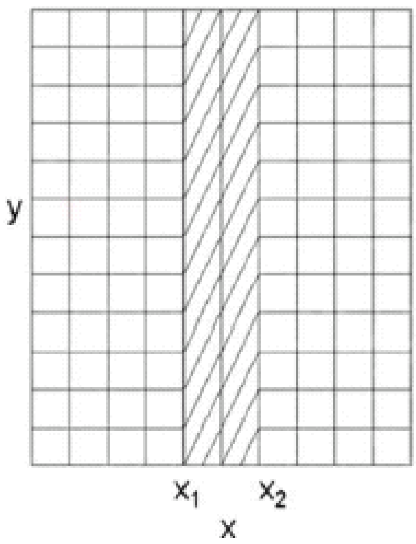
Point source apparently rotated to inner observer



Simpler constructions

- Layered system composing of isotropic materials

Shifting by t units



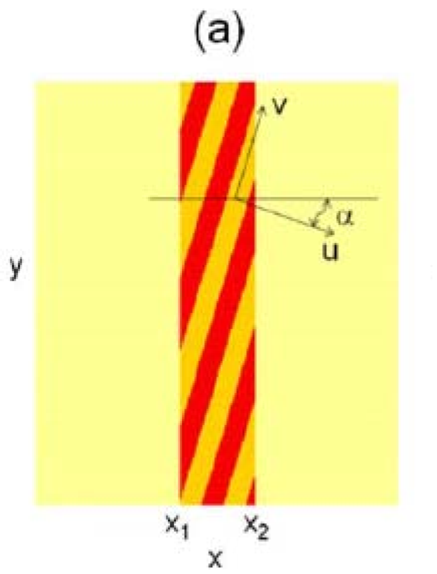
$$x' = x,\ z' = z \text{ and } y' = y \text{ (for } x < x_1\text{)},$$
$$x' = x,\ z' = z \text{ and } y' = y + t(x - x_1) \text{ (for } x_1 < x < x_2\text{)},$$
$$x' = x,\ z' = z \text{ and } y' = y + t(x_2 - x_1) \text{ (for } x > x_2\text{)},$$

$$\overleftrightarrow{\varepsilon}' = |\det(\overleftrightarrow{\Lambda})|^{-1}\, \overleftrightarrow{\Lambda}\overleftrightarrow{\varepsilon}\overleftrightarrow{\Lambda}^T,$$
$$\overleftrightarrow{\mu}' = |\det(\overleftrightarrow{\Lambda})|^{-1}\, \overleftrightarrow{\Lambda}\overleftrightarrow{\mu}\overleftrightarrow{\Lambda}^T,$$

$For \quad x_1 < x < x_2$

$$\overleftrightarrow{\varepsilon}' = \begin{bmatrix} 1 & 0 \\ t & 1 \end{bmatrix} \begin{bmatrix} \varepsilon_a & 0 \\ 0 & \varepsilon_b \end{bmatrix} \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$$

Oblique layered system: each layer isotropic



Layer thickness << λ

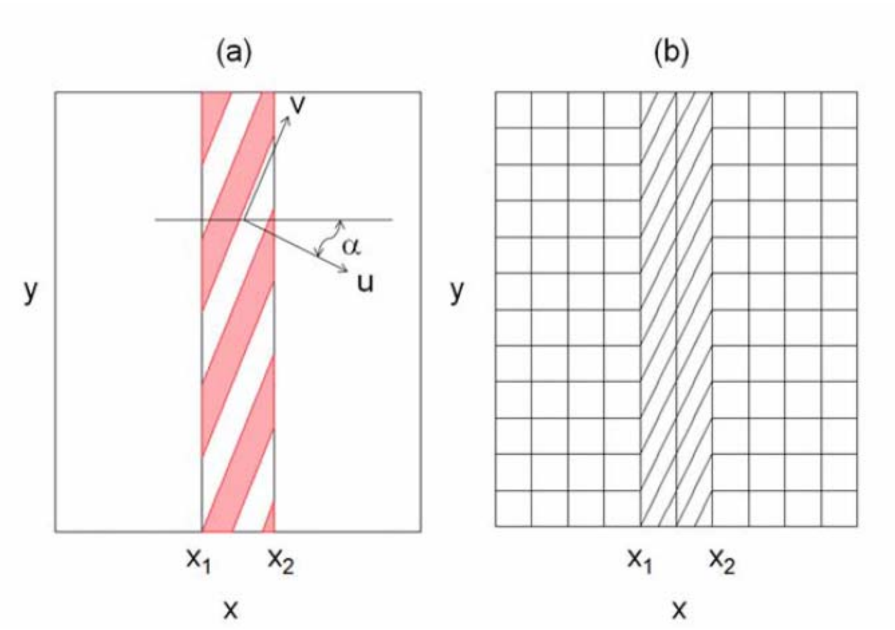
Effectively anisotropic

$$\overleftrightarrow{\varepsilon}_{normal} = \begin{bmatrix} \dfrac{2\varepsilon_1\varepsilon_2}{\varepsilon_1+\varepsilon_2} & 0 \\ 0 & \dfrac{\varepsilon_1+\varepsilon_2}{2} \end{bmatrix}$$

$$\overleftrightarrow{\varepsilon}_{oblique} = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} \varepsilon_u & 0 \\ 0 & \varepsilon_v \end{bmatrix} \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$$

Comparing equations, there is a correspondence between an oblique layered system and the transformation media

Correspondence between an oblique layered system and the transformation media:



$$\varepsilon_v = (2 + t^2 - t\sqrt{t^2 + 4})/2$$

$$\varepsilon_u = (2 + t^2 + t\sqrt{t^2 + 4})/2$$

$$\alpha = -\tau/2$$

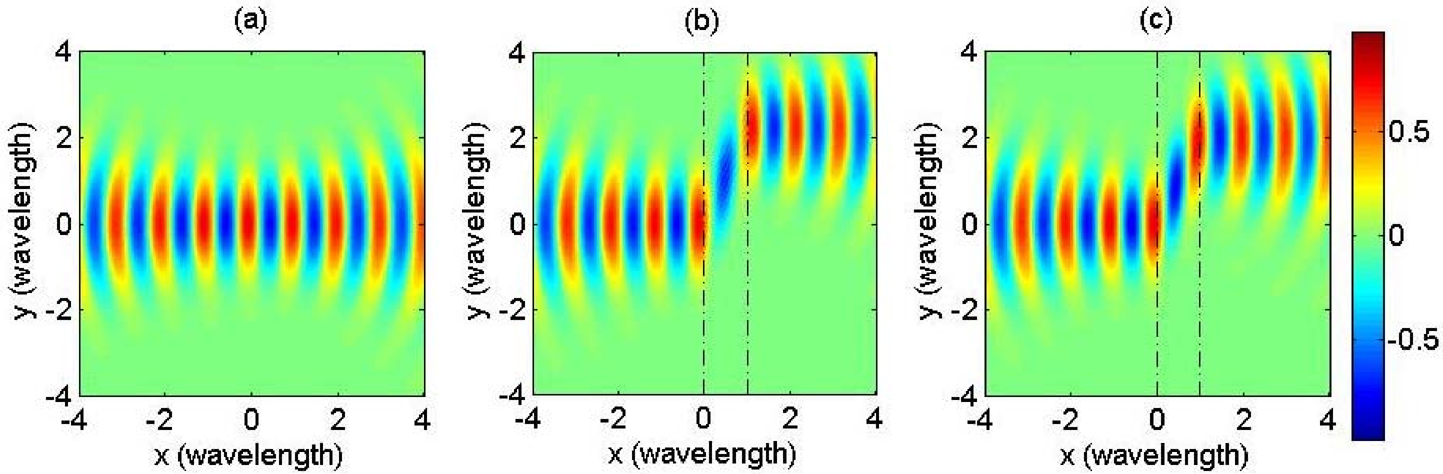
$$\cos\tau = \frac{t}{\sqrt{t^2 + 4}} \qquad \sin\tau = -\frac{2}{\sqrt{t^2 + 4}}$$

An oblique layered system is mathematically identical to a transformation media with a shift of coordinates

TE polarized (magnetic field along the z-direction)

H.Y. Chen, C.T. Chan, Phys. Rev. B 78, 054204 (2008).

Oblique layer as optical element: It shifts a beam laterally



(b)The beam passes through the alternative oblique layer system.

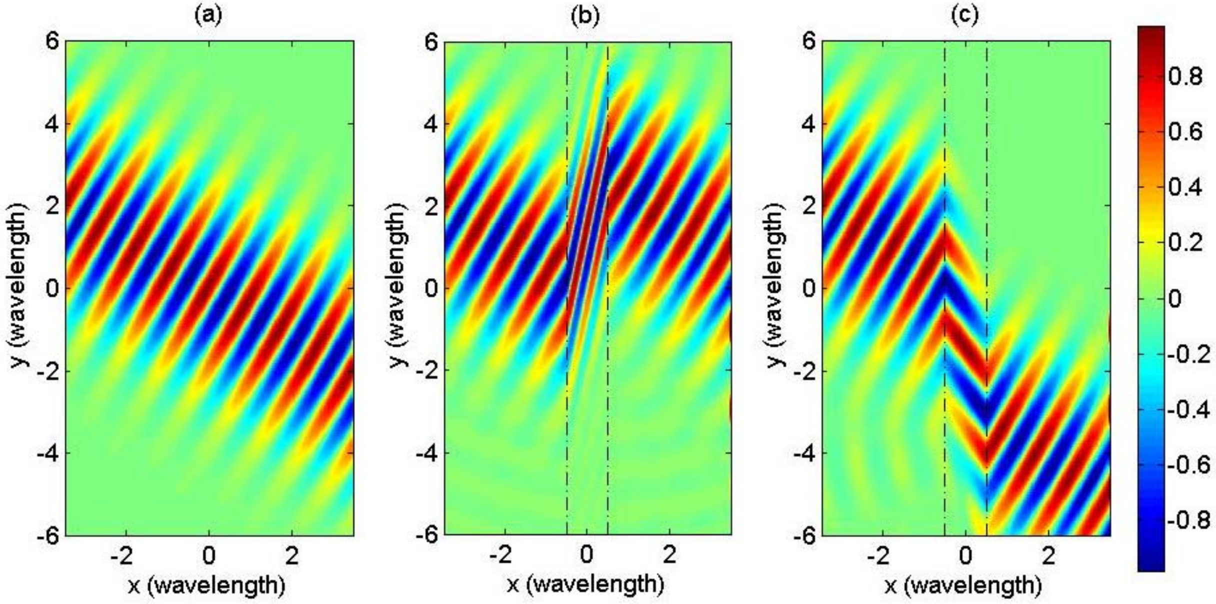
(c) The beam passes through the corresponding transformation media. The black dotted-dash lines denote the boundary of the shifters.

(They should be identical in the effective medium limit)

H.Y. Chen, C.T. Chan, Phys. Rev. B 78, 054204 (2008).
See also, M. Rahm, et al., Phys. Rev. Lett. 100, 063903 (2008).

Finite element simulation results of the magnetic field distribution near the center of the oblique incident Gaussian beam.

- (a) The beam is in free space.
- (b) The beam passes through the wave shifter with a positive shift parameter.
- (c) The beam passes through the wave shifter with a negative shift parameter.



The black dotted-dashed lines indicate the boundaries of the shifters.

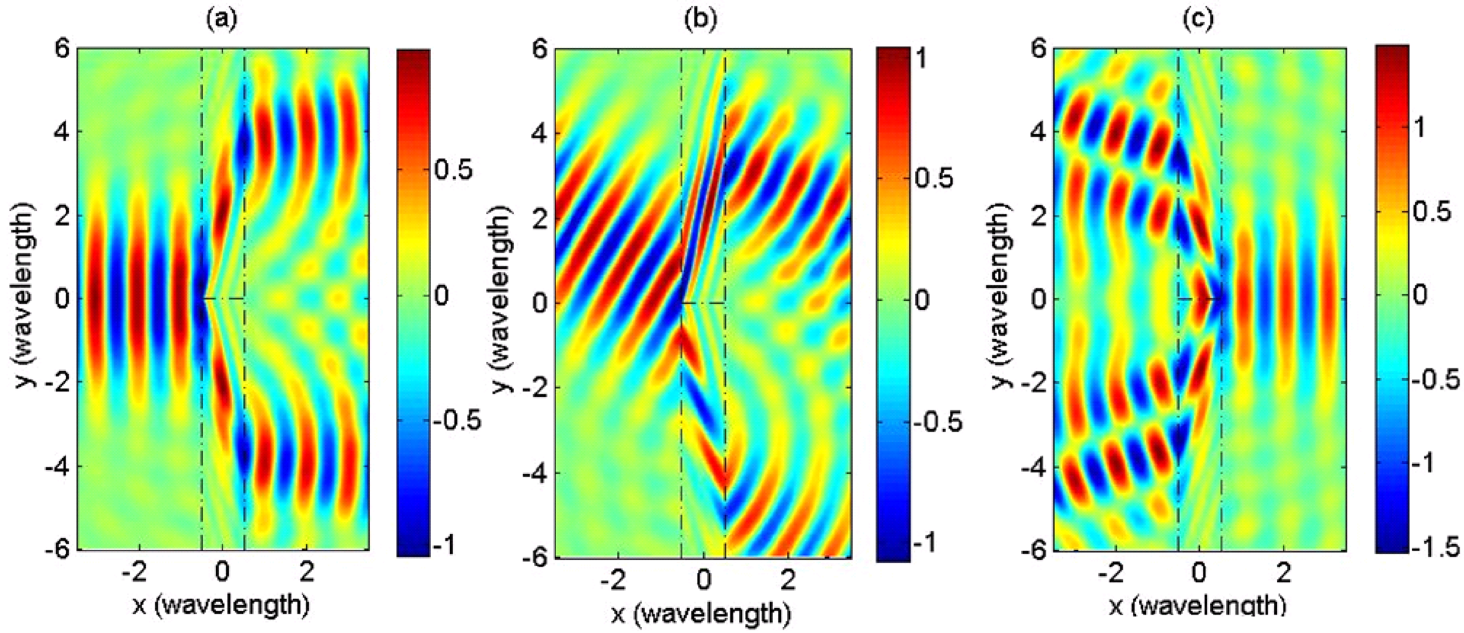
Finite element simulation results of the magnetic field distribution near the center of the incident Gaussian beam(s).

(a) The beam is normally incident propagating through the wave splitter.

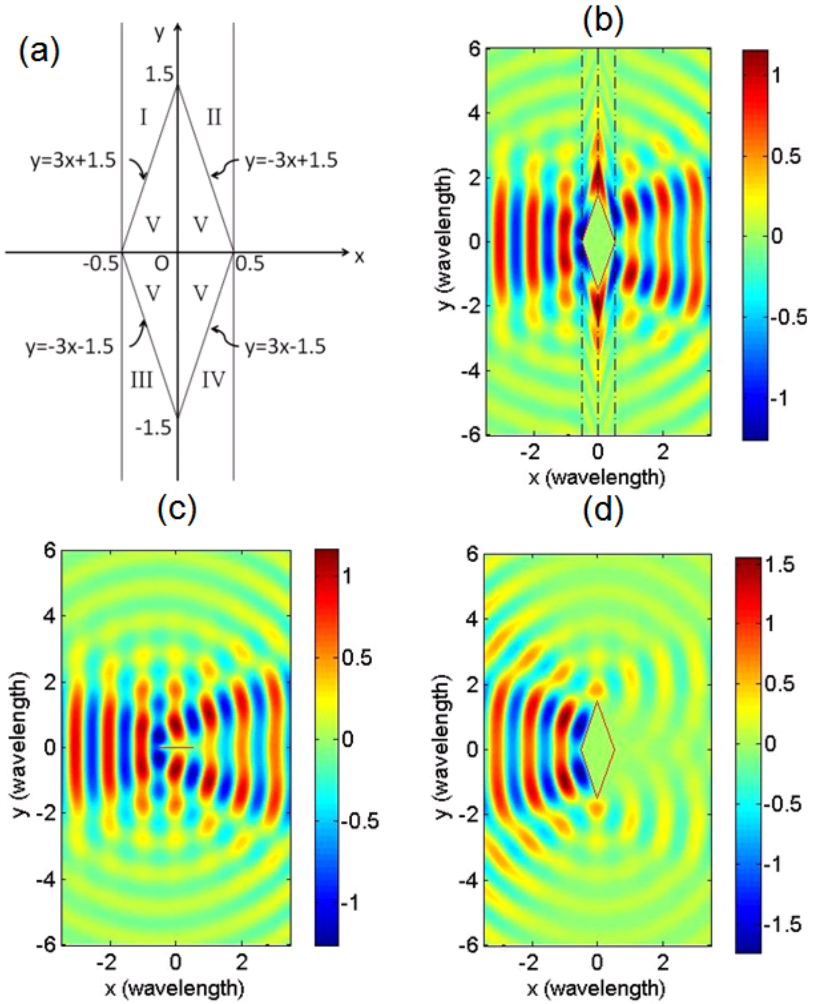
(b) The beam is obliquely incident propagating through the wave splitter.

(c) Two beams are normally incident propagating through the wave combiner.

The black dotted-dashed lines indicate the boundaries of the shifters.



“one dimensional” cloak: Reduced cross section



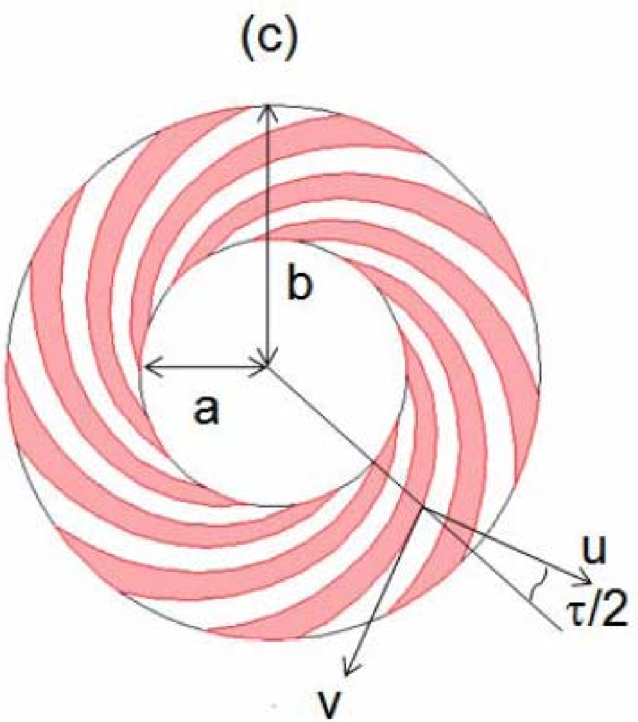
Cloak reduces the cross section of the diamond-shaped domain to that of a thin-plate

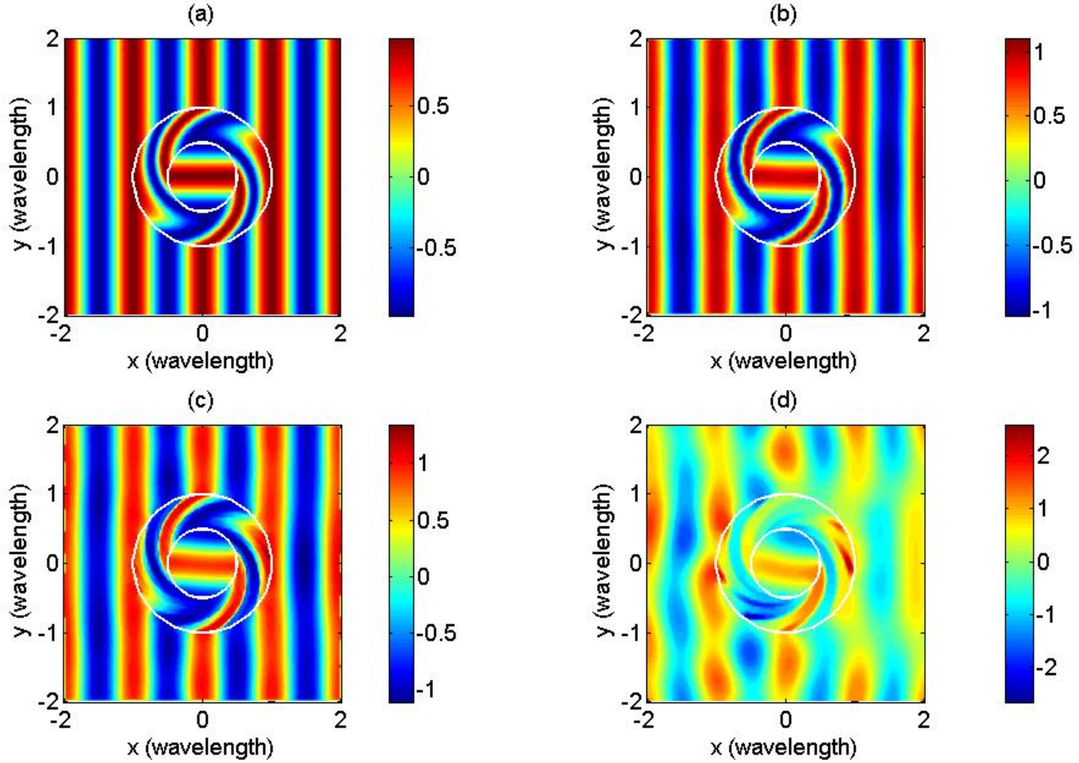
(independent of what you put inside the diamond-shaped domain)

If there is no cloak

Similar ideas: see e.g. Jensen Li and J. Pendry,
http://arxiv.org/abs/0806.4396

Use layered media as (invisible) wave rotator



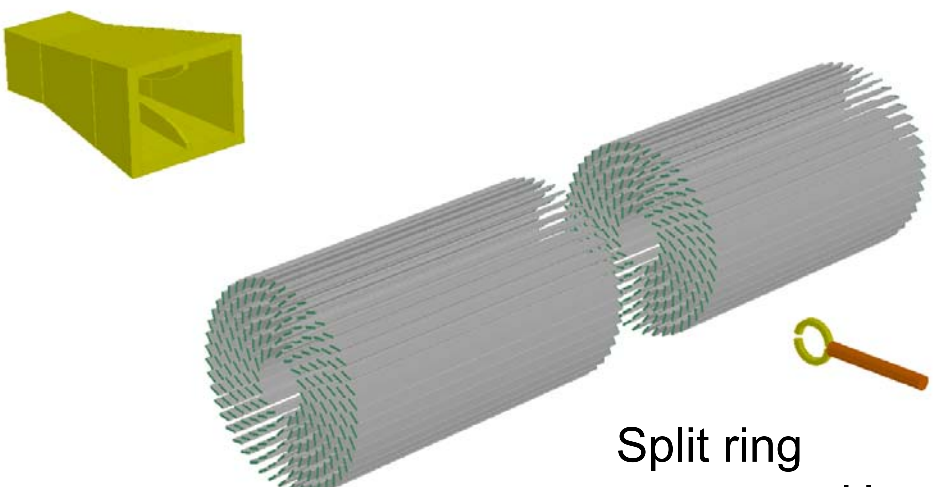


(a) perfect rotation cloak (transformation media).
(b) Layered rotation cloak with 72 layers.
(c) with 36 layers. (d) with 18 layers.

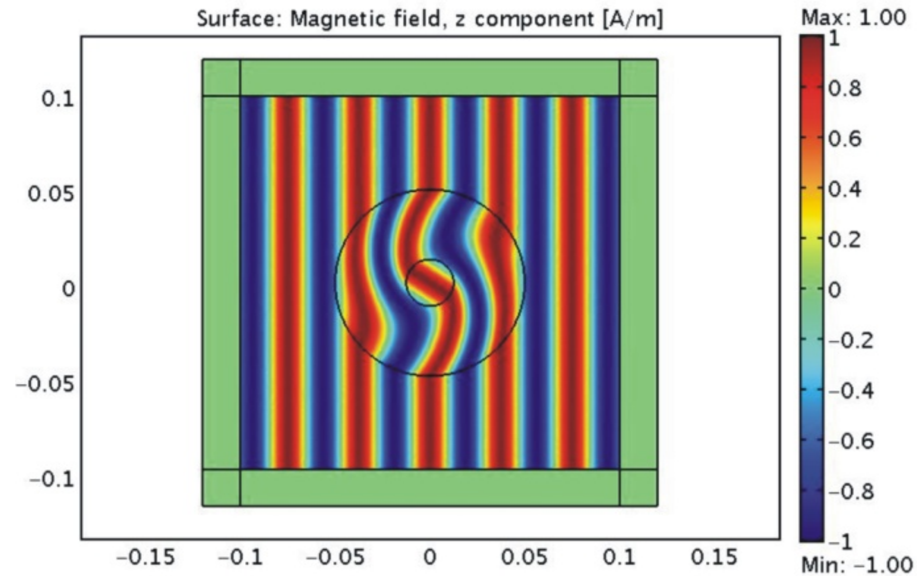
H.Y. Chen, C.T. Chan, Phys. Rev. B 78, 054204 (2008).

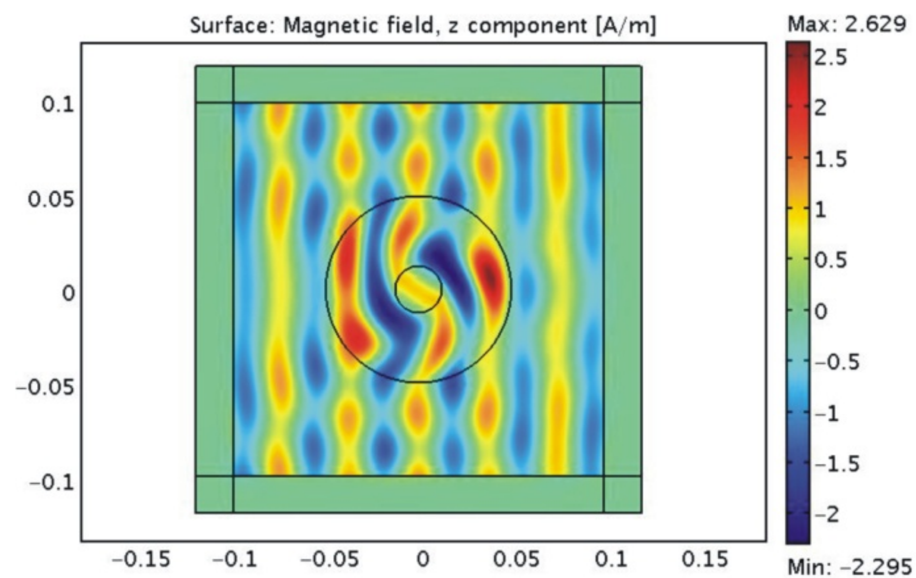
Experiment: Reduced rotator

166 rectangular Al 6x1 mm^2 plates

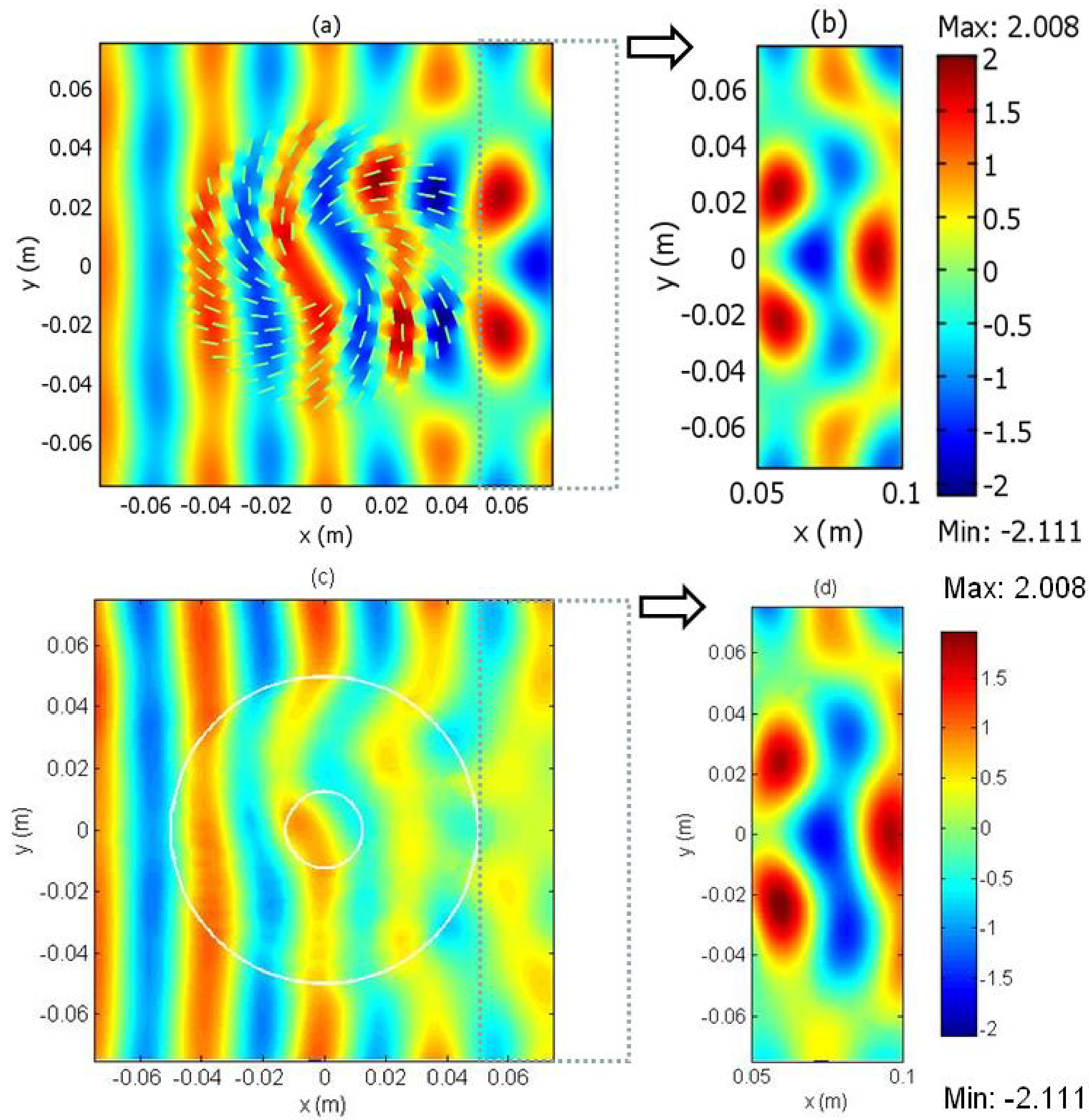


Split ring measures Hz





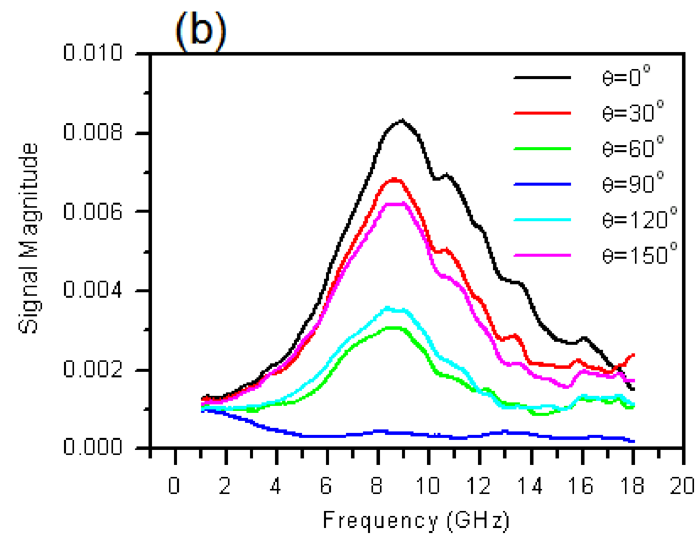
Measurements at 8 GHz

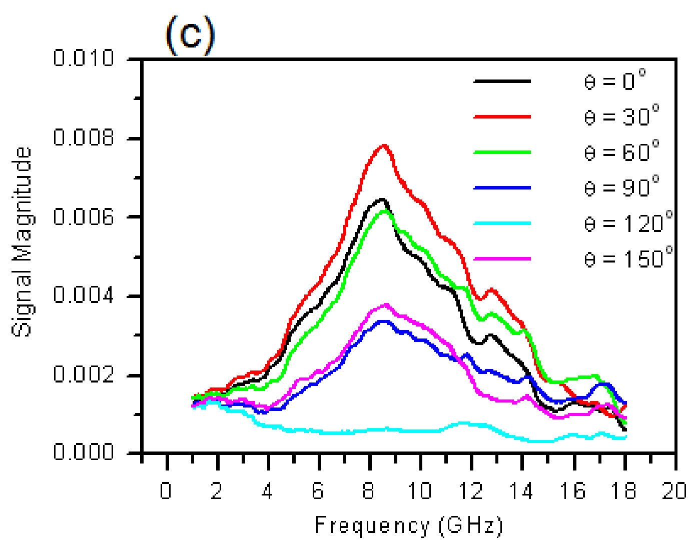


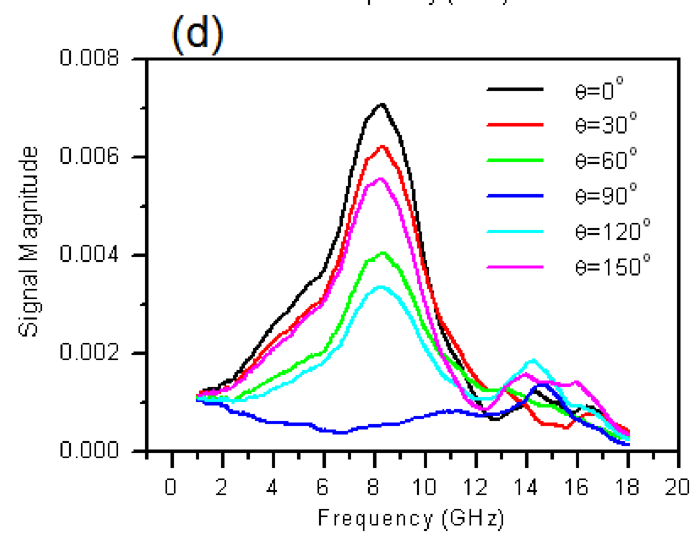
Using a dipole antenna

(a)

Note the broad band width rotation functionality







Acoustic cloak

- Can we make an object invisible to sound?

$$j\omega\rho_\phi v_\phi = -\frac{1}{r}\frac{\partial p}{\partial \phi}, \quad \Longleftrightarrow \quad j\omega\mu_r(-H_r) = -\frac{1}{r}\frac{\partial(-E_z)}{\partial \phi},$$

$$j\omega\rho_r v_r = -\frac{\partial p}{\partial r}, \qquad j\omega\mu_\phi H_\phi = -\frac{\partial(-E_z)}{\partial r},$$

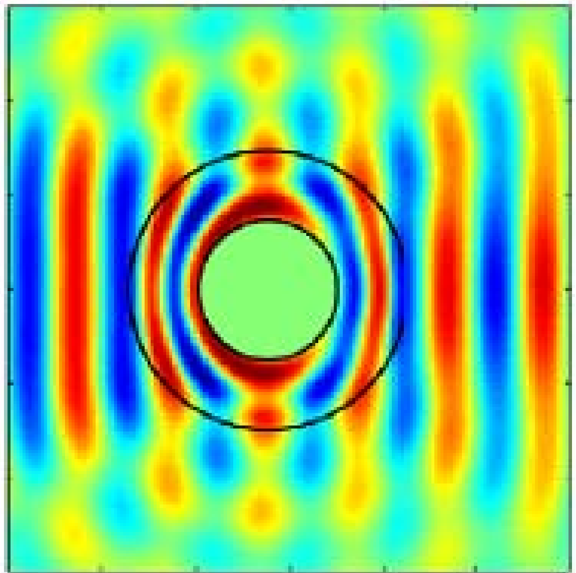
$$j\omega\frac{1}{\lambda}p = -\frac{1}{r}\frac{\partial(r v_r)}{\partial r} - \frac{1}{r}\frac{\partial v_\phi}{\partial \phi}. \qquad j\omega\epsilon_z(-E_z) = -\frac{1}{r}\frac{\partial(rH_\phi)}{\partial r} - \frac{1}{r}\frac{\partial(-H_r)}{\partial \phi}.$$

$$[p, v_r, v_\phi, \rho_r, \rho_\phi, \lambda^{-1}] \leftrightarrow [-E_z, H_\phi, -H_r, \mu_\phi, \mu_r, \epsilon_z].$$

2D EM cloaking $\Rightarrow$ 2D acoustic cloaking:

$$r' = a + r(b-a)/b, \ \ \phi' = \phi, \ \ z' = z$$

$$\rho_r = \frac{r}{r-a}, \ \rho_\phi = \frac{r-a}{r}, \ \lambda = \frac{(b-a)^2}{b^2}\frac{r}{r-a}$$



S. A. Cummer and D. Schurig, New J. Phys **9**, 45 (2007).

DC conductivity equations:

$$\nabla\cdot(\sigma(x)\nabla V(x)) = f(x) \qquad \nabla'\cdot(\sigma'(x')\nabla' V'(x')) = f'(x')$$

$$\sigma'(x') = A\sigma(x)A^T/\det A \qquad f'(x') = f(x)/\det A$$

Acoustic eqns at fixed frequency :

$$\nabla\cdot\left(\frac{1}{\rho(x)}\nabla p(x)\right) = -\frac{\omega^2}{\lambda(x)}p(x)$$

$$\nabla'\cdot\left(\frac{1}{\rho'(x')}\nabla' p'(x')\right) = -\frac{\omega^2}{\lambda'(x')}p'(x')$$

$$\frac{1}{\rho'(x')} = A\frac{1}{\rho(x)}A^T/\det A$$

$$-\frac{\omega^2}{\lambda'(x')} = -\frac{\omega^2}{\lambda(x)}/\det A$$

$$\lambda'(x') = \lambda(x)\det A$$

A. Greenleaf, M. Lassas and G. Uhlmann, Physiol. Meas. **24**, 413 (2003).

G. W. Milton, M. Briane and J. R. Wills, New J. Phys **8**, 248 (2006).

$$[V(x), \sigma(x), f(x)] \leftrightarrow \left[p(x), \frac{1}{\rho(x)}, -\frac{\omega^2}{\lambda(x)}p(x)\right]$$

H. Chen and C. T. Chan
Appl. Phys. Lett. 91, 183518 (2007).

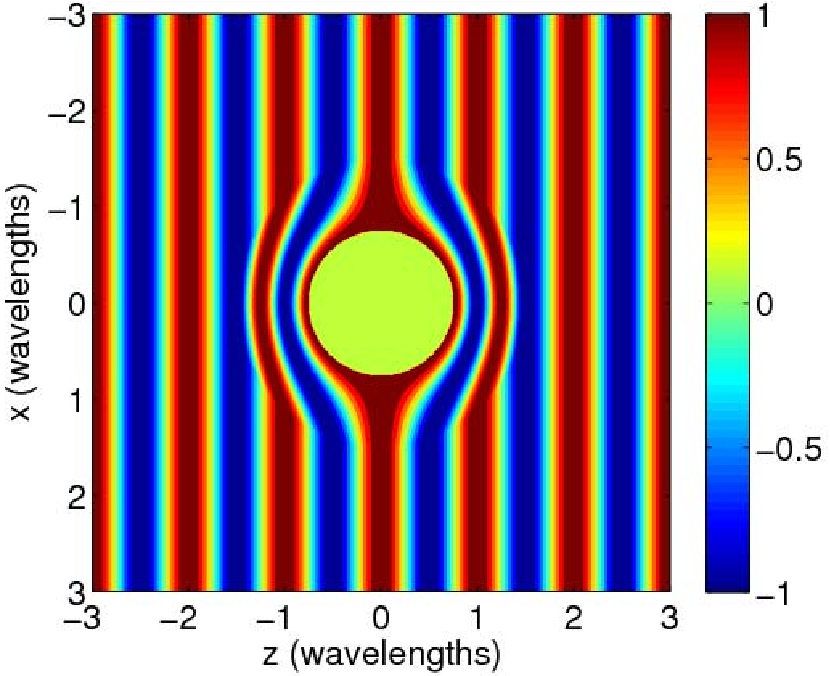
Extension to acoustic wave: 3D acoustic cloaking

$$r' = a + r(b-a)/b,\ \ \theta' = \theta,\ \ \varphi' = \varphi$$

$$\rho_r = \frac{b-a}{b}\frac{r^2}{(r-a)^2},\ \rho_\theta = \rho_\varphi = \frac{b-a}{b},\ \lambda = \frac{(b-a)^3}{b^3}\frac{r^2}{(r-a)^2}$$

H. Chen and C. T. Chan, APL (2007).

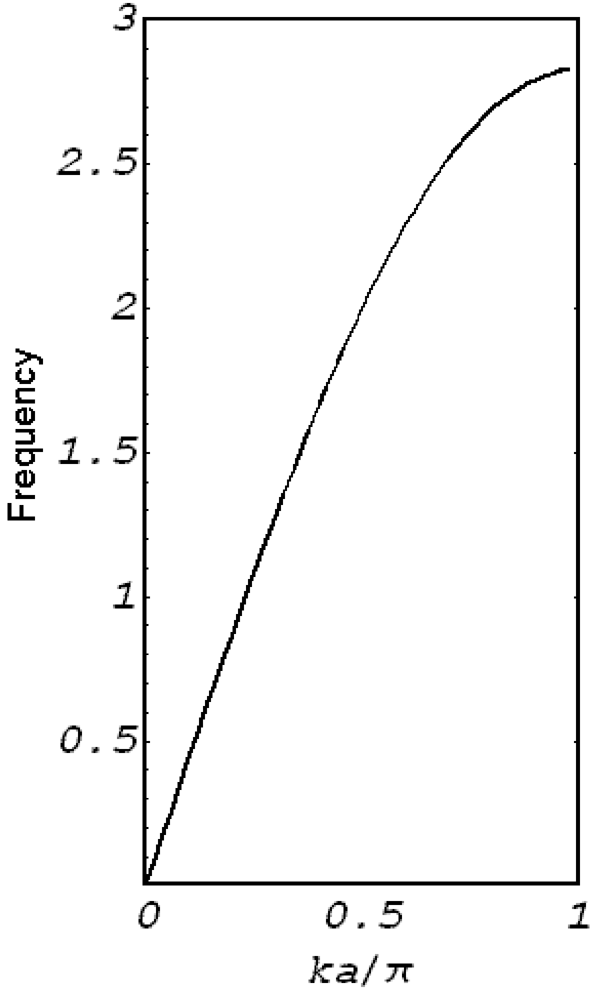
See, also S. A. Cummer et al., Phys. Rev. Lett. 100, 024301 (2008).



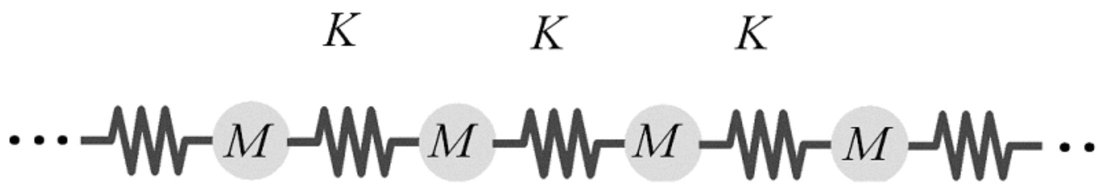
Acoustic metamaterials

- Man made material with embedded sub-wavelength mechanical resonators, which give effectively any value of density tensor and modulus
  - Negative, small, big values are allowed
  - Anisotropy allowed

J. Li and C. T. Chan, Phys. Rev. E **70**, 055602 (2004).



Simple ball-and-spring model



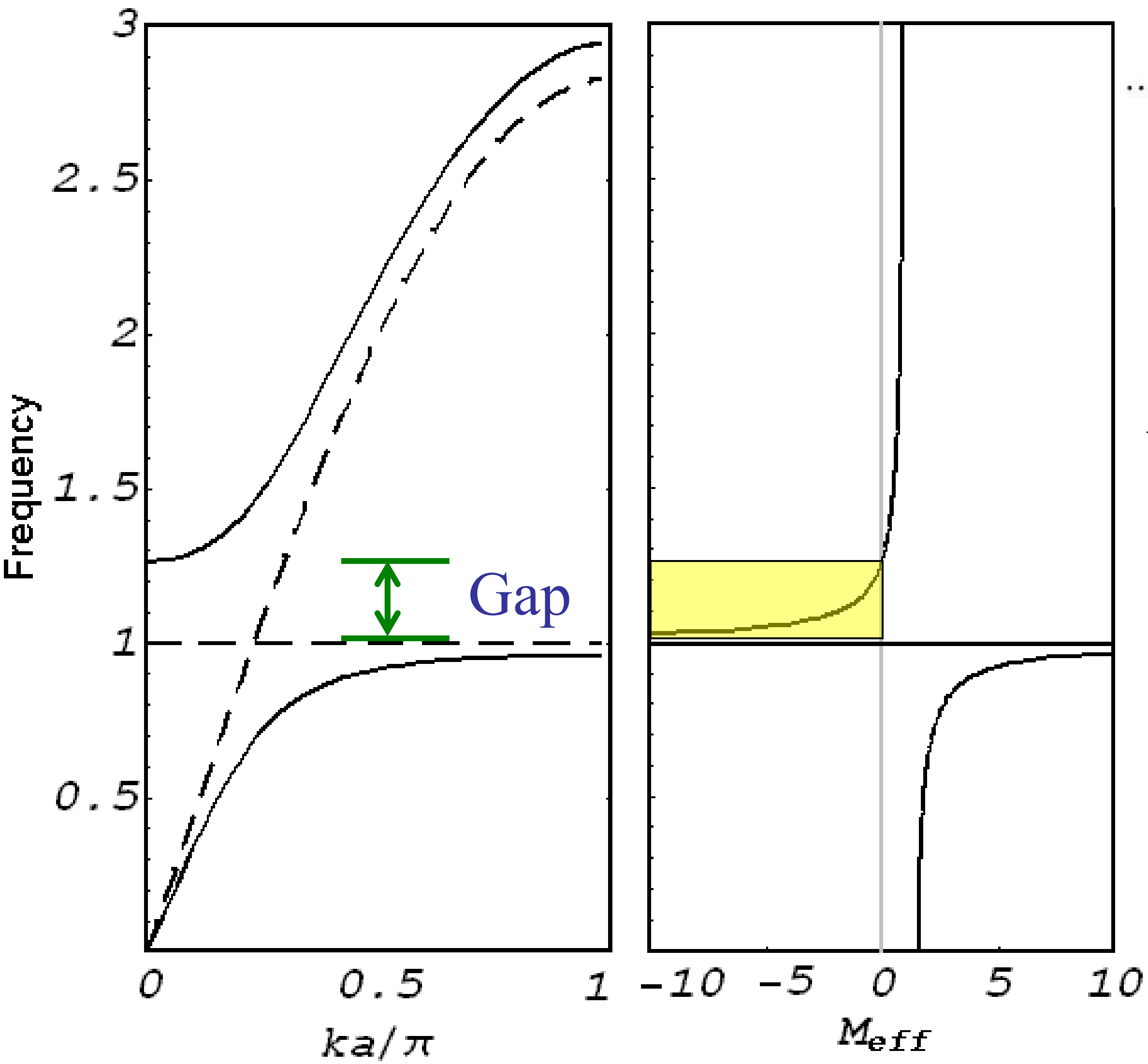
Mechanical: $$\omega^2 = \frac{4K}{M}\sin^2(\frac{ka}{2})$$

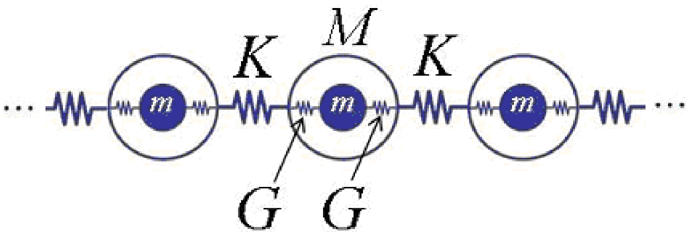
EM: $$k^2 = (\varepsilon\mu)(\varepsilon_0\mu_0)\omega^2$$

Positive M and K

What if K and M are negative?

Resonance gives "negative mass"





$$\omega^2 = \frac{4K}{M_{eff}} \sin^2\left(\frac{ka}{2}\right)$$

where

$$M_{eff} = M + \frac{m\omega_0^2}{\omega_0^2 - \omega^2}$$

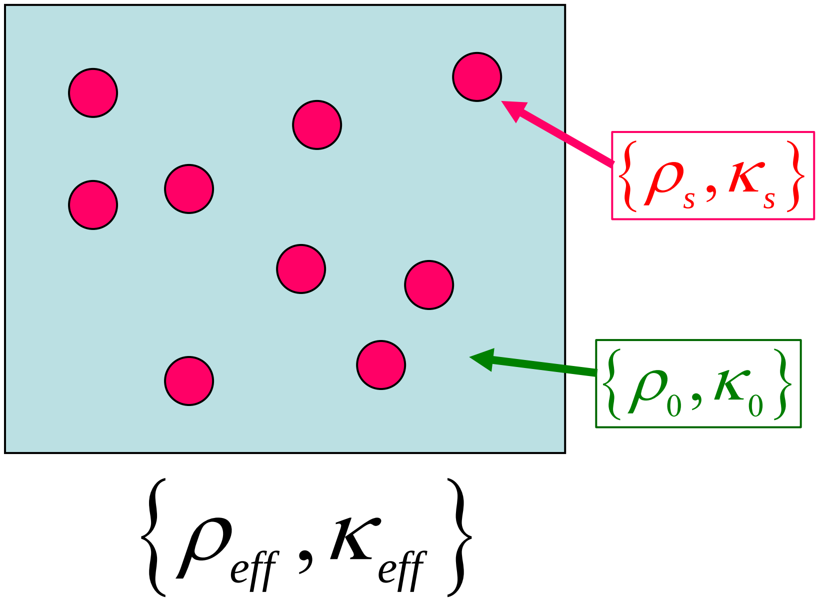
and $$\omega_0 = \sqrt{\frac{2G}{m}}$$

Negative M by resonance

Standard effective medium for spherical particles composite

$$\frac{1}{\kappa_{eff}} = \frac{f}{\kappa_s} + \frac{1-f}{\kappa_0},$$

$$\frac{\rho_{eff} - \rho_0}{2\rho_{eff} + \rho_0} = f\,\frac{\rho_s - \rho_0}{2\rho_s + \rho_0}.$$



J. G. Berryman, *J. Acoust. Soc. Am.* **68** 1809 (1980).

Effective modulus and density in a composite cannot be negative from standard theory

Effective medium with resonances

- The standard effective medium equations are correct only in “linear dispersion” regime.
- New theory needed to take care of resonances:

$$-1+\frac{\kappa_0}{\kappa_{eff}} = -f\frac{3i}{(k_0 r_s)^3}\frac{D_0}{1+D_0}$$

$$\frac{\rho_{eff}-\rho_0}{2\rho_{eff}+\rho_0} = -f\frac{3i}{(k_0 r_s)^3}\frac{D_1}{1+D_1}$$

$k_0 = \omega\sqrt{\rho_0/\kappa_0}$

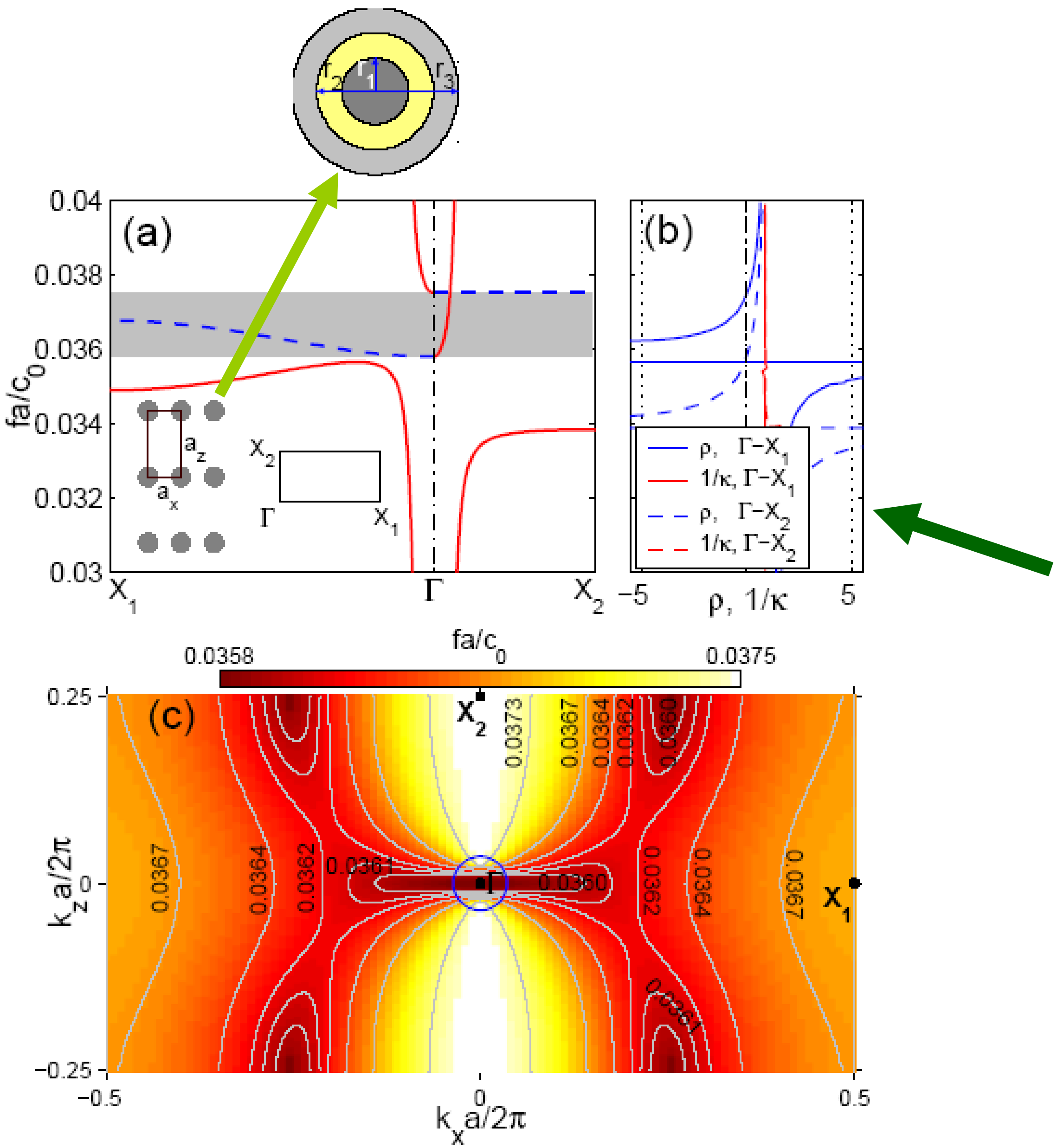
D_l = Mie scattering coefficient

f = volume-filling ratio

With resonance: “arbitrary values” of modulus and density can be realized

Jensen Li, CT Chan, Phys. Rev. E **70**, 055602(R) (2004)

Anisotropy



By arranging isotropic resonators in an anisotropic lattice (e.g. rectangle), we can get anisotropic response.

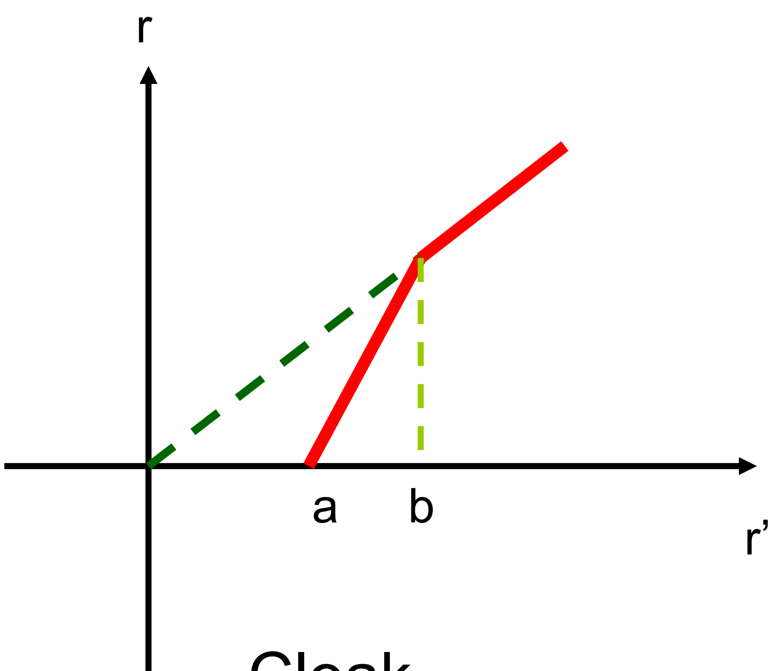
For example, the effective density is different along X and Y direction.

If…

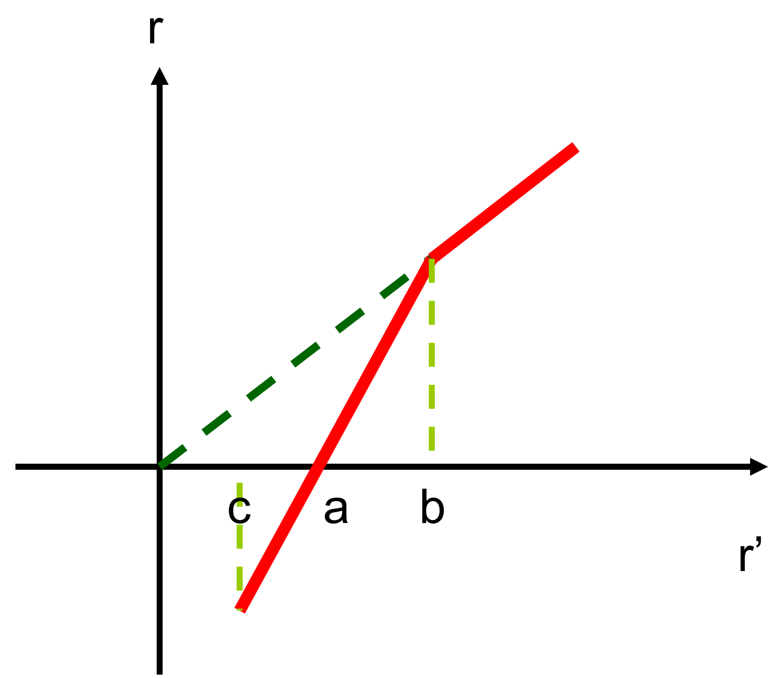
- If EM cloak can make an object invisible to radar, the acoustic cloak can make an submerged object invisible to sonar

Anti-cloak

- Can transformation media cloak hide everything inside?

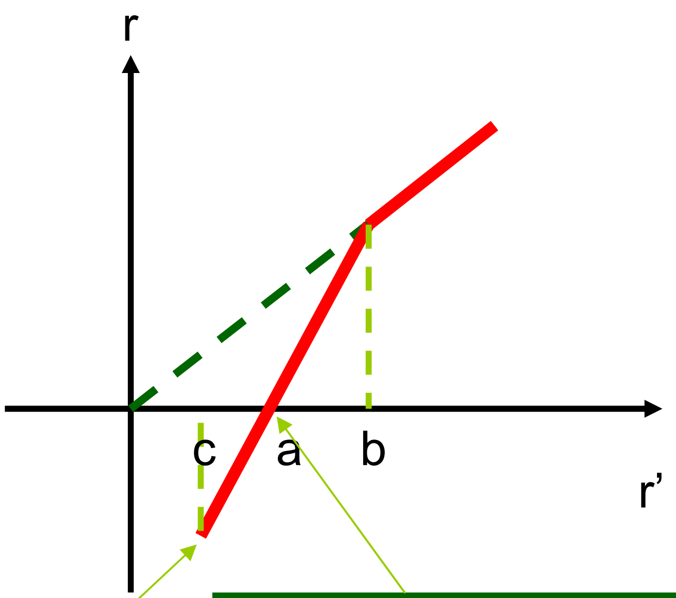


Cloak



What is this?

Anti-cloak



This gives "zero" cross section

Continuation of the line:

This gives back some cross section ?

Material properties:

- Between [c,a] the ε,μ are negative
- They cancel some effect of the perfect cloak

There exist something that "transformation"cloak cannot hide

2D cylindrical system

Two dimensional situations , consider E field along z axis:

$$\left\{\frac{1}{\varepsilon_z}\frac{1}{r}\frac{\partial}{\partial r}\left(\frac{r}{\mu_\theta}\frac{\partial}{\partial r}\right)+\frac{1}{\varepsilon_z}\frac{1}{r^2}\frac{\partial}{\partial \theta}\left(\frac{1}{\mu_r}\frac{\partial}{\partial \theta}\right)+k_0^2\right\}E_z=0$$

Transformation:

$$(f(r),\theta,z)$$

$$\frac{\mu_\theta}{\mu_0}=r\frac{f'}{f}$$

$$\frac{\mu_r}{\mu_0}=\frac{f}{rf'}$$

$$\frac{\varepsilon_z}{\varepsilon_0}=\frac{1}{r}ff'$$

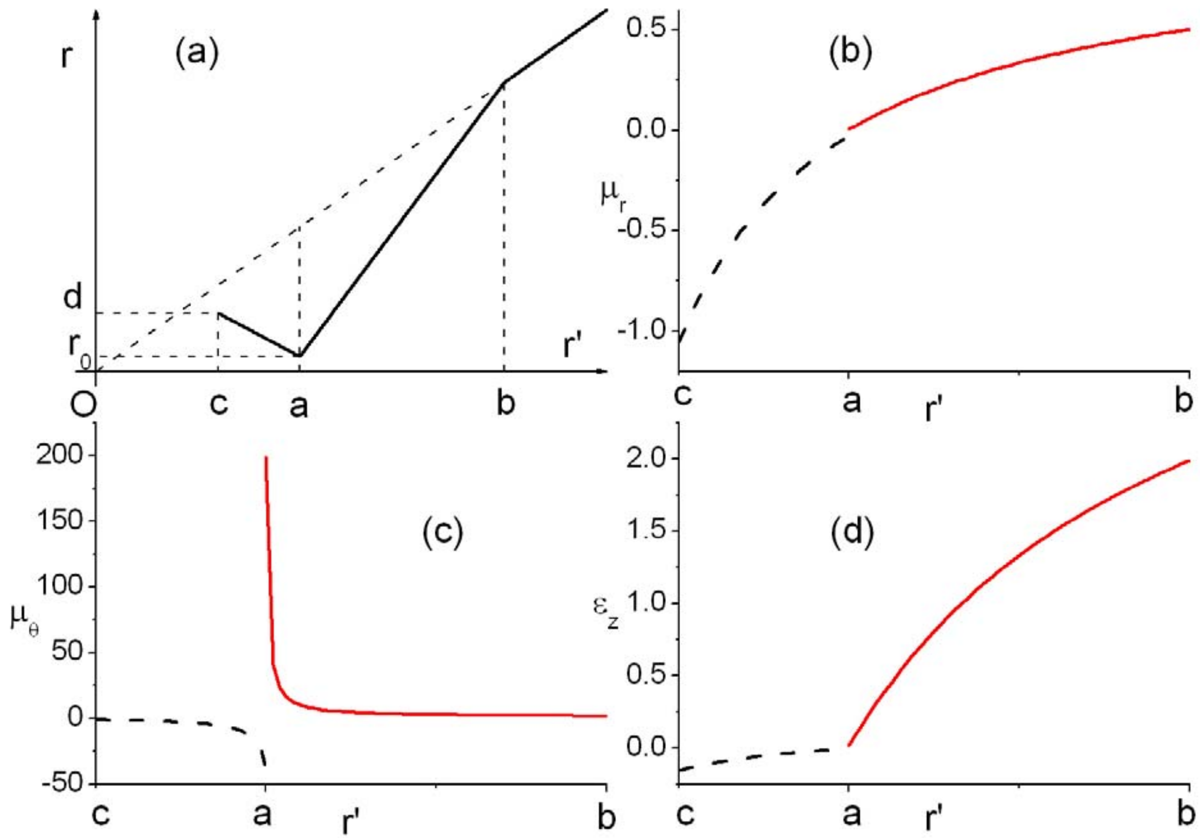
Either *f* or *f'=df/dr* is negative implies negative index material

Anti-cloak

- Can we avoid dealing with a “negative” radius?
- Avoid the divergence as the function passes through zero…
  - Bend the function up

TM: (Hx, Hy, Ez)

Anti-cloak



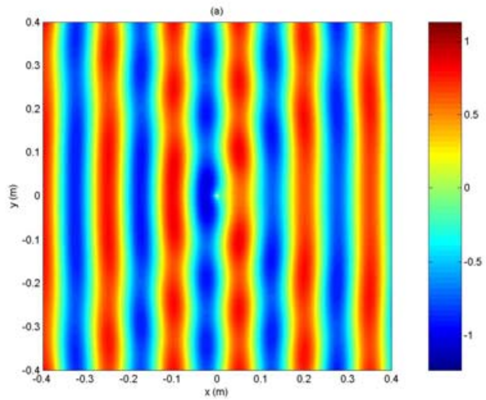
"folded geometry"

See, e.g. Leonhardt et al,

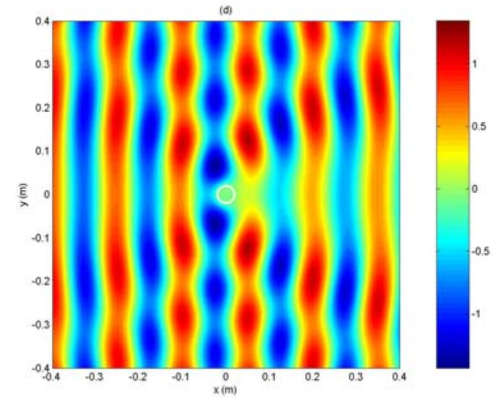
Milton et al.

(a) The coordinate transformation of the cloak and anti-cloak: [a,b] is the cloak [c,a] is the "anti-cloak". They are in direct contact. Note that "anti-cloak region has negative parameters

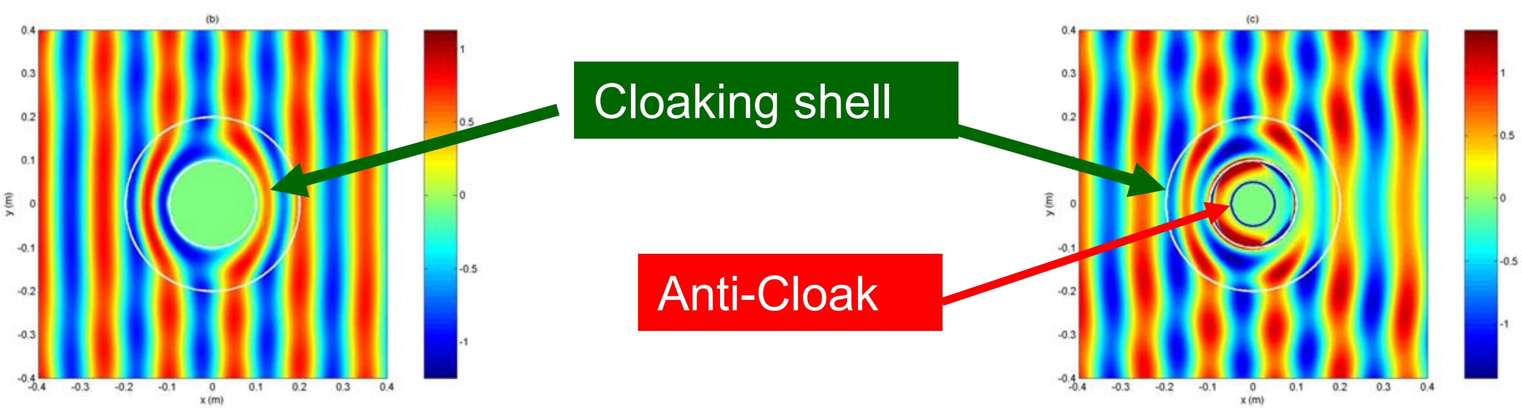
H.Y. Chen, X. Luo, H. Ma, C.T. Chan   Opt. Express 16, 14603 (2008)



A tiny PEC cylinder with a radius r_o



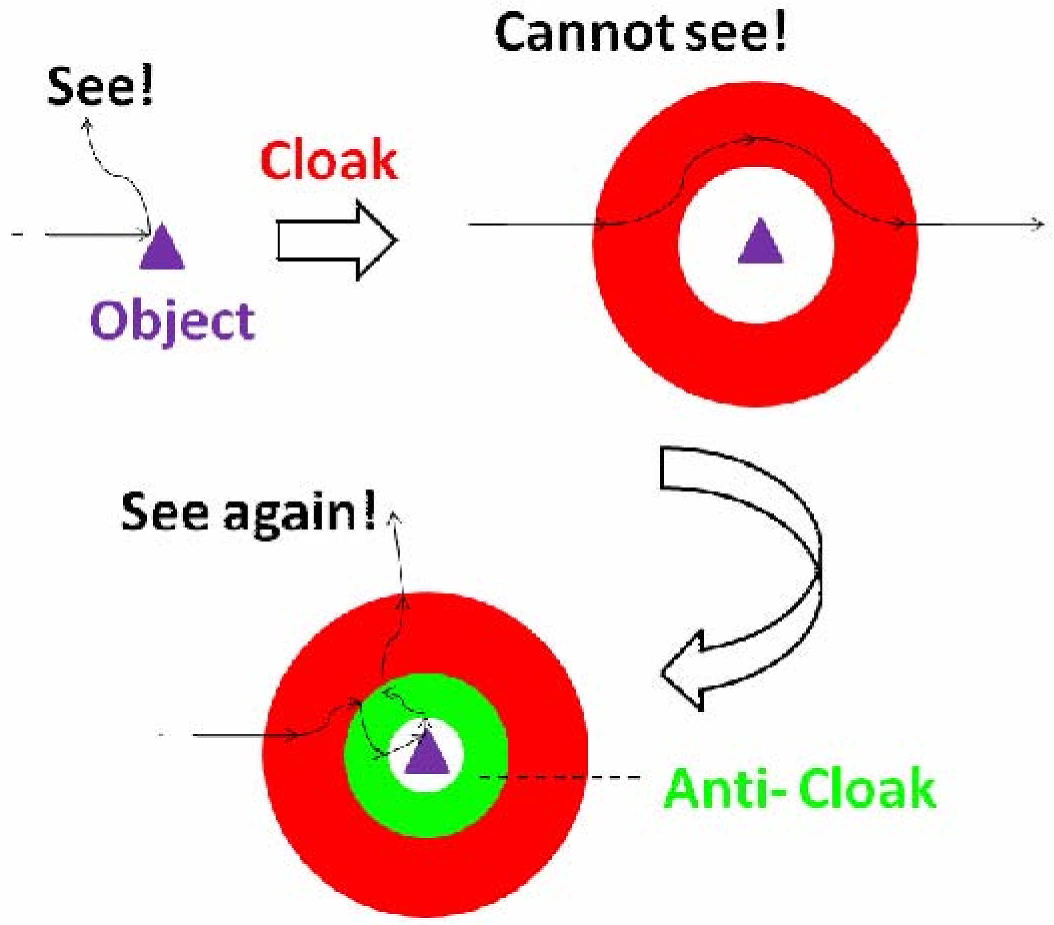
A PEC cylinder with a radius d



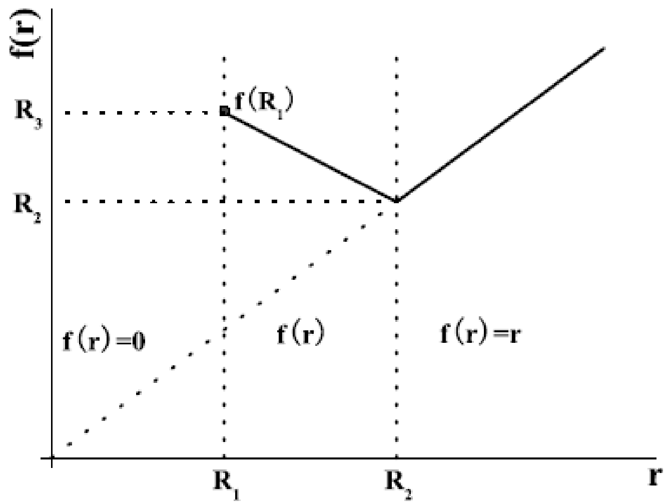
A PEC cylinder with a radius a wearing a partial cloak, reducing the Cross section to r_o

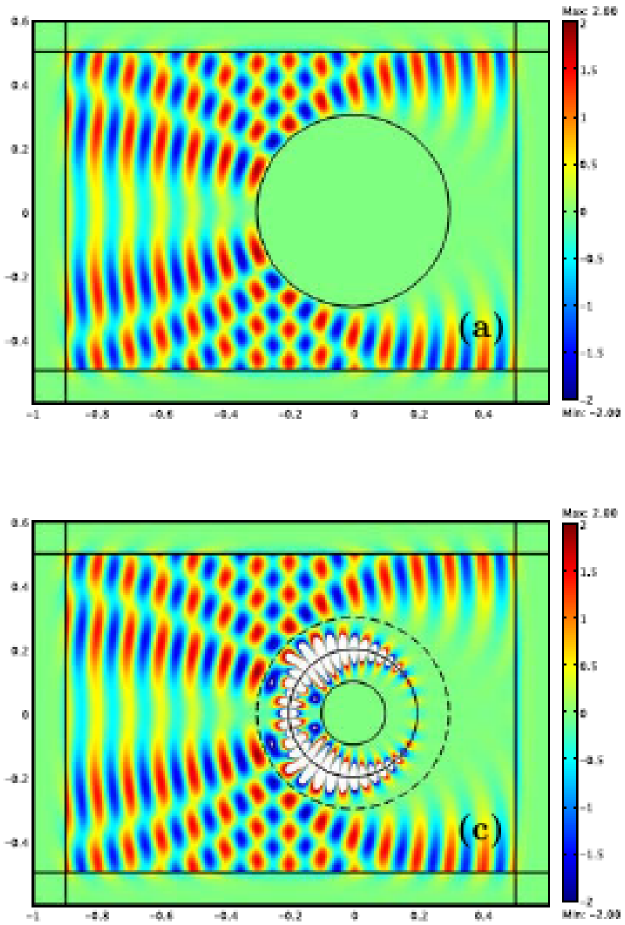
PEC cylinder with a radius c wearing a partial cloak in contact with an anti-cloak. Cross section is same as d

Cloak and Anti-cloak



Large cross section





Tao Yang et. al. http://arxiv.org/abs/0807.5038
Yu Luo et al. http://arxiv.org/abs/0808.021

More general transformations

$$\begin{cases} r'(r,\theta) = \dfrac{b-a}{b-c}(r-b)+b = \dfrac{b-a}{b-c}r + \dfrac{a-c}{b-c}b, \\ \theta' = \theta, \quad 0 \le \theta < 2\pi \\ z' = z, \quad \square \end{cases},$$

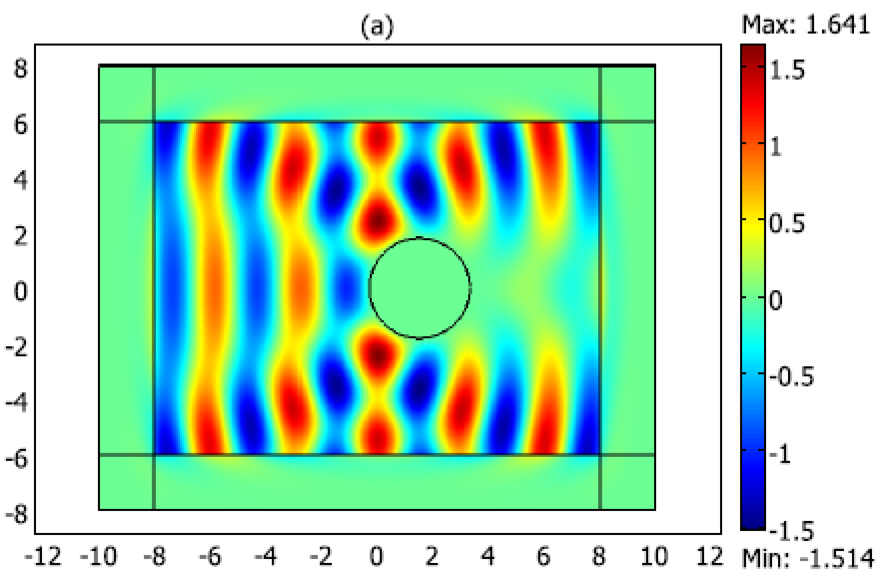
$$a = \rho_1(\theta), b = \rho_2(\theta), c = \rho_3(\theta),$$

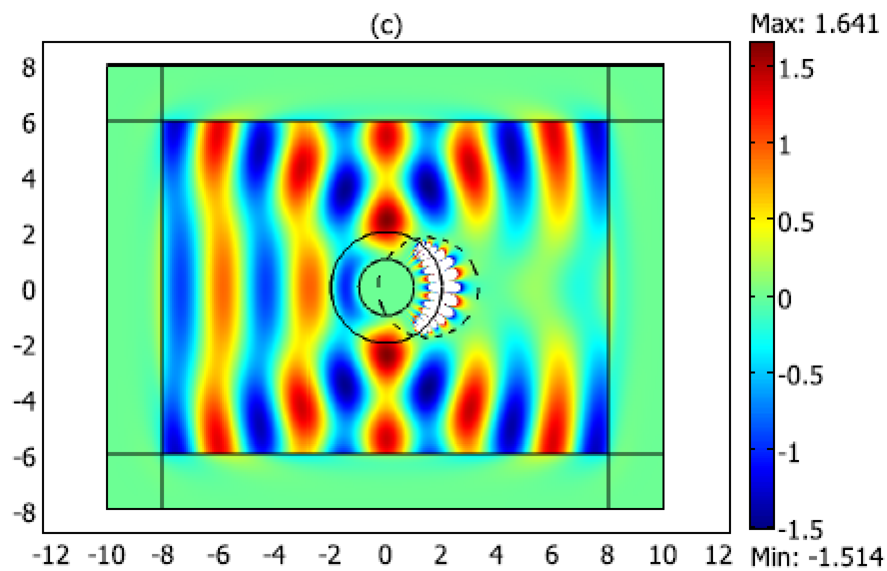
$$\mu_{rr} = \frac{(\frac{\partial r'}{\partial r})^2 + (\frac{\partial r'}{r\partial\theta})^2}{\frac{\partial r'}{\partial r}\frac{r'}{r}}, \qquad \mu_{r\theta} = \mu_{\theta r} = \frac{\frac{\partial r'}{r\partial\theta}}{\frac{\partial r'}{\partial r}}, \qquad \mu_{\theta\theta} = \frac{\frac{r'}{r}}{\frac{\partial r'}{\partial r}}, \qquad \varepsilon_{zz} = \frac{1}{\frac{\partial r'}{\partial r}\frac{r'}{r}}.$$

"Mirage cylinders": Transformation media can make observer see a cylinder with a different size, different shape, at a different position

$$c = \rho_3(\theta) = x_0 \cos\theta + \sqrt{a^2 + x_0^2 \cos^2\theta},$$

A circle centered at x_o





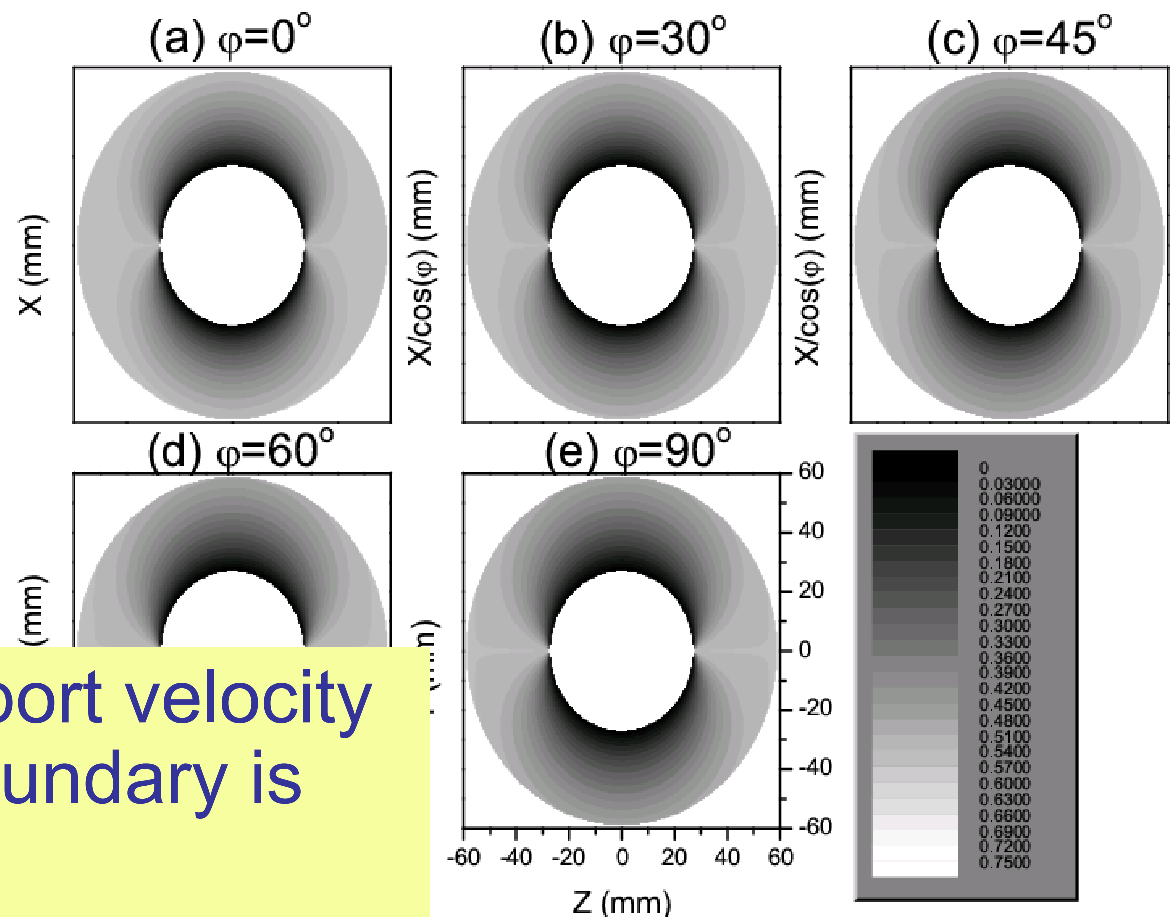
PEC cylinder with radius a and centered at x_o

A cylinder with a different radius at origin, coated with the transformation media shell

Some subtle properties of cloaks

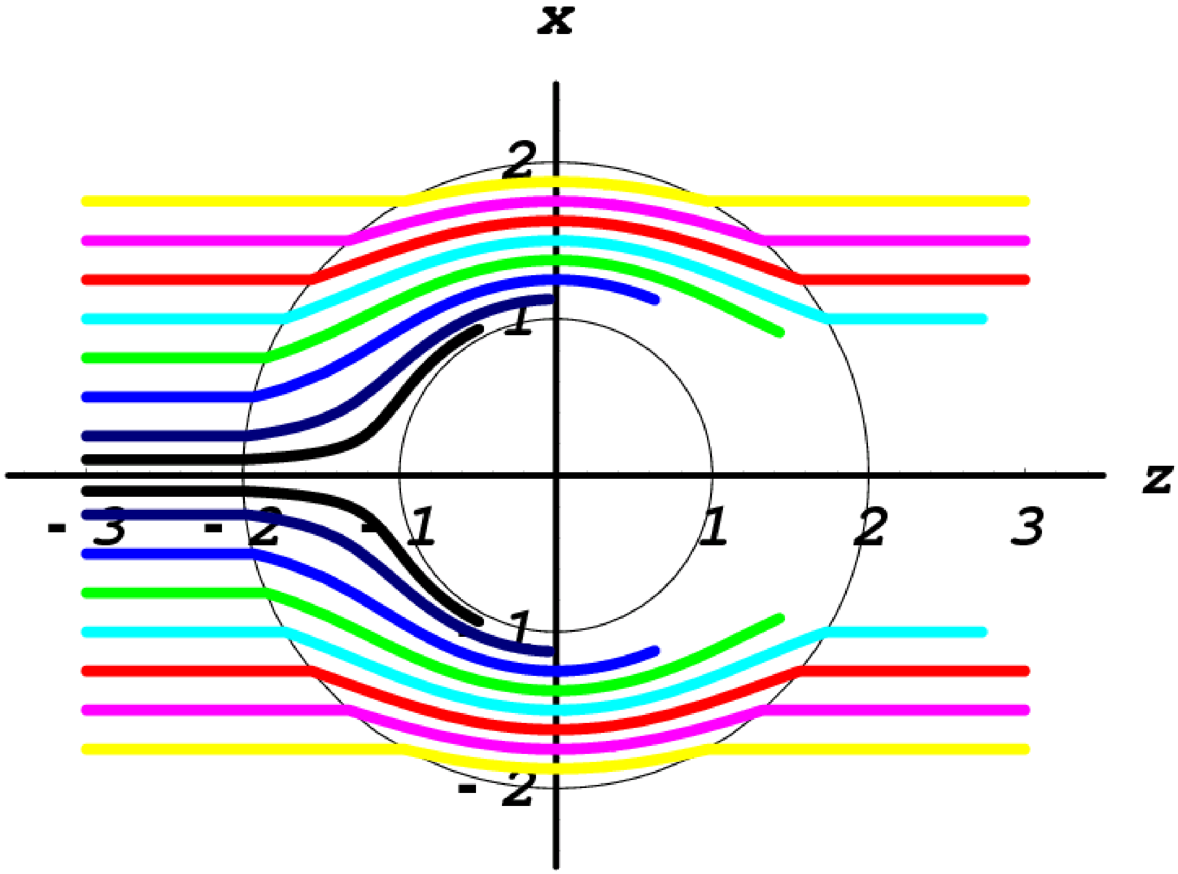
Energy transport velocity of 3D EM cloak

$$\frac{v_e}{c}=\frac{2[((\frac{b}{b-a})^2(\frac{r-a}{r})^2\cos\theta)^2+((\frac{b}{b-a})^2(\frac{r-a}{r})\sin\theta)^2]^{1/2}}{(\frac{b}{b-a}\sin\theta)^2[(\varepsilon_r+\omega\frac{d\varepsilon_r}{d\omega})(\cos\varphi)^2+(\mu_r+\omega\frac{d\mu_r}{d\omega})(\sin\varphi)^2]+\frac{b}{b-a}(\frac{b}{b-a}\frac{r-a}{r}\cos\theta)^2+\frac{b}{b-a}(\frac{b}{b-a}\frac{r-a}{r})^2}.$$



- the energy transport velocity near the inner boundary is very small

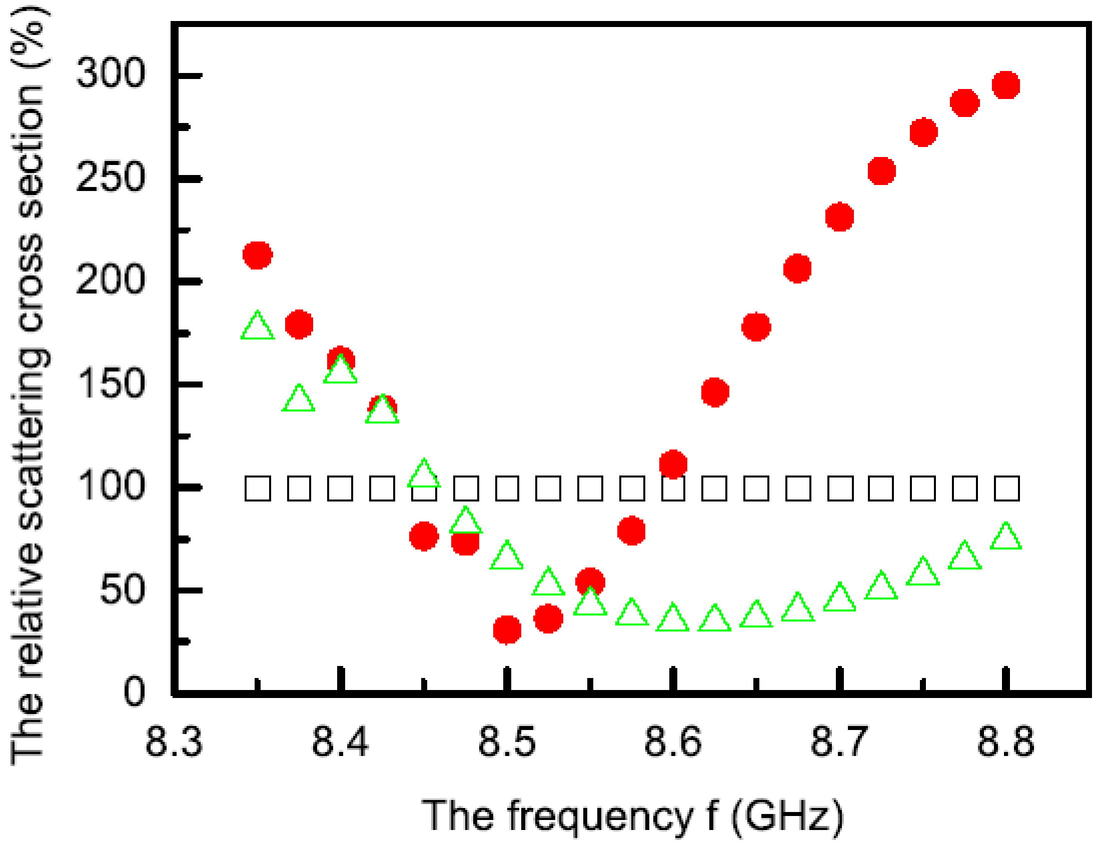
Time delay



- Ray tracing for beams incident at different positions
- A beam pointing at the origin of the cloak will take infinite time to pass through the cloak

H.Y. Chen, C.T. Chan J. Appl. Phys. 104, 033113 (2008).

Extension to broader frequency range



- Transformation media equations can be realized at one single frequency, but it cannot be made to be compatible with the causality over an extended range of frequencies
- By a simple adaptation, the form transformation media equations can be made to be compatible with the causality requirements, which then leads to a simple way of designing a reduced-cross-section cloak for a finite range of frequencies.

H.Y. Chen, Z. Liang, P. Yao, X. Jiang, H. Ma, C.T. Chan, PRB 76, 241104(R) (2007).

Summary

- Don't believe your eyes or ears (if they react to only one frequency)
  - If you don't see anything, it may still be there
  - If don't hear a sound, it does not mean it is not there
  - If I am facing you, I may not be
  - It may be bigger (smaller) than it looks
  - If it is here, it may actually be there


Thank You

Invisibility cloaking

$$\overleftrightarrow{\varepsilon} = \begin{vmatrix} \varepsilon_{xx} & \varepsilon_{xy} \\ \varepsilon_{xy} & \varepsilon_{yy} \end{vmatrix}$$

$$\varepsilon_{xx} = \varepsilon_r \cos^2(\theta) + \varepsilon_\theta \sin^2(\theta)$$
$$\varepsilon_{xy} = (\varepsilon_r - \varepsilon_\theta)\sin(\theta)\cos(\theta)$$
$$\varepsilon_{yy} = \varepsilon_r \sin^2(\theta) + \varepsilon_\theta \cos^2(\theta)$$
$$\mu_z = \left(\frac{b}{b-a}\right)^2 \left(\frac{r-a}{r}\right)$$

Rotation cloaking

$$\varepsilon_{xx} = \varepsilon_u \cos^2(\theta + \tau/2) + \varepsilon_v \sin^2(\theta + \tau/2)$$
$$\varepsilon_{xy} = (\varepsilon_u - \varepsilon_v)\sin(\theta + \tau/2)\cos(\theta + \tau/2)$$
$$\varepsilon_{yy} = \varepsilon_u \sin^2(\theta + \tau/2) + \varepsilon_v \cos^2(\theta + \tau/2)$$

$$\mu_z = 1$$