Near-Cloaking by Change of Variables at Finite Frequency, I: An Approach using Lossy Layers

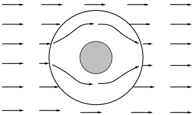
Robert V. Kohn
Courant Institute, NYU

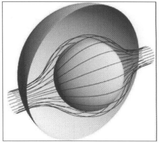
CSCAMM, September 2008

Collaborators: D. Onofrei, M. Vogelius, M. Weinstein

This talk: framework and theory
Onofrei: examples and numerics

What is cloaking?





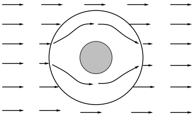
cloaked region can have any shape

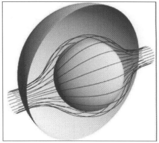
- the cloaked region should be invisible
- even the cloak itself should be invisible
- our cloaks will be coatings with heterogeneous, anisotropic dielectric properties

In what sense invisible?

- this talk: Helmholtz at fixed frequency

What is cloaking?





cloaked region can have any shape

- the cloaked region should be invisible
- even the cloak itself should be invisible
- our cloaks will be coatings with heterogeneous, anisotropic dielectric properties

In what sense invisible?

- this talk: Helmholtz at fixed frequency

Outline

(1) Cloaking by change of variables

- The basic idea
- Approximate cloaks and inclusion problems

(2) Does it work?

- At frequency 0: yes
- At frequency $\neq 0$: problem due to resonance
- Resolution: damping

(3) How well does it work?

- 2D case (is $1/|\log \rho|$ small?)
- 3D case (much better)

Change-of-variable scheme introduced by:

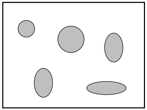
- Greenleaf, Lassas, Uhlmann (2003, freq 0 = impedance tomography)
- Pendry, Schurig, Smith (2006, finite freq = electromag scattering)

Just one approach to cloaking; others include

- anomalous localized resonance (Milton, Nicorovici)
- optical conformal mapping (Leonhardt)

Basic definitions

$$\sum \frac{\partial}{\partial x_i}\left(A_{ij}(x)\frac{\partial u}{\partial x_j}\right) + \omega^2 q(x) u = 0 \quad \text{in } \Omega$$

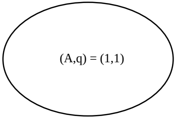


Neumann-to-Dirichlet map characterizes "boundary measurements" (invertible if ω^2 is not an eigenvalue)

$$\Lambda_{A,q} \; : \; (A\nabla u)\cdot\nu|_{\partial\Omega} \rightarrow u|_{\partial\Omega}$$

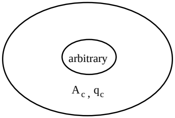
Same DN map $\Leftrightarrow$ same scattering data.

Cloaking in this setting: $A_c(x)$ and $q_c(x)$, defined on $\Omega \setminus D$, cloak D if resulting bdry measurments "look uniform," indep of content of D.



uniform case: $\Lambda_{1,1}$

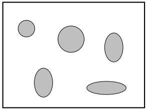
$$A(x), q(x) = \begin{cases} A_c(x), q_c(x) & \text{for } x \in \Omega \setminus D \\ \text{arbitrary} & \text{for } x \in D \end{cases}$$



same: $\Lambda_{A,q} = \Lambda_{1,1}$

Getting used to the definitions

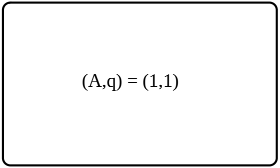
Scattering seeks knowledge of interior properties, based on response to plane waves.



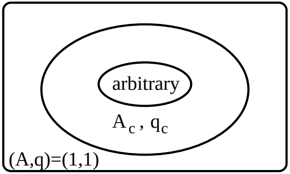
Exterior sees Ω only via Cauchy data (“bdry meas” or “DN map”).

We say $A_c(x), q_c(x)$ (defined in $\Omega \setminus D$) cloaks D if the Cauchy data at $\partial\Omega$ are (a) indep of content of D, and (b) same as for uniform case $A = q = 1$.

Name is apt, since extn of A_c, q_c by 1 to larger domain is also a cloak.



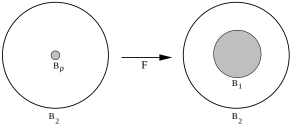
unif case: $\Lambda_{1,1}$



same as unif case: $\Lambda_{A,q} = \Lambda_{1,1}$

Invariance under change of variables

Basic observation: bdry meas determine material properties at most "up to change of variables."



If $F : \Omega \to \Omega$ is invertible and $F(x) = x$ on $\partial\Omega$ then A, q and F_*A, F_*q produce the same boundary measurements, where

$$F_*A(y) = \frac{1}{\det(DF)(x)} DF(x)A(x)(DF(x))^T \quad F_*q(y) = \frac{1}{\det(DF)(x)} q(x)$$

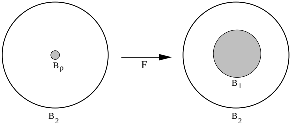
with $y = F(x)$.

Sketch: write PDE in weak form, then change variables.

- weak form: $\int_\Omega \langle A\nabla_x u, \nabla_x \phi\rangle - \omega^2 qu\phi\, dx = 0$ if $\phi = 0$ near $\partial\Omega$
- change vars: $\int_\Omega \langle F_*(A)\nabla_y u, \nabla_y \phi\rangle - \omega^2 F_*(q)u\phi\, dy = 0$
- $F = \text{id}$ at bdry $\Rightarrow$ chg of vars doesn't affect bdry data

Invariance under change of variables

Basic observation: bdry meas determine material properties at most "up to change of variables."



If $F : \Omega \to \Omega$ is invertible and $F(x) = x$ on $\partial\Omega$ then A, q and F_*A, F_*q produce the same boundary measurements, where

$$F_*A(y) = \frac{1}{\det(DF)(x)} DF(x)A(x)(DF(x))^T \quad F_*q(y) = \frac{1}{\det(DF)(x)} q(x)$$

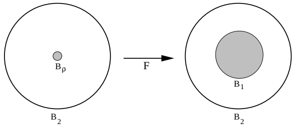
with $y = F(x)$.

Sketch: write PDE in weak form, then change variables.

- weak form: $\int_\Omega \langle A\nabla_x u, \nabla_x \phi\rangle - \omega^2 qu\phi \, dx = 0$ if $\phi = 0$ near $\partial\Omega$
- change vars: $\int_\Omega \langle F_*(A)\nabla_y u, \nabla_y \phi\rangle - \omega^2 F_*(q)u\phi \, dy = 0$
- $F = \text{id}$ at bdry $\Rightarrow$ chg of vars doesn't affect bdry data

Invariance under change of variables

Basic observation: bdry meas determine material properties at most “up to change of variables.”



If $F : \Omega \to \Omega$ is invertible and $F(x) = x$ on $\partial\Omega$ then A, q and F_*A, F_*q produce the same boundary measurements, where

$$F_*A(y) = \frac{1}{\det(DF)(x)} DF(x) A(x) (DF(x))^T \quad F_*q(y) = \frac{1}{\det(DF)(x)} q(x)$$

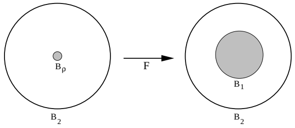
with $y = F(x)$.

Sketch: write PDE in weak form, then change variables.

- weak form: $\int_\Omega \langle A\nabla_x u, \nabla_x \phi\rangle - \omega^2 q u \phi \, dx = 0$ if $\phi = 0$ near $\partial\Omega$
- change vars: $\int_\Omega \langle F_*(A)\nabla_y u, \nabla_y \phi\rangle - \omega^2 F_*(q) u \phi \, dy = 0$
- $F = \text{id}$ at bdry $\Rightarrow$ chg of vars doesn't affect bdry data

Invariance under change of variables

Basic observation: bdry meas determine material properties at most "up to change of variables."



If $F : \Omega \to \Omega$ is invertible and $F(x) = x$ on $\partial\Omega$ then A, q and F_*A, F_*q produce the same boundary measurements, where

$$F_*A(y) = \frac{1}{\det(DF)(x)} DF(x)A(x)(DF(x))^T \quad F_*q(y) = \frac{1}{\det(DF)(x)} q(x)$$

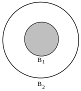
with $y = F(x)$.

Sketch: write PDE in weak form, then change variables.

- weak form: $\int_\Omega \langle A\nabla_x u, \nabla_x \phi\rangle - \omega^2 qu\phi \, dx = 0$ if $\phi = 0$ near $\partial\Omega$
- change vars: $\int_\Omega \langle F_*(A)\nabla_y u, \nabla_y \phi\rangle - \omega^2 F_*(q)u\phi \, dy = 0$
- $F = \mathrm{id}$ at bdry $\Rightarrow$ chg of vars doesn't affect bdry data

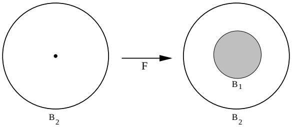
The singular change-of-variable-based cloak

Radial version, for simplicity only:
domain is B_2, cloaked region is B_1.



Choose properties of the cloak to be $A_c = F_* 1$ and $q_c = F_* 1$, where F "blows up" the origin to B_1:

$$F(x) = \left(1 + \tfrac{1}{2}|x|\right) \frac{x}{|x|}$$



Formally B_1 is cloaked. In fact, if

$$(A(y), q(y)) = \begin{cases} F_*(1,1) & \text{for } y \in B_2 \setminus B_1 \\ \text{arbitrary} & \text{for } y \in B_1 \end{cases}$$

we have, using F^{-1} as our change of variable,

$$\int_{B_2} \langle A(y)\nabla_y u, \nabla_y \phi\rangle - \omega^2 q(y) u\phi \, dy = \int_{B_2} \langle \nabla_x u, \nabla_x \phi\rangle - \omega^2 u\phi \, dx$$

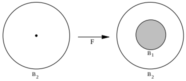
since F^{-1} shrinks B_1 (the region being cloaked) to a point.

Is this correct? F and F^{-1} are very singular.

Remarks on the singular cloak

- This scheme requires exotic materials. Recall that

$$(A_c(y), q_c(y)) = F_*(1, 1) \quad \text{at } y = F(x)$$



where F blows up a point to the region being cloaked. The material is anisotropic and singular: as $|y| \downarrow 1$, $A_c(y)$ has

  - radial eigenvector with eigenvalue $\sim (|y| - 1)^{n-1}$
  - tangential eigenspace with eigenvalue $\sim (|y| - 1)^{n-3}$,

and $q_c(y) \sim (|y| - 1)^{2(n-1)}$.

- Analysis is possible, but requires suitable notion of "weak solution" (Greenleaf, Kurylev, Lassas, Uhlmann, CMP 2008).

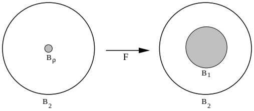
- The singular cloak makes me uncomfortable. We usually deal with singularities by smoothing them. Why not here?

A regularized version

Same idea, with more regular F. Domain B_2, cloaked region B_1.

Approx cloak uses $(A_c, q_c) = F_*(1,1)$, where $F = F_\rho$ is less singular:

- F is cont's and piecewise smooth
- it expands B_ρ to B_1 while preserving B_2
- $F(x) = x$ at the outer bdry $|x| = 2$.



Impact of contents of B_1 on bndry data becomes, via change of vars, effect of small inclusion with uncontrolled properties. In fact, if

$$(A(y), q(y)) = \begin{cases} F_*(1,1) & \text{for } y \in B_2 \setminus B_1 \\ A_D(y), q_D(y) & \text{for } y \in B_1 \end{cases}$$

then, using F^{-1} as change of variable,

$$\int_{B_2} \langle A(y)\nabla_y u, \nabla_y \phi\rangle - \omega^2 q(y) u\phi \, dy = \int_{B_2 \setminus B_\rho} \langle \nabla_x u, \nabla_x \phi\rangle - \omega^2 u\phi \, dx + \int_{B_\rho} \langle F_*^{-1}(A_D)\nabla_x u, \nabla_x \phi\rangle - \omega^2 F_*^{-1}(q_D) u\phi \, dx.$$

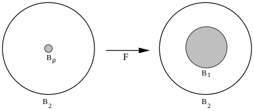
Approximate cloaking $\Leftrightarrow$ small inclusion with uncontrolled content has little effect on bndry meas.

A regularized version

Same idea, with more regular F. Domain B_2, cloaked region B_1.

Approx cloak uses $(A_c, q_c) = F_*(1,1)$, where $F = F_\rho$ is less singular:

- F is cont's and piecewise smooth
- it expands B_ρ to B_1 while preserving B_2
- $F(x) = x$ at the outer bdry $|x| = 2$.



Impact of contents of B_1 on bndry data becomes, via change of vars, effect of small inclusion with uncontrolled properties. In fact, if

$$(A(y), q(y)) = \begin{cases} F_*(1,1) & \text{for } y \in B_2 \setminus B_1 \\ A_D(y), q_D(y) & \text{for } y \in B_1 \end{cases}$$

then, using F^{-1} as change of variable,

$$\int_{B_2} \langle A(y)\nabla_y u, \nabla_y \phi\rangle - \omega^2 q(y) u\phi \, dy = \int_{B_2 \setminus B_\rho} \langle \nabla_x u, \nabla_x \phi\rangle - \omega^2 u\phi \, dx + \int_{B_\rho} \langle F_*^{-1}(A_D)\nabla_x u, \nabla_x \phi\rangle - \omega^2 F_*^{-1}(q_D) u\phi \, dx.$$

Approximate cloaking $\Leftrightarrow$ small inclusion with uncontrolled content has little effect on bndry meas.

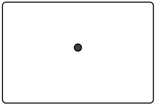
Frequency 0 is OK

Singular cloak works at frequency 0 (Greenleaf, Lassas, Uhlmann 2003)
Explanation via regularization (Kohn, Shen, Vogelius, Weinstein 2008):

$$\nabla \cdot (A \nabla u) = 0 \quad \text{in } \Omega, \quad \Lambda_A = \text{DN map}$$

Theorem: *If* $A \equiv 1$ *outside* B_ρ*, then*

$$\|\Lambda_A - \Lambda_1\| \leq C\rho^n \quad \textit{in space dim n.}$$



- Use operator norm, $\Lambda_A : H^{-1/2}(\partial\Omega) \to H^{1/2}(\partial\Omega)$. Natural choice, since finite-energy solutions of $\nabla \cdot (A\nabla u) = 0$ have Dirichlet data in $H^{1/2}$ and Neumann data in $H^{-1/2}$.
- Estimate is well-known when inclusion has constant conductivity – even for the extreme cases, when $A = 0$ or $A = \infty$ in B_ρ.
- Variational principle implies that effect of any inclusion is bracketed by effect of extreme inclusions.

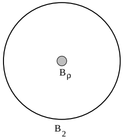
So: our regularized scheme almost cloaks B_1, if ρ is small.

Finite frequency is different

Recall: approx cloaking achieved $\Leftrightarrow$ small inclusion with uncontrolled content has little effect on bndry meas.

But: at finite frequency a small inclusion can have huge effect, due to resonance. Consider radial setting:

$$(A, q) = \begin{cases} (1,1) & \text{in } B_2 \setminus B_\rho \\ (A_\rho, q_\rho) & \text{in } B_\rho \end{cases}$$



Separate variables:

$$u = \sum \alpha_k J_k \left(\omega r \sqrt{q_\rho / A_\rho}\right) e^{ik\theta} \quad \text{for } r < \rho$$

$$u = \sum \left(\beta_k J_k(\omega r) + \gamma_k H_k^{(1)}(\omega r)\right) e^{ik\theta} \quad \text{for } \rho < r < 2$$

At freq k: 3 unknowns $\alpha_k, \beta_k, \gamma_k$ and 3 eqns:

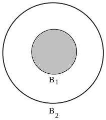
1 eqn at $r = 2$ to match Neumann data
2 eqns at $r = \rho$ to impose transmission bdry cond

Hence unique solution if eqns are not redundant. But eqns are redundant at special A_ρ, q_ρ (resonances).

Similar issue for singular cloak

Greenleaf, Kurylev, Lassas, Uhlmann (CMP 2008) studied cloaking for 3D Helmholtz by (singular) change of variables. Their conclusion: if

$$(A, q) = \begin{cases} F_*(1,1) & \text{in } \Omega \setminus D \\ (A_D, q_D) & \text{in } D \end{cases}$$



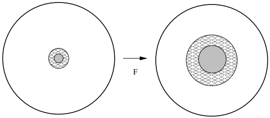
then $\nabla \cdot (A \nabla u) + \omega^2 q u = 0$ exactly when

- outside the cloaked region, $u(y) = v(x)$ where $y = F(x)$ and $\Delta v + \omega^2 v = 0$ in Ω.
- inside the cloaked region, u solves given PDE with Neumann data 0

Indicates cloaking (since v is indep of inclusion). But clearly problematic if Neumann problem for cloaked region has a resonance.

Resolution: include a lossy layer

Before mapping: uncontrolled inclusion of size ρ coated by isotropic lossy shell of width ρ



After mapping: uncontrolled inclusion of size $\frac{1}{2}$ coated by isotropic lossy shell of width $\frac{1}{2}$

$$A, q = \begin{cases} (1,1) & \text{for } |x| > 2\rho \\ (1, 1+i\beta) & \text{for } \rho < |x| < 2\rho \\ \text{arbitrary} & \text{for } |x| < \rho \end{cases} \qquad A, q = \begin{cases} F_*(1,1) & \text{for } |y| > 1 \\ F_*(1, 1+i\beta) & \text{for } \frac{1}{2} < |y| < 1 \\ \text{arbitrary} & \text{for } |y| < \frac{1}{2} \end{cases}$$

Successful $\Leftrightarrow$ presence of inclusion has little effect on DN map, regardless of inclusion contents.

Our results:

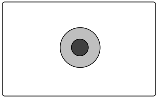
- best choice of damping is $\beta \sim \rho^{-2}$
- effect of inclusion is $1/|\log \rho|$ in 2D, $\sqrt{\rho}$ in 3D.

Suboptimal in 3D? Intuition and numerics suggest ρ not $\sqrt{\rho}$.

Results for 2D Helmholtz

Claim: an arbitrary but small inclusion, coated by a lossy layer, has little effect on bdry meas, if loss parameter is $\beta \sim \rho^{-2}$.

$$A, q = \begin{cases} (1, 1 + i\rho^{-2}) & \text{for } \rho < |x| < 2\rho \\ \text{arbitrary pos} & \text{for } |x| < \rho \end{cases}$$



Theorem. When embedded in a uniform medium ($A = 1$, $q = 1$), the effect of such an inclusion is bounded by

$$\|\Lambda_{A,q} - \Lambda_{1,1}\| \leq C_\omega / |\log \rho|.$$

LHS is operator norm from $H^{-1/2}$ to $H^{1/2}$ (natural norms for Neumann and Dirichlet data of finite-energy solutions). If $f = \sum a_k e^{ik\theta}$,

$$\|f\|^2_{H^{-1/2}} = \sum |k|^{-1} |a_k|^2, \quad \|f\|^2_{H^{1/2}} = \sum |k| |a_k|^2.$$

3D is better

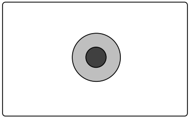
For 2D Helmholtz, cloaking error was $C/|\log \rho|$.
Linked to fund soln of Laplacian.

For 3D Helmholtz, obvious guess is $C\rho$. Supported by numerics.
However our method gives only $C\sqrt{\rho}$: for

$$\nabla \cdot (A \nabla u_\rho) + \omega^2 q u_\rho = 0 \text{ in } \Omega \subset R^3$$

with

$$\begin{cases} A = 1, q = 1 & \text{in } \Omega \setminus B_{2\rho} \\ A = 1, q = 1 + i\rho^{-2} & \text{in } B_{2\rho} \setminus B_\rho \\ \text{arbitrary real, positive} & \text{in } B_\rho. \end{cases}$$



we get

$$\|\Lambda_{A,q} - \Lambda_{1,1}\| \leq C_\omega \sqrt{\rho}.$$

Overview of analysis

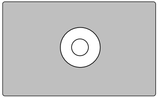
Recall eqn:

$$\nabla \cdot (A\nabla u_\rho) + \omega^2 q u_\rho = 0 \text{ in } \Omega$$

where

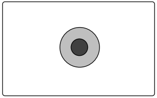
$$\begin{cases} A = 1, q = 1 & \text{in } \Omega \setminus B_{2\rho} \\ A = 1, q = 1 + i\beta & \text{in } B_{2\rho} \setminus B_\rho \\ \text{arbitrary real, positive} & \text{in } B_\rho. \end{cases}$$

I. Compare Helmholtz in shell $\Omega \setminus B_{2\rho}$ to Helmholtz in Ω.



Show that inclusion has little effect on boundary measurements, unless something wild is happening at $\partial B_{2\rho}$.

II. Obtain global control using lossiness of $B_{2\rho} \setminus B_\rho$.



Make good choice of lossiness ($\beta \sim \rho^{-2}$). Show that nothing wild can happen at $\partial B_{2\rho}$, regardless of content of B_ρ.

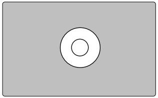
Estimate holds even when lossless problem is resonant.

Outline of step I

I. Compare Helmholtz in shell $\Omega \setminus B_{2\rho}$ to Helmholtz in Ω.

Consider

$$\begin{aligned} \Delta u_0 + \omega^2 u_0 &= 0 \text{ in } \Omega \\ \Delta u_\rho + \omega^2 u_\rho &= 0 \text{ in } \Omega \setminus B_{2\rho} \end{aligned}$$



with same Neumann data ψ at $\partial\Omega$, and Dir data ϕ for u_ρ at $\partial B_{2\rho}$. Then

$$\|u_\rho - u_0\|_{H^{1/2}(\partial\Omega)} \leq Ce(\rho) \left(\|\psi\|_{H^{-1/2}(\partial\Omega)} + \|\phi(2\rho\, \cdot)\|_{H^{-1/2}(\partial B_1)} \right)$$

where

$$e(\rho) = \begin{cases} 1/|\log \rho| & \text{in dim 2} \\ \rho & \text{in dim 3.} \end{cases}$$

Main idea: if behavior at inclusion edge is uniform, then effect is like a small hole with a Dirichlet bdry condition.

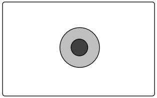
If behavior at inclusion edge is oscillatory in θ, influence decays faster.

Outline of step II

II. Control u_ρ on $\partial B_{2\rho}$, if annulus $\rho < |x| < 2\rho$ is lossy. Let

$$\nabla \cdot (A \nabla u_\rho) + \omega^2 q u_\rho = 0 \text{ in } \Omega,$$

$$\begin{cases} A = 1, q = 1 & \text{for } x \in \Omega \setminus B_{2\rho} \\ A = 1, q = 1 + i\beta & \text{for } \rho < |x| < 2\rho \\ \text{any real, pos values} & \text{for } |x| < \rho. \end{cases}$$



using Neumann data ψ at $\partial\Omega$. Then (in dim n)

$$\|u_\rho(2\rho\,\cdot)\|_{H^{-1/2}(\partial B_1)} \leq C(1 + (1+\beta)\rho^2) \frac{1}{\rho^{n/2}\sqrt{\beta}} \left(\|\psi\|_{H^{-1/2}(\partial\Omega)} + \|u_\rho\|_{H^{1/2}(\partial\Omega)} \right)$$

Main ideas:

1) Imaginary part of energy identity gives

$$\omega^2 \beta \int_{B_{2\rho} \setminus B_\rho} |u_\rho|^2 \leq \left(\|\psi\|_{H^{-1/2}(\partial\Omega)} + \|u_\rho\|_{H^{1/2}(\partial\Omega)} \right)^2$$

2) Elliptic estimate for $\Delta u + \omega^2(1 + i\beta)u = 0$ on $B_{2\rho} \setminus B_\rho$ gives:

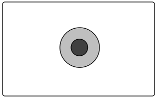
$$\|u_\rho(2\rho\,\cdot)\|^2_{H^{-1/2}(\partial B_1)} \leq C(1 + (1+\beta)\rho^2)^2 \rho^{-n} \int_{B_{2\rho} \setminus B_\rho} |u_\rho|^2$$

Outline of step II

II. Control u_ρ on $\partial B_{2\rho}$, if annulus $\rho < |x| < 2\rho$ is lossy. Let

$$\nabla \cdot (A \nabla u_\rho) + \omega^2 q u_\rho = 0 \text{ in } \Omega,$$

$$\begin{cases} A = 1, q = 1 & \text{for } x \in \Omega \setminus B_{2\rho} \\ A = 1, q = 1 + i\beta & \text{for } \rho < |x| < 2\rho \\ \text{any real, pos values} & \text{for } |x| < \rho. \end{cases}$$



using Neumann data ψ at $\partial\Omega$. Then (in dim n)

$$\|u_\rho(2\rho\,\cdot)\|_{H^{-1/2}(\partial B_1)} \le C(1 + (1+\beta)\rho^2)\frac{1}{\rho^{n/2}\sqrt{\beta}}\left(\|\psi\|_{H^{-1/2}(\partial\Omega)} + \|u_\rho\|_{H^{1/2}(\partial\Omega)}\right)$$

Main ideas:

1) Imaginary part of energy identity gives

$$\omega^2 \beta \int_{B_{2\rho}\setminus B_\rho} |u_\rho|^2 \le \left(\|\psi\|_{H^{-1/2}(\partial\Omega)} + \|u_\rho\|_{H^{1/2}(\partial\Omega)}\right)^2$$

2) Elliptic estimate for $\Delta u + \omega^2(1 + i\beta)u = 0$ on $B_{2\rho} \setminus B_\rho$ gives:

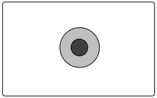
$$\|u_\rho(2\rho\,\cdot)\|^2_{H^{-1/2}(\partial B_1)} \le C(1 + (1+\beta)\rho^2)^2 \rho^{-n} \int_{B_{2\rho}\setminus B_\rho} |u_\rho|^2$$

Outline of step II

II. Control u_ρ on $\partial B_{2\rho}$, if annulus $\rho < |x| < 2\rho$ is lossy. Let

$$\nabla \cdot (A \nabla u_\rho) + \omega^2 q u_\rho = 0 \text{ in } \Omega,$$

$$\begin{cases} A = 1, q = 1 & \text{for } x \in \Omega \setminus B_{2\rho} \\ A = 1, q = 1 + i\beta & \text{for } \rho < |x| < 2\rho \\ \text{any real, pos values} & \text{for } |x| < \rho. \end{cases}$$



using Neumann data ψ at $\partial\Omega$. Then (in dim n)

$$\|u_\rho(2\rho\,\cdot)\|_{H^{-1/2}(\partial B_1)} \leq C(1 + (1+\beta)\rho^2) \frac{1}{\rho^{n/2}\sqrt{\beta}} \left(\|\psi\|_{H^{-1/2}(\partial\Omega)} + \|u_\rho\|_{H^{1/2}(\partial\Omega)} \right)$$

Main ideas:

1) Imaginary part of energy identity gives

$$\omega^2 \beta \int_{B_{2\rho} \setminus B_\rho} |u_\rho|^2 \leq \left(\|\psi\|_{H^{-1/2}(\partial\Omega)} + \|u_\rho\|_{H^{1/2}(\partial\Omega)} \right)^2$$

2) Elliptic estimate for $\Delta u + \omega^2(1 + i\beta)u = 0$ on $B_{2\rho} \setminus B_\rho$ gives:

$$\|u_\rho(2\rho\,\cdot)\|^2_{H^{-1/2}(\partial B_1)} \leq C(1 + (1+\beta)\rho^2)^2 \rho^{-n} \int_{B_{2\rho} \setminus B_\rho} |u_\rho|^2$$

Putting it together

Goal: compare solutions of

$$\Delta u_0 + \omega^2 u_0 = 0 \quad \text{and} \quad \nabla(A\nabla u_\rho) + \omega^2 q u_\rho = 0 \text{ in } \Omega$$

with same Neumann data ψ at $\partial\Omega$.

Step 1 gave $\|u_\rho - u_0\|_{H^{1/2}(\partial\Omega)} \le Ce(\rho)\left(\|\psi\|_{H^{-1/2}(\partial\Omega)} + \|u_\rho(2\rho\,\cdot)\|_{H^{-1/2}(\partial B_1)}\right)$.

Step 2 with $\beta \sim \rho^{-2}$ gives

$$\|u_\rho(2\rho\,\cdot)\|_{H^{-1/2}(\partial B_1)} \le C\rho^{1-\frac{n}{2}}\left(\|\psi\|_{H^{-1/2}(\partial\Omega)} + \|u_\rho\|_{H^{1/2}(\partial\Omega)}\right)$$

Combining gives

$$\|u_\rho - u_0\|_{H^{1/2}(\partial\Omega)} \le Ce(\rho)\left(\rho^{1-\frac{n}{2}}\|\psi\|_{H^{-1/2}(\partial\Omega)} + \rho^{1-\frac{n}{2}}\|u_\rho\|_{H^{1/2}(\partial\Omega)}\right)$$

Eliminate last RHS term using $\|u_\rho\| \le \|u_\rho - u_0\| + \|u_0\|$ to get

$$\|u_\rho - u_0\|_{H^{1/2}(\partial\Omega)} \le Ce(\rho)\rho^{1-\frac{n}{2}}\|\psi\|_{H^{-1/2}(\partial\Omega)}$$

Thus: perturbation of boundary operator is at most

$$\le Ce(\rho)\rho^{1-\frac{n}{2}} = \begin{cases} C/|\log\rho| & n=2 \\ C\sqrt{\rho} & n=3 \end{cases}$$

Putting it together

Goal: compare solutions of

$$\Delta u_0 + \omega^2 u_0 = 0 \quad \text{and} \quad \nabla(A\nabla u_\rho) + \omega^2 q u_\rho = 0 \text{ in } \Omega$$

with same Neumann data ψ at $\partial\Omega$.

Step 1 gave $\|u_\rho - u_0\|_{H^{1/2}(\partial\Omega)} \le Ce(\rho)\left(\|\psi\|_{H^{-1/2}(\partial\Omega)} + \|u_\rho(2\rho\,\cdot)\|_{H^{-1/2}(\partial B_1)}\right).$

Step 2 with $\beta \sim \rho^{-2}$ gives

$$\|u_\rho(2\rho\,\cdot)\|_{H^{-1/2}(\partial B_1)} \le C\rho^{1-\frac{n}{2}}\left(\|\psi\|_{H^{-1/2}(\partial\Omega)} + \|u_\rho\|_{H^{1/2}(\partial\Omega)}\right)$$

Combining gives

$$\|u_\rho - u_0\|_{H^{1/2}(\partial\Omega)} \le Ce(\rho)\left(\rho^{1-\frac{n}{2}}\|\psi\|_{H^{-1/2}(\partial\Omega)} + \rho^{1-\frac{n}{2}}\|u_\rho\|_{H^{1/2}(\partial\Omega)}\right)$$

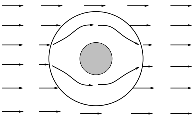
Eliminate last RHS term using $\|u_\rho\| \le \|u_\rho - u_0\| + \|u_0\|$ to get

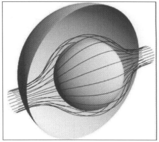
$$\|u_\rho - u_0\|_{H^{1/2}(\partial\Omega)} \le Ce(\rho)\rho^{1-\frac{n}{2}}\|\psi\|_{H^{-1/2}(\partial\Omega)}$$

Thus: perturbation of boundary operator is at most

$$\le Ce(\rho)\rho^{1-\frac{n}{2}} = \begin{cases} C/|\log\rho| & n=2 \\ C\sqrt{\rho} & n=3 \end{cases}$$

Conclusions





How well does the change-of-variable-based cloaking scheme work?

- Equivalent to: how much can a small inclusion affect bdry meas?
- At freq 0: error estimate ρ^n in dim n (no damping)
- At freq $\neq 0$:
  - complete failure if object to be cloaked is resonant
  - difficulty fixed by introducing lossy shell
  - error estimate $1/|\log \rho|$ in 2D, $\sqrt{\rho}$ in 3D.

Examples and numerics to be presented by Onofrei.