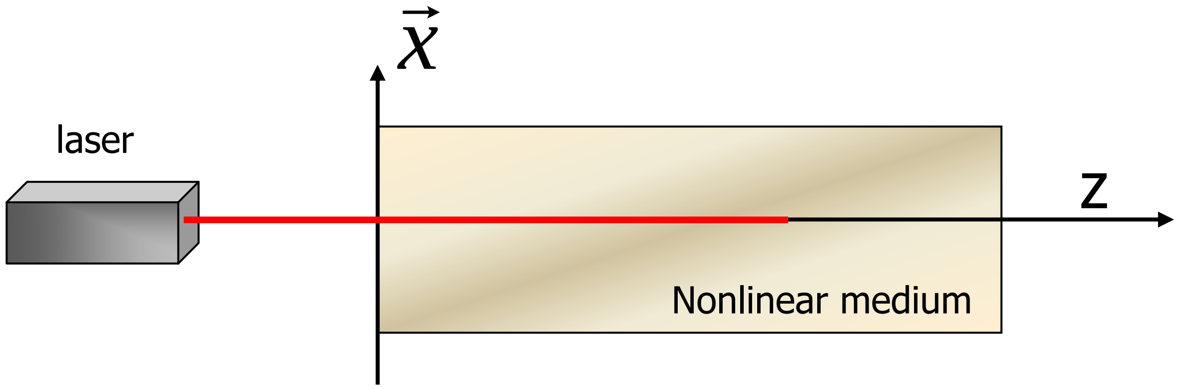
Stability and instability of solitons in inhomogeneous media

Yonatan Sivan, Tel Aviv University, Israel
now at Purdue University, USA

- G. Fibich, Tel Aviv University, Israel
- M. Weinstein, Columbia University, USA
- B. Ilan, UC Merced, USA

Motivation



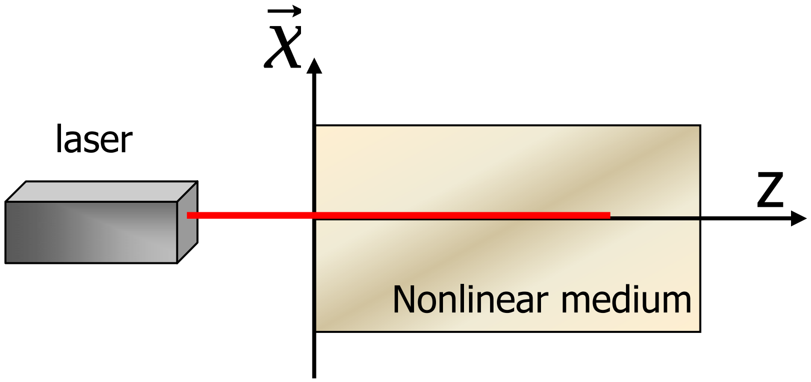
- Light propagation in a nonlinear medium
  - z is the direction of propagation
  - transverse coordinate $\vec{x} = (x_1, \ldots, x_d), \quad d = 1, 2, 3$
- Look for solitons – waves that maintain their shape along the propagation in z

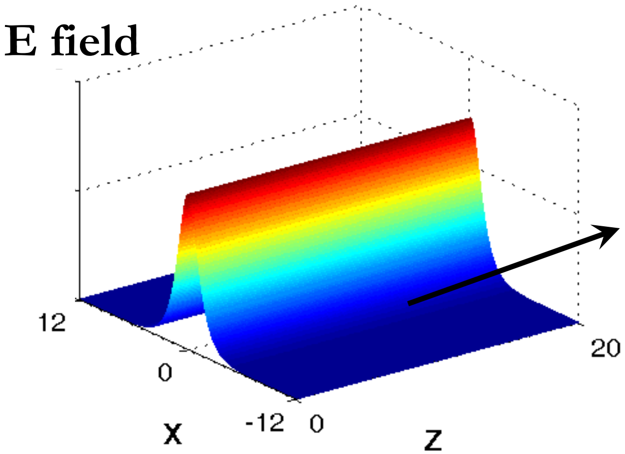
Solitons

- Appear in
  - Nonlinear optics
  - Cold atoms – Bose-Einstein Condensation (BEC)
  - Solid state
  - Water waves
  - Plasma physics
- Applications – communications, quantum computing, …

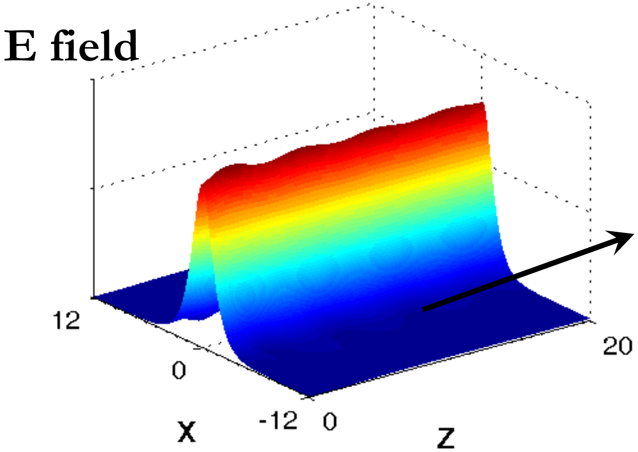
Soliton stability

- Ideally – get same soliton at the other end





- In practice, soliton must be stable (robust) under perturbations



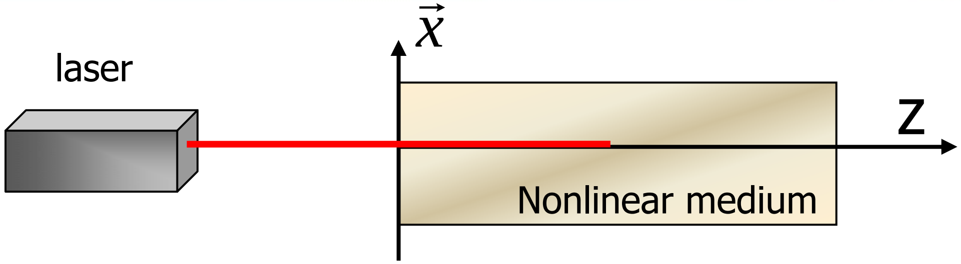
Stability analysis

- KEY question – is the soliton stable?
  - hundreds of papers…
- Typical answer – **yes** (stable)/**no** (unstable)
  - What is the instability dynamics?
- In this talk, a different approach
  - **Qualitative approach:** characterize instability dynamics
  - **Quantitative approach:** quantify strength of stability/instability

Outline of the talk

- Solitons in Nonlinear Schrödinger (NLS) Eq.
- Stability theory
- Qualitative approach
- Quantitative approach

Paraxial propagation – NLS model



$$iA_z(z,\vec{x}) + \underbrace{\nabla^2 A}_{diffraction} + \underbrace{F(|A|^2)A}_{\text{focusing nonlinearity}} = 0$$

- A – amplitude of electric field
- Initial condition: $A(z=0,\vec{x}) = A_0(\vec{x})$
  - – z is a "time"-like coordinate
- Competition of diffraction $(\nabla^2 = \partial_{x_1}^2 + \cdots + \partial_{x_d}^2)$ and focusing nonlinearity (F)

Typical (self-) focusing nonlinearities

$$iA_z(z,\vec{x}) + \nabla^2 A + F(|A|^2)A = 0$$

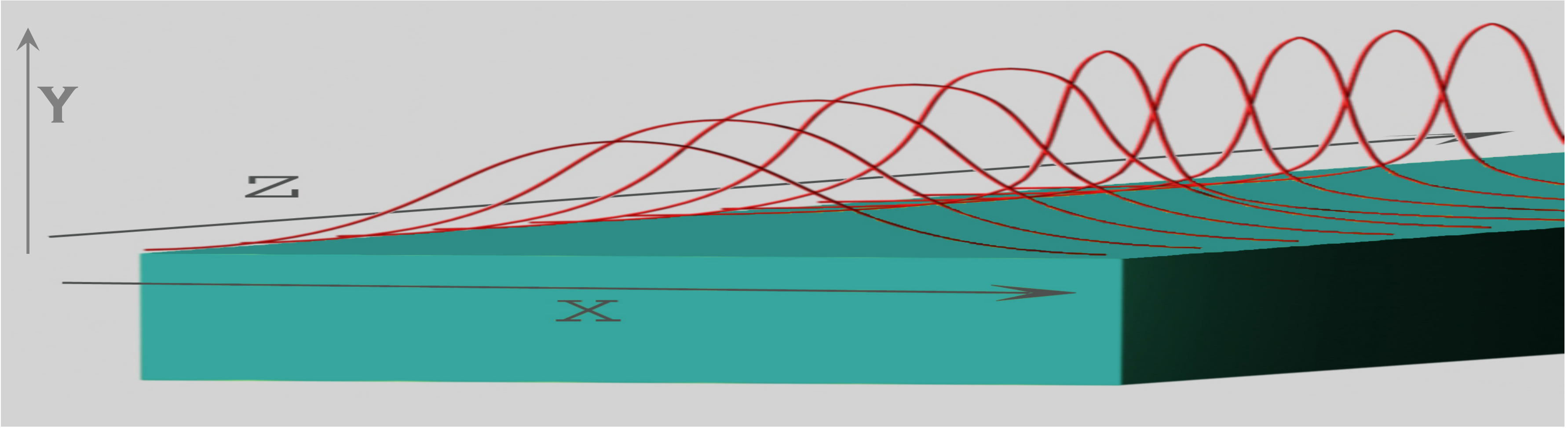
$$n = n_0 + F(|A|^2)$$

- Cubic (Kerr) nonlinearity $\quad F(|A|^2) = |A|^2$
- Cubic-quintic nonlinearity $\quad F(|A|^2) = |A|^2 - \gamma |A|^4$
- Saturable nonlinearity $\quad F(|A|^2) = \dfrac{|A|^2}{1+\gamma |A|^2}$

Physical configurations – slab/planar waveguide

$$iA_z(z,x) + \underbrace{A_{xx}}_{diffraction} + \underbrace{F(|A|^2)A}_{nonlinearity} = 0$$

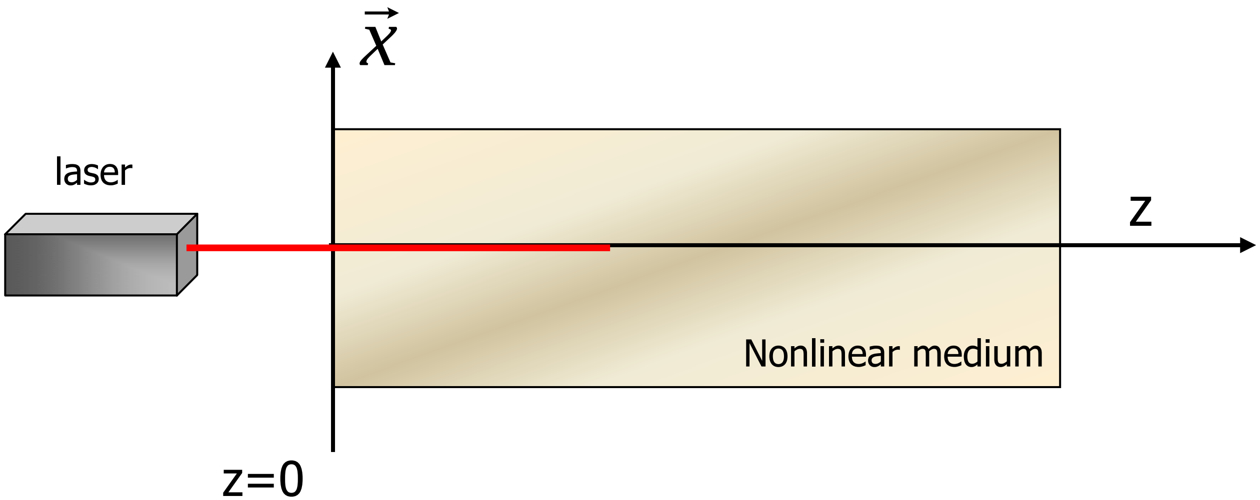
- $\vec{x} = x, \quad d = 1$
- no dynamics in y direction
- 1+1 dimensions (x,z)



Physical configurations – bulk medium

$$iA_z(z,x,y)+\underbrace{\nabla^2 A}_{diffraction}+\underbrace{F(|A|^2)A}_{nonlinearity}=0$$

- $\vec{x}=(x,y), \quad d=2$
- 2+1 dimensions (x+y,z)



Physical configurations – pulses in bulk medium

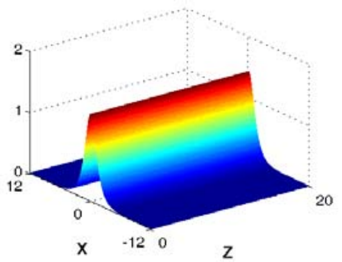
$$iA_z(z,x,y,t) + \underbrace{\nabla^2 A}_{diffraction} - \underbrace{\beta_2 A_{tt}}_{dispersion} + \underbrace{F(|A|^2)A}_{nonlinearity} = 0$$

- $\vec{x} = (x, y, t), \quad d = 3$
- $\beta_2 < 0$, anomalous group velocity dispersion (GVD)
- 3+1 dimensions (x+y+t,z)
- Spatio-temporal soliton = "Light bullets"

Eq. for solitons

$$iA_z(z,\vec{x}) + \nabla^2 A + F(|A|^2)A = 0$$

- Solitons are of the form



$$A(z,\vec{x}) = e^{i\nu z}u(\vec{x};\nu), \quad \nabla^2 u(\vec{x};\nu) + F(u^2)u - \nu u = 0$$

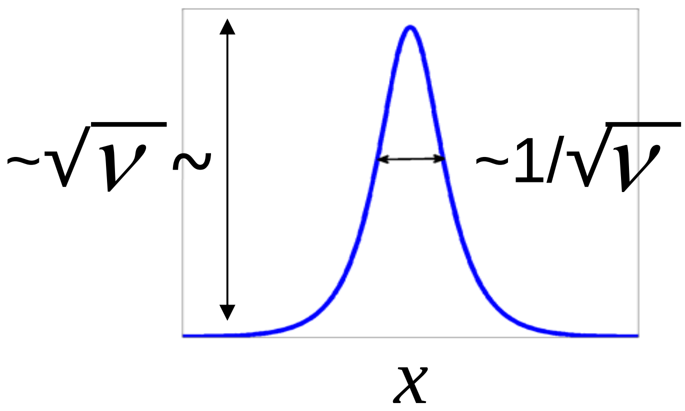
- Do not change their shape during propagation (in z)
- Exhibit perfect balance between diffraction and nonlinearity

Example – 1D solitons in a Kerr medium

$$iA_z(z,x) + A_{xx} + \underbrace{|A|^2 A}_{\substack{Kerr\,(cubic)\\ nonlinearity}} = 0$$

$$A(z,x) = e^{i\nu z}u(x;\nu), \quad \nabla^2 u(x;\nu) + u^3 - \nu u = 0$$

- Explicit solution $u(x;\nu) = \sqrt{2\nu}\,\mathrm{sech}(\sqrt{\nu}\,x)$
- ν (propagation const.) proportional to
  - amplitude
  - inverse width



Outline of the talk

- Solitons in Nonlinear Schrödinger Eq.
- Stability theory
- Qualitative approach
- Quantitative approach

Stability theory

- Vakhitov-Kolokolov (1973): necessary condition for stability of

$$A = e^{i\nu z} u(\vec{x}, \nu)$$

is the **slope condition**

$$\frac{\partial P(\nu)}{\partial \nu} > 0, \qquad \underbrace{P(\nu) = \int u^2(\vec{x};\nu)\, d\vec{x}}_{soliton \;\; power}$$

Example: homogeneous Kerr medium

- $P(\nu) = \int u^2(\vec{x};\nu)\,d\vec{x} = C(d)\nu^{\frac{2-d}{2}}$

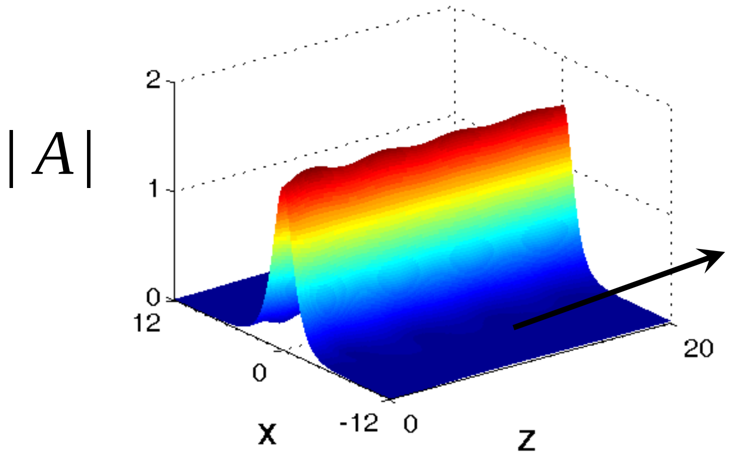
$$d = 1, \quad \frac{\partial P}{\partial \nu} > 0 \quad \Rightarrow \quad \text{stability}$$

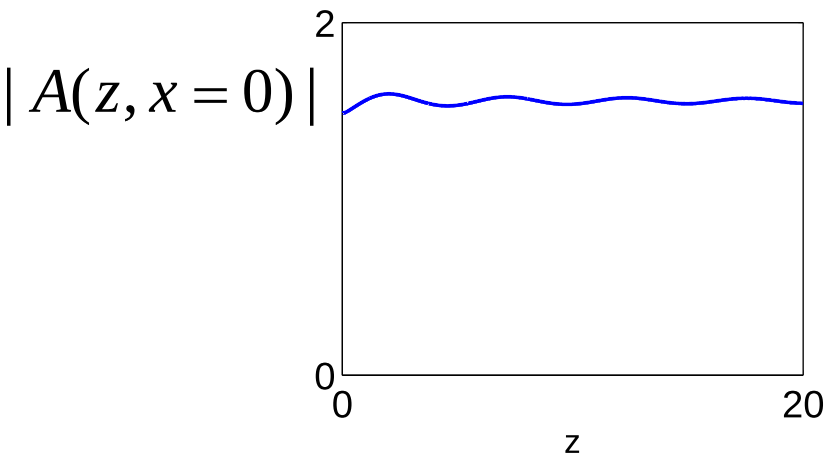
$$\left.\begin{aligned} d = 2, \quad \frac{\partial P}{\partial \nu} = 0 \\ d = 3, \quad \frac{\partial P}{\partial \nu} < 0 \end{aligned}\right\} \quad \Rightarrow \quad \text{instability}$$

Stability in a d=1 Kerr medium

$$A(z=0,x) = (1+0.05)\underbrace{\sqrt{2\nu}\,\mathrm{sech}(\sqrt{\nu}\,x)}_{u(x)}$$

- Incident beam is a perturbed d=1 soliton

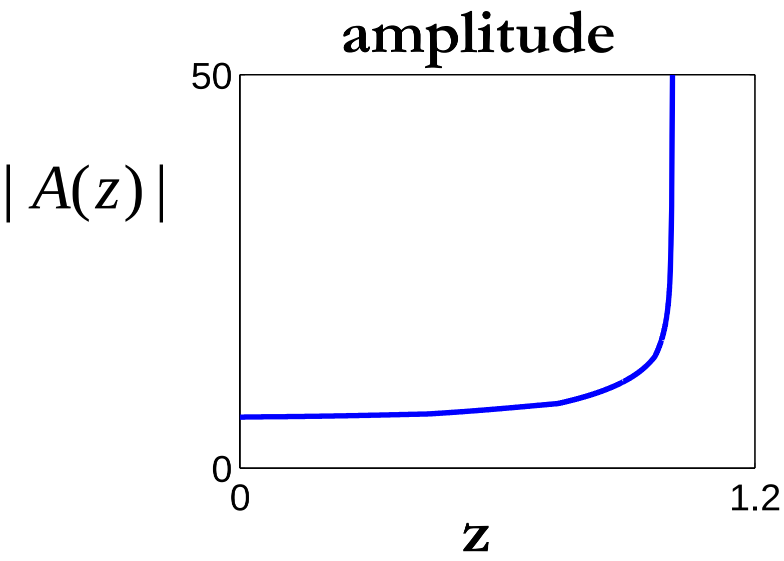


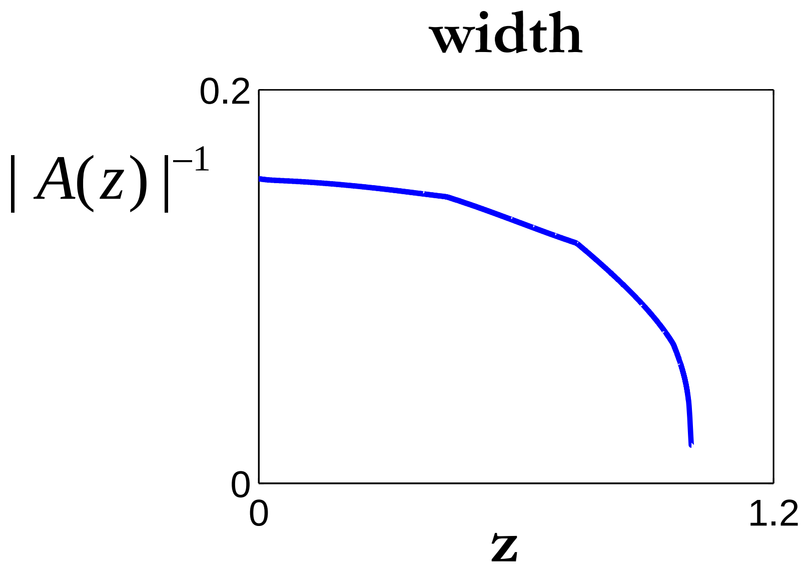


- Solution stays close to the soliton
- 1D Solitons are stable!

Instability in a d=2 Kerr medium

- Incident beam is a perturbed d=2 soliton
- $A(z=0)=(1+0.02)u(x,y)$





- collapse at a finite distance!
- 2D Solitons are unstable

- Vakhitov-Kolokolov (1973): Slope condition is **necessary** for stability
- Is it also sufficient?

Rigorous stability theory (u>0)

- Weinstein (1985-6), Grillakis, Shatah, Strauss (1987-9):
  1. **Slope (VK) condition**
  2. **Spectral condition:** the operator

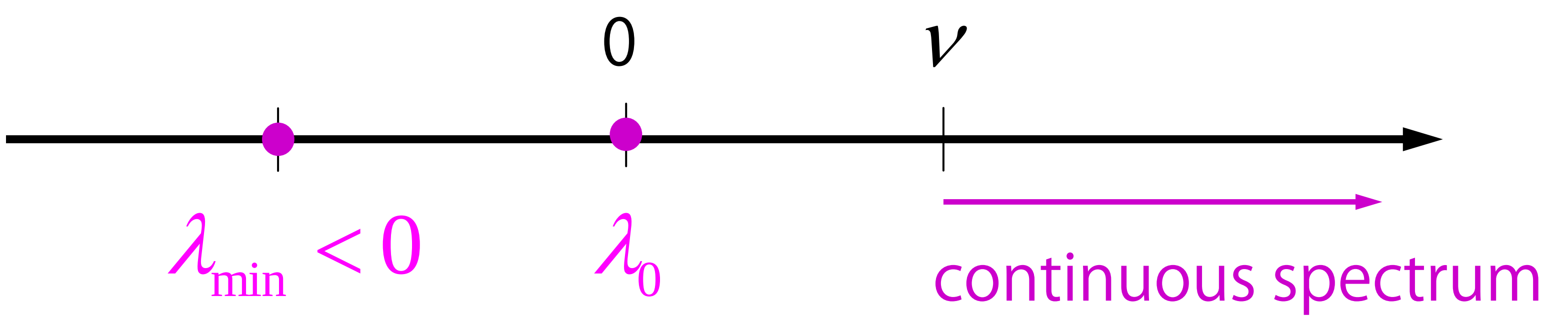
$$L_+ = -\nabla^2 + \nu - G(u^2)$$

must have only one negative eigenvalue

- Two conditions are necessary and sufficient for stability

Spectrum of L_+

$$L_+ = -\nabla^2 + \nu - G(u^2)$$



- Only one negative eigenvalue λ_{min}
- Spectral condition is satisfied

Summary - stability in homogeneous medium

- Spectral condition is always satisfied
- Stability determined by slope (VK) condition

$$\frac{\partial P(\nu)}{\partial \nu} > 0, \qquad \underbrace{P(\nu) = \int u^2(\vec{x};\nu)\, d\vec{x}}_{soliton \quad power}$$

Inhomogeneous media

Light propagation in inhomogeneous medium

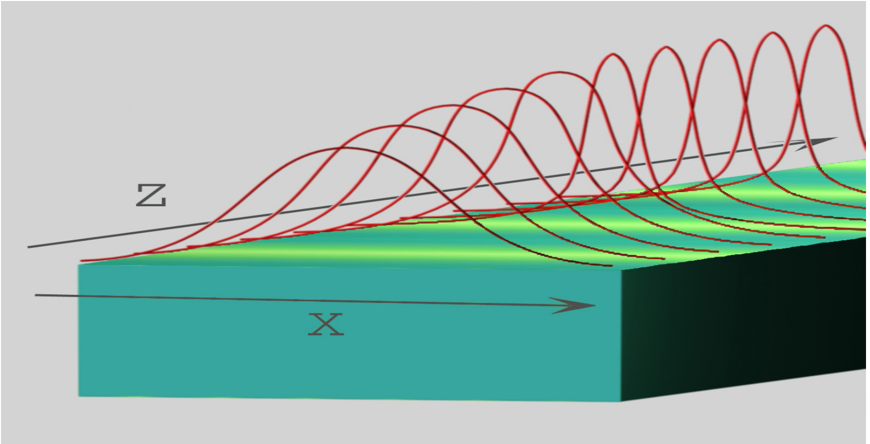
- Since late 1980's: interest in inhomogeneous media
  - E. Yablonovitch, S. John (1986) – photonic crystals
  - Christodoulides & Joseph (1988) – discrete solitons
  - ...
- Goals:
  - Stabilize beams in high dimensions ("Light bullets")
  - Applications – communications (switching, routing, …)

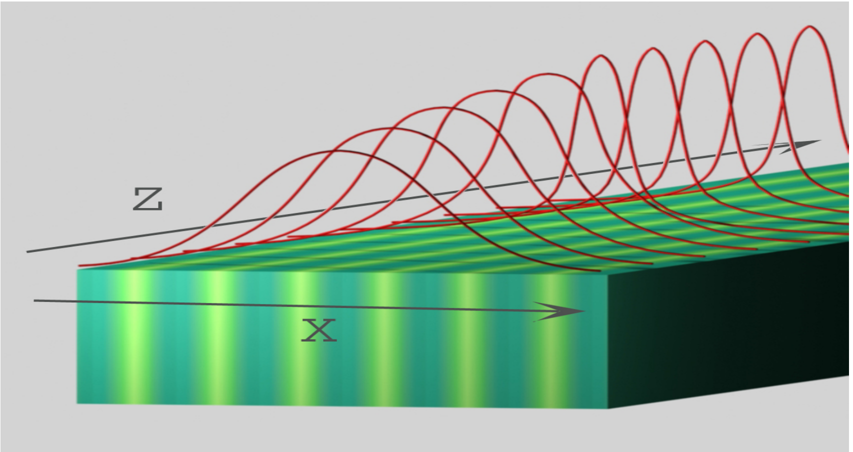
Inhomogeneous media

- Varying **linear** refractive index
  - Waveguide arrays / photonic lattices

$$n = n_0(z) + n_2 |A|^2$$

$$n = n_0(x) + n_2 |A|^2$$

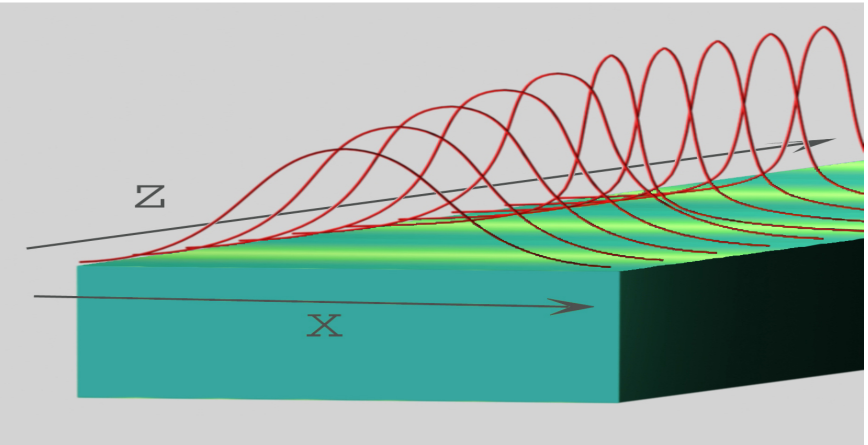


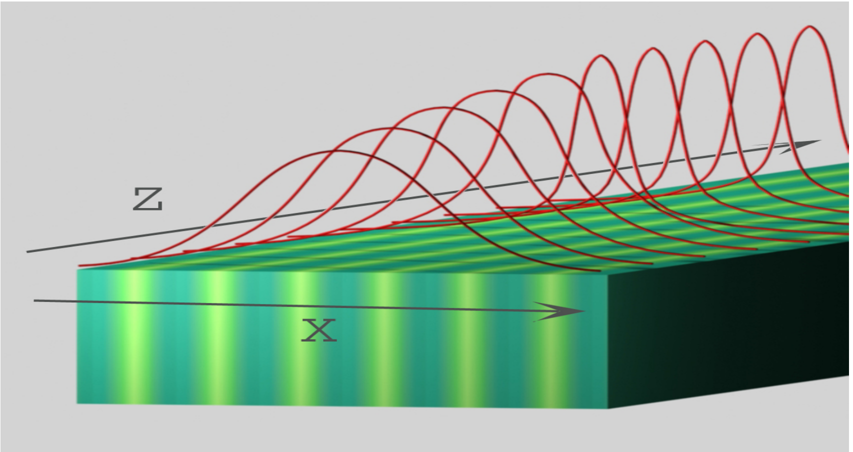


Inhomogeneous media

- Varying **nonlinear** refractive index
  - Novel materials

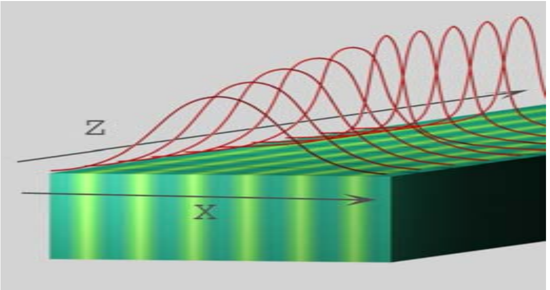
$$n = n_0 + n_2(z)\,|A|^2 \qquad\qquad n = n_0 + n_2(x)\,|A|^2$$



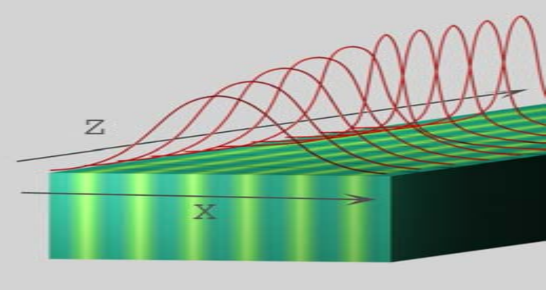


NLS in inhomogeneous media

- This study – modulation in $\vec{x}$ only
- Refractive index $=$ $-$ Potential



NLS in inhomogeneous media



- This study – modulation in $\vec{x}$ only
- Refractive index $=$ **–** Potential

$$iA_z(z,\vec{x}) + \nabla^2 A + \left(1 - V_{nl}(\vec{x})\right) F(|A|^2) A - V_l(\vec{x}) A = 0$$

- – arbitrary potentials (V_{nl}, V_l)
  - periodic/disordered potentials, periodic potentials with defects, single/multi-waveguide potentials etc.
- – any nonlinearity F
- – any dimension d

Inhomogeneities in BEC

- Same equation (Gross-Pitaevskii) in BEC

$$iA_t(t,\vec{x}) + \nabla^2 A + \left(1 - V_{nl}(\vec{x})\right) F(|A|^2)A - V_l(\vec{x})A = 0$$

- Dynamics in time (not z)
- $\vec{x} = (x, y, z), \;\; \vec{x} = (x, y), \;\; \vec{x} = x$
- Inhomogeneities created by
  - Magnetic traps
  - Feshbach resonance
  - Optical lattices

How do inhomogeneities affect stability?

"Applied" approach	Rigorous approach
• Slope (VK) condition • Numerics • Ignore spectral condition	• Slope (VK) condition • Spectral condition $L_{+}^{(V)} = L_{+} + V_{nl}(\vec{x})G(\|A\|^{2}) + V_{l}(\vec{x})$

Typical result – soliton stable (yes)/unstable (no)

Qualitative approach

- Characterize the instability dynamics
- Key observation: instability dynamics depends on which condition is violated
  - Look at each condition separately
- Results for ground state solitons only (u>0)

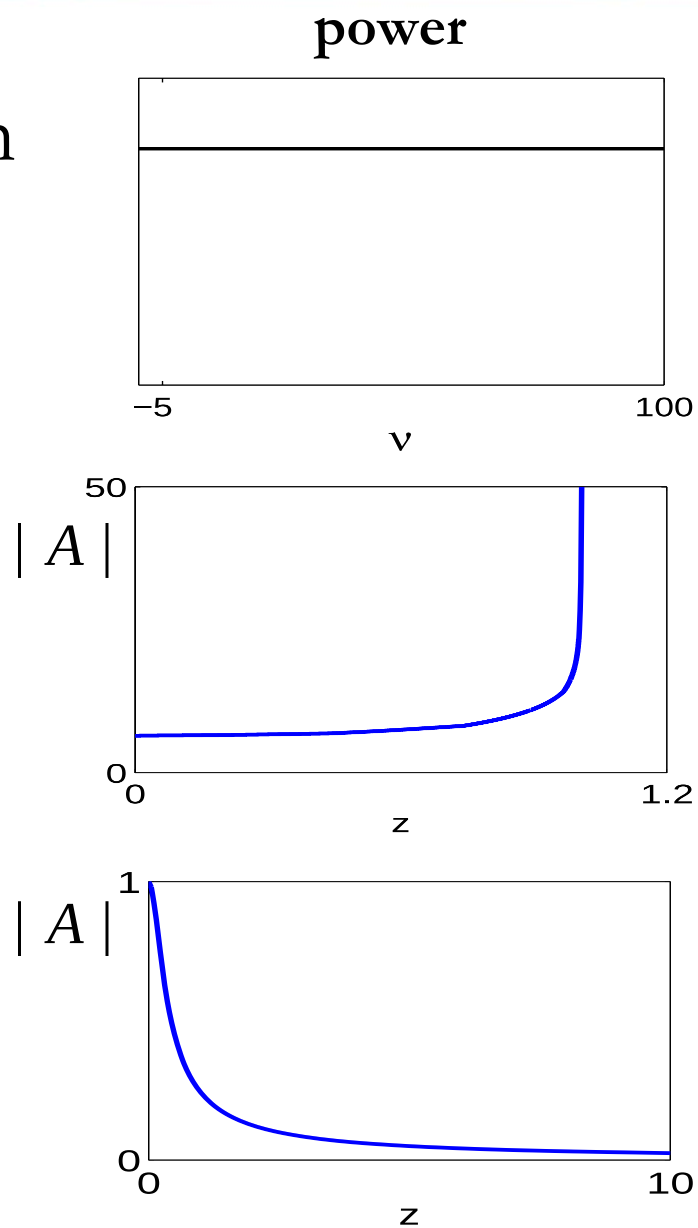
Outline of the talk

- Solitons in Nonlinear Schrödinger Eq.
- Stability theory
- Qualitative approach
  - Slope condition
  - Spectral condition
- Quantitative approach

Instability due to violation of **slope** condition

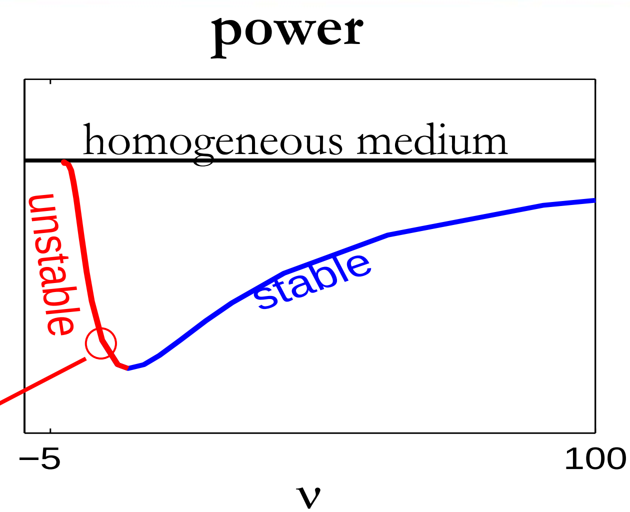
Violation of slope condition

- d=2 homogeneous Kerr medium
  - $P(\nu) = Const$
  - Slope = 0, i.e., instability
  - $A(z=0) = (1+0.02)u(x,y)$
    - Collapse
  - $A(z=0) = (1-0.02)u(x,y)$
    - Total diffraction



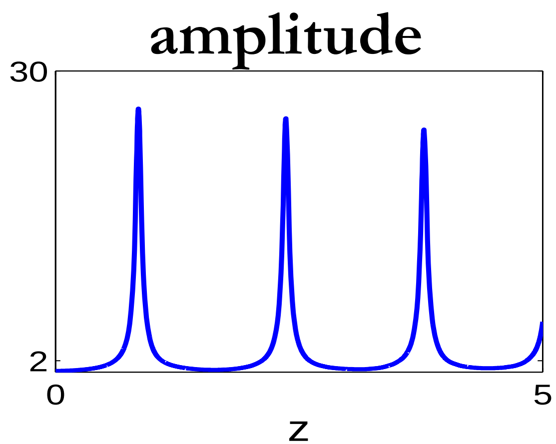
Violation of slope condition – cont.

- d=2 Kerr medium with potential
  - Stable and unstable branches

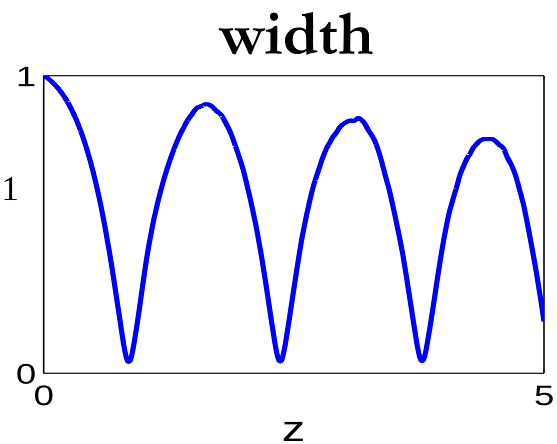


  - $A(z=0)=(1+0.01)u(x,y)$
  - 1% perturbation $\Longrightarrow$ width decrease by factor of 15

$|A(z)|$

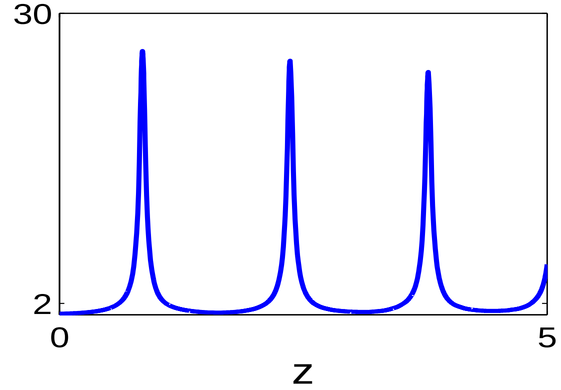


$|A(z)|^{-1}$



Violation of slope condition – cont.

Conclusion:
violation of slope condition $\Longrightarrow$ focusing instability



Instability due to violation of **spectral** condition

Spectral condition

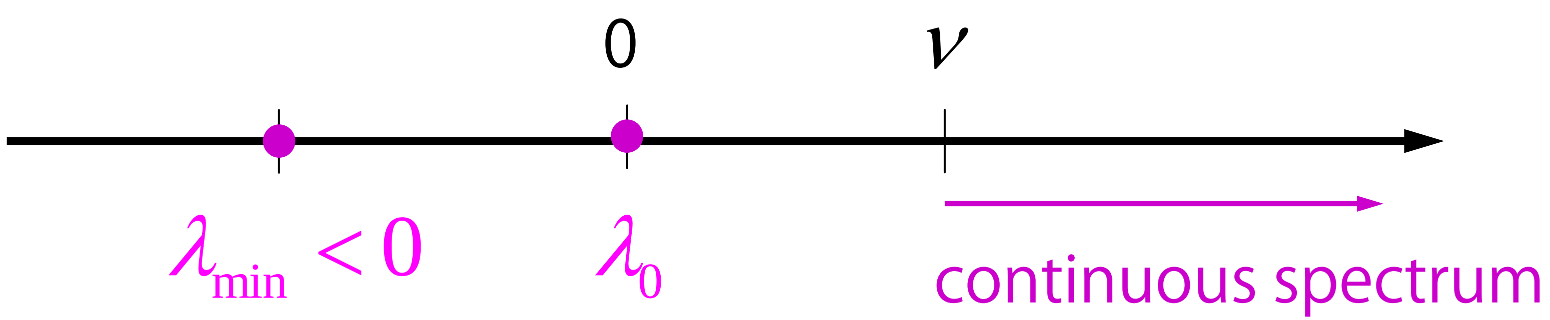
$$A(z,\vec{x}) = e^{i\nu z} u(\vec{x}), \quad u > 0$$

- The operator

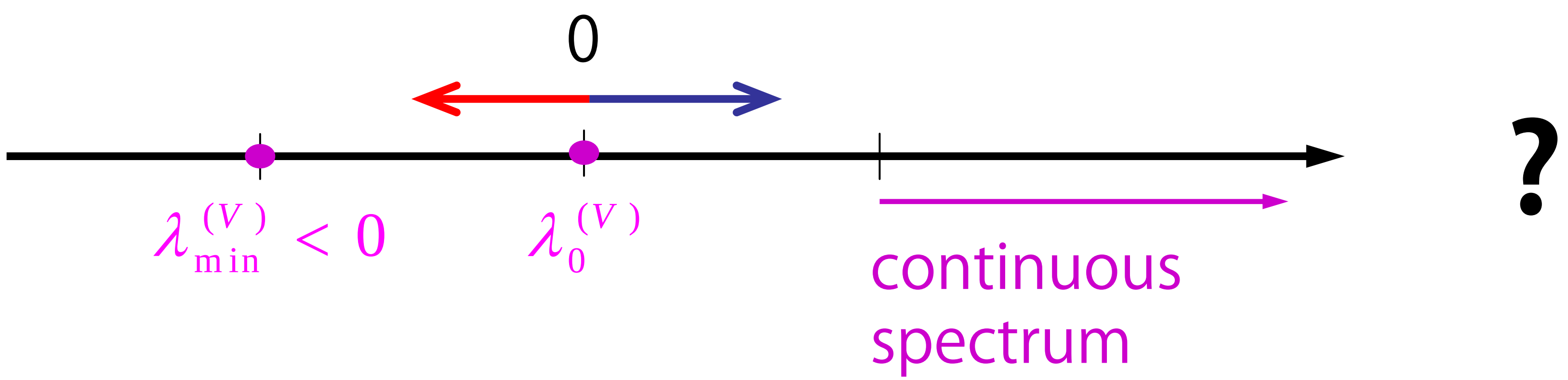
$$L_+^{(V)} = -\nabla^2 + \nu - (1 - V_{nl}(\vec{x}))G(u^2)u + V_l(\vec{x})$$

must have only one negative eigenvalue

- No potential $\left(L_+^{(V)} = L_+\right)$: spectral condition satisfied

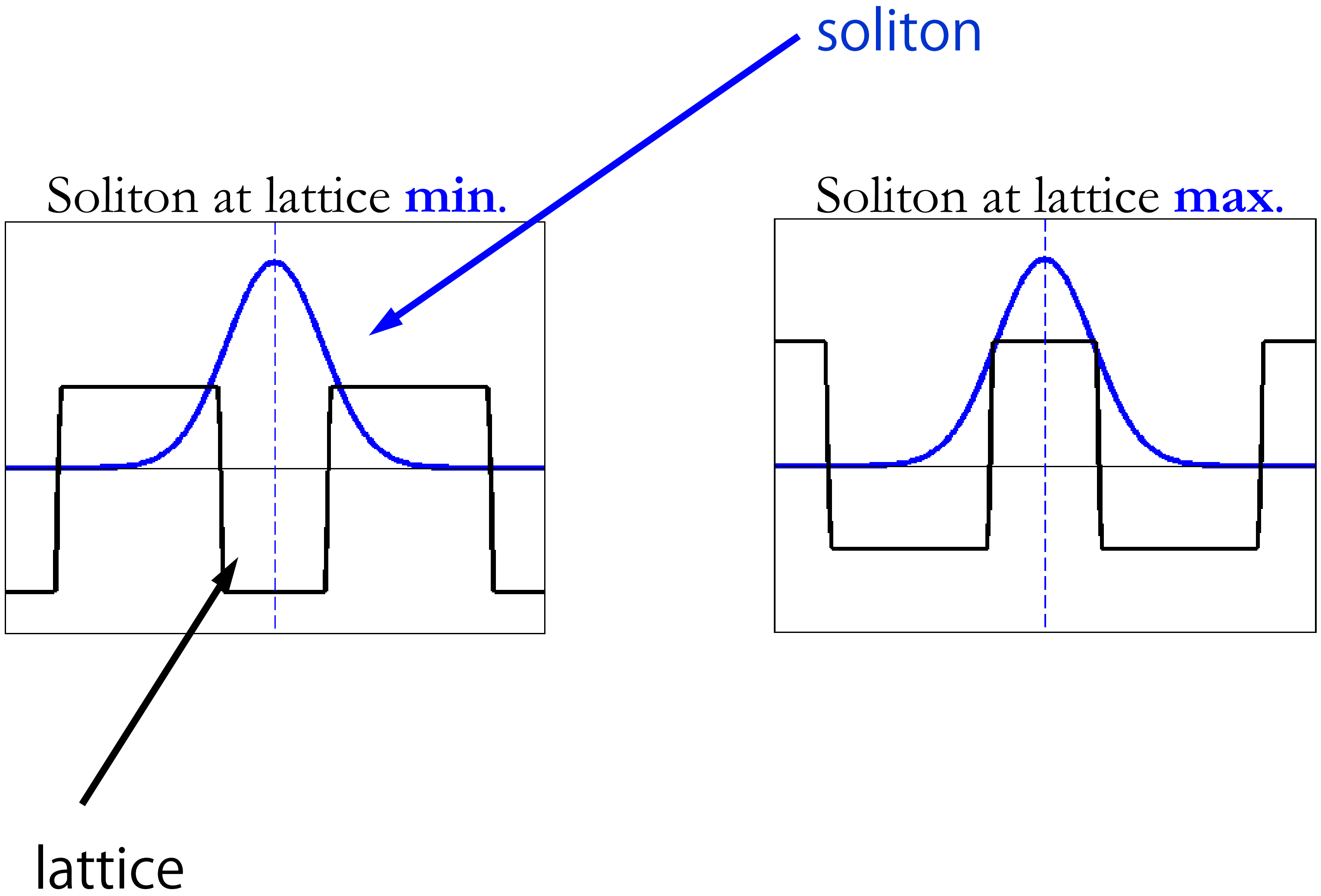


Spectral condition – cont.



- With potential:
  - $\lambda_{\min}^{(V)}$ remains negative
  - continuous spectrum remains positive
  - only $\lambda_0^{(V)}$ can become negative
- Spectral condition determined by $\lambda_0^{(V)}$
- Spectral condition not automatically satisfied

Generic families of solitons

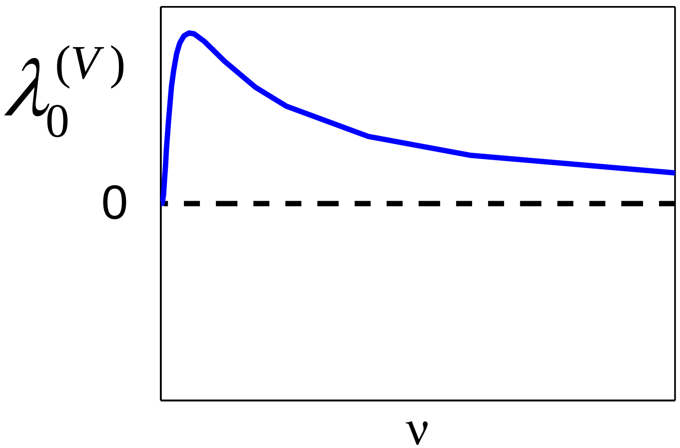


Sign of $\lambda_0^{(V)}$

- Numerical/asymptotic/analytic observation

$\lambda_0^{(V)}$ **>** 0 for solitons at a lattice **min.** (potential well)

spectral condition **satisfied**



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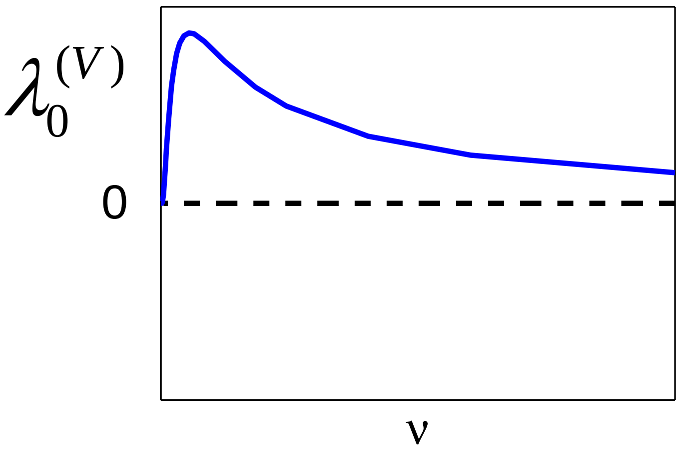
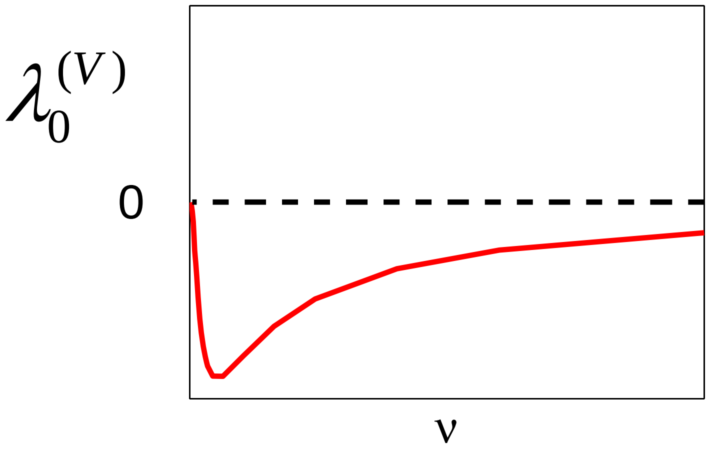


$\lambda_0^{(V)}$ **<** 0 for solitons at a lattice **max.** (potential barrier)

spectral condition **violated**



Physical intuition

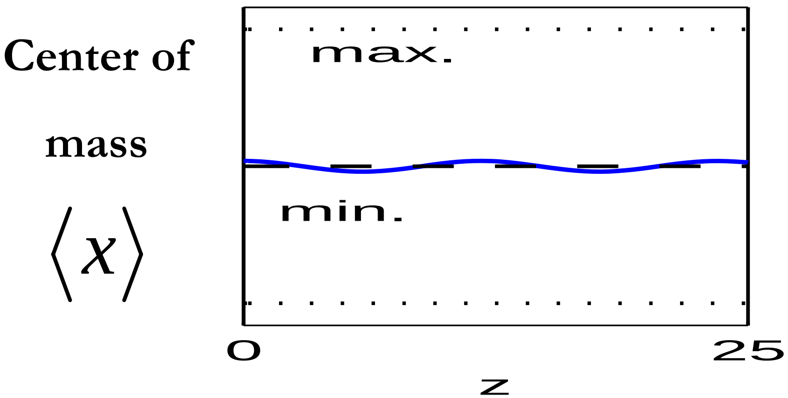
- Stability only at potential min. – solitons are more "comfortable" at a potential min. (well) than at a potential max. (barrier)
  - stay near potential min.
  - tend to move from potential max. to potential min.
- Different type of instability
  - Lateral location rather width

Numerical demonstration

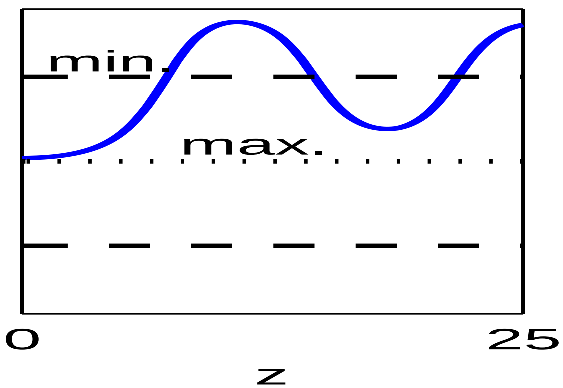
Input beam: $A(z=0,x) = u(x - \underbrace{\delta}_{shift})$

"Center of mass": $\langle x \rangle = \int x|A|^2$

- Soliton centered at a lattice min.
  - stays at lattice min.
  - lateral stability

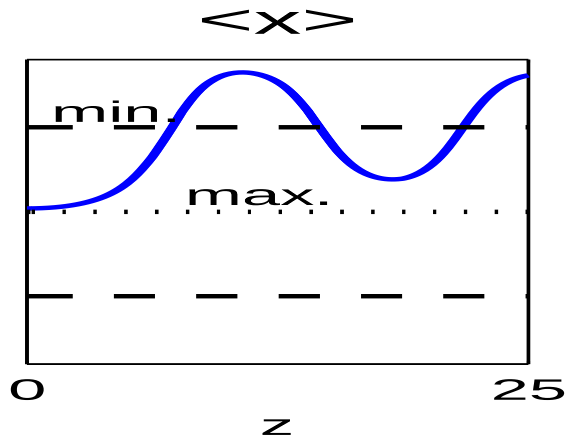


- Soliton centered at a lattice max.
  - moves from lattice max. to lattice min.
  - **drift instability**



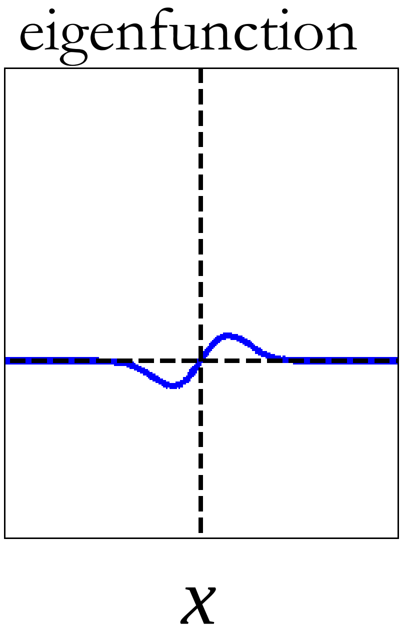
Conclusion:

violation of spectral condition $\longrightarrow$ drift instability



Mathematical intuition

- Eigenfunction associated with $\lambda_0^{(V)}$ is odd



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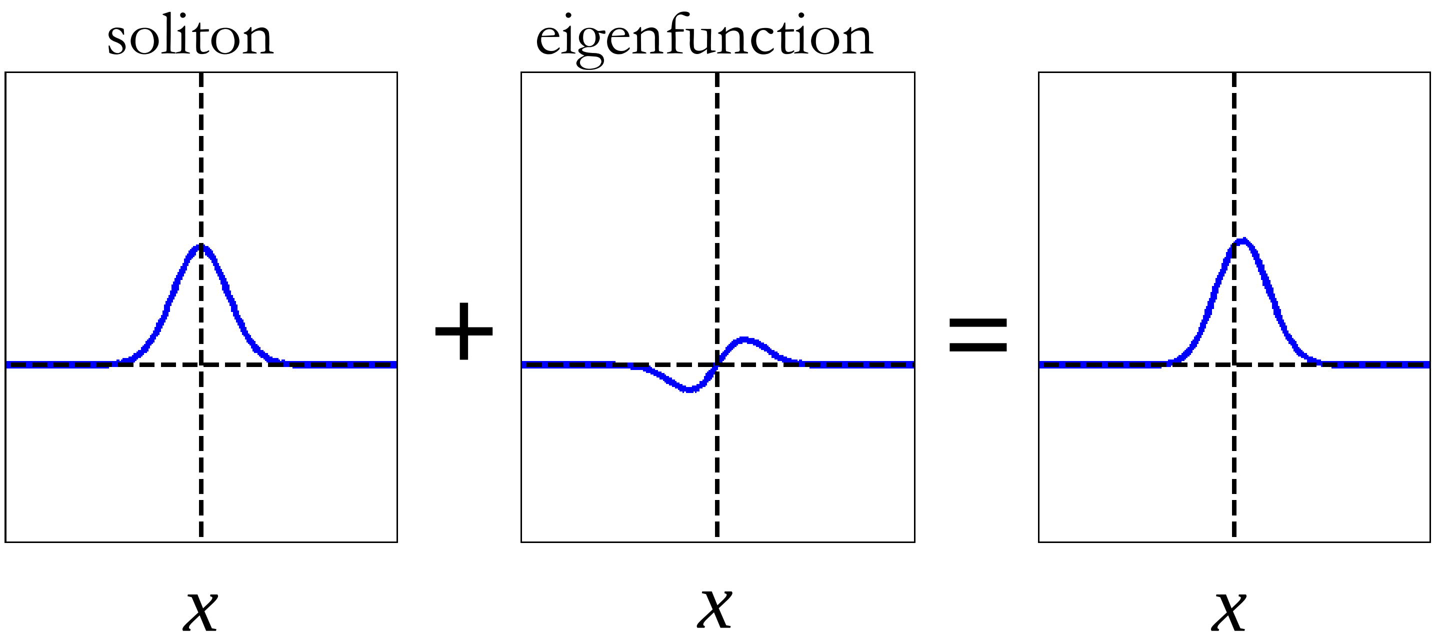
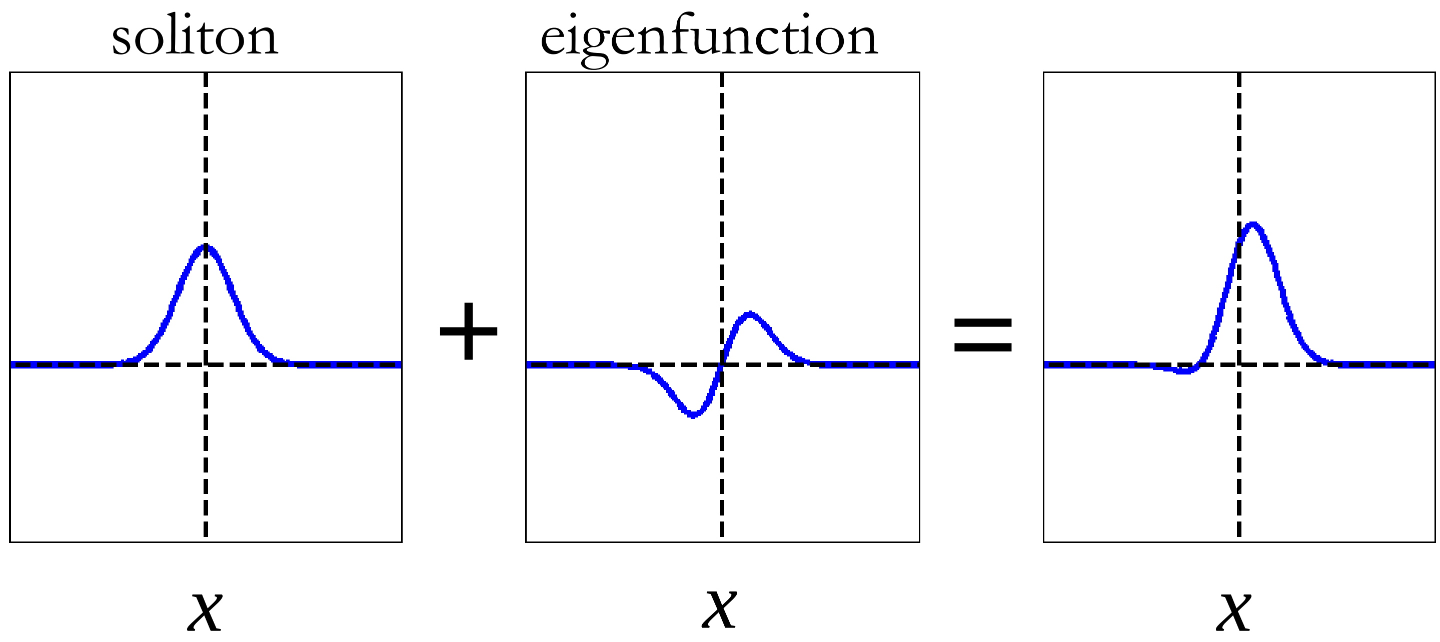


Mathematical intuition

- Eigenfunction associated with $\lambda_0^{(V)}$ is odd
- Its growth causes lateral shift of beam center



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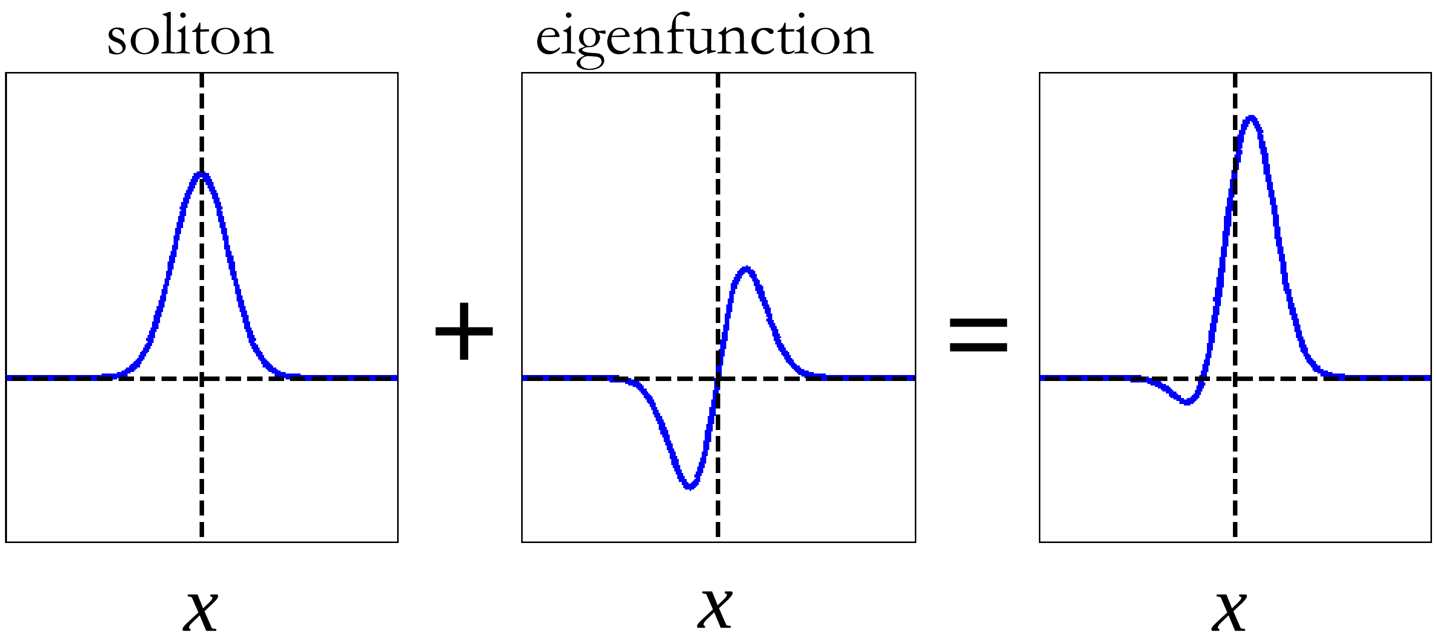
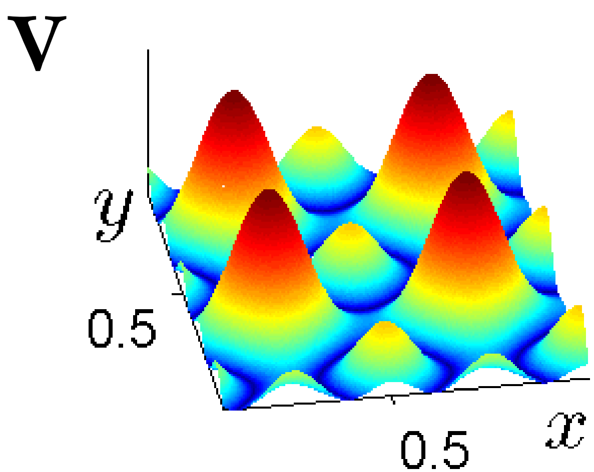


Qualitative approach – summary

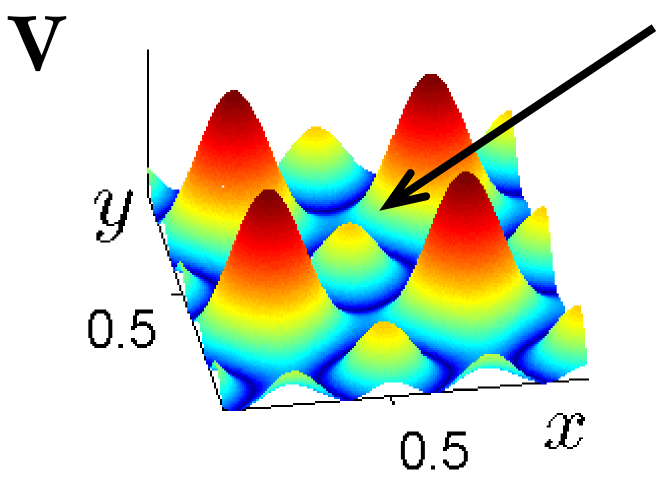
- Characterization of instabilities
  - Violation of **slope** condition $\Longrightarrow$ focusing instability
  - Violation of **spectral** condition $\Longrightarrow$ drift instability
- Distinction between instabilities is useful for more complex lattices

Example



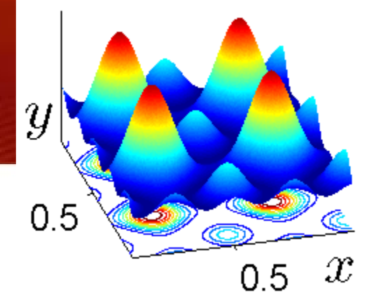
- Created experimentally by optical induction

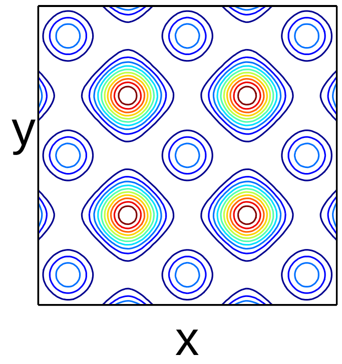
Example



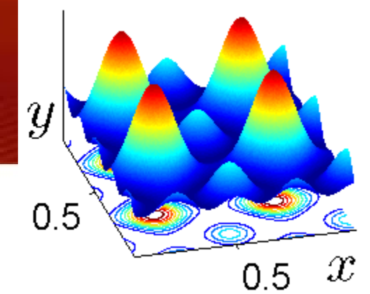
- Created experimentally by optical induction
- Consider solitons centered at a shallow local maximum

Example

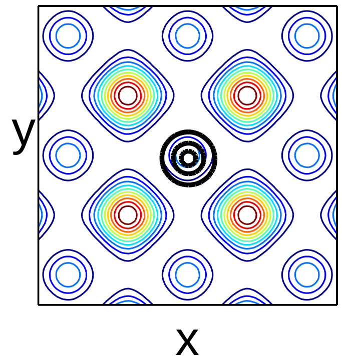




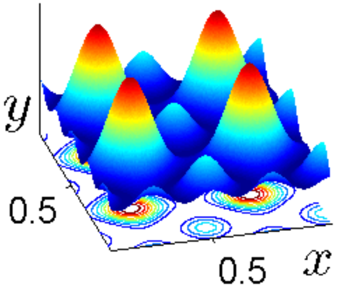
Example



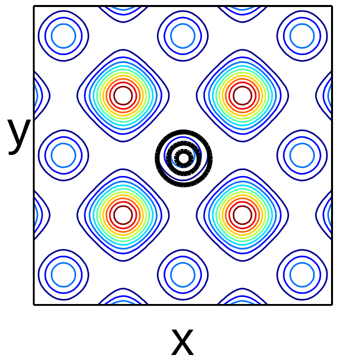
soliton centered at a lattice max.



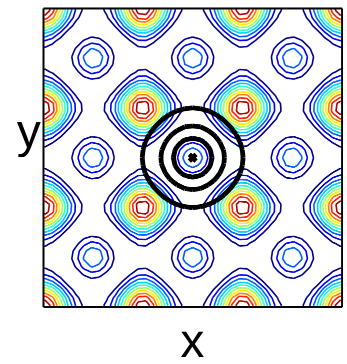
Example



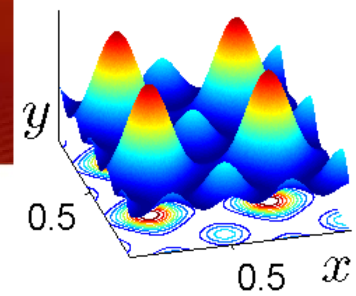
soliton centered at a lattice max.



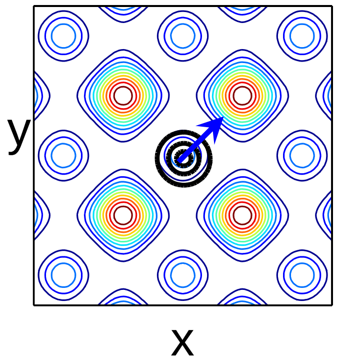
soliton centered at a lattice max.



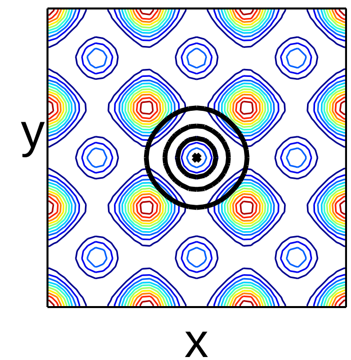
Example



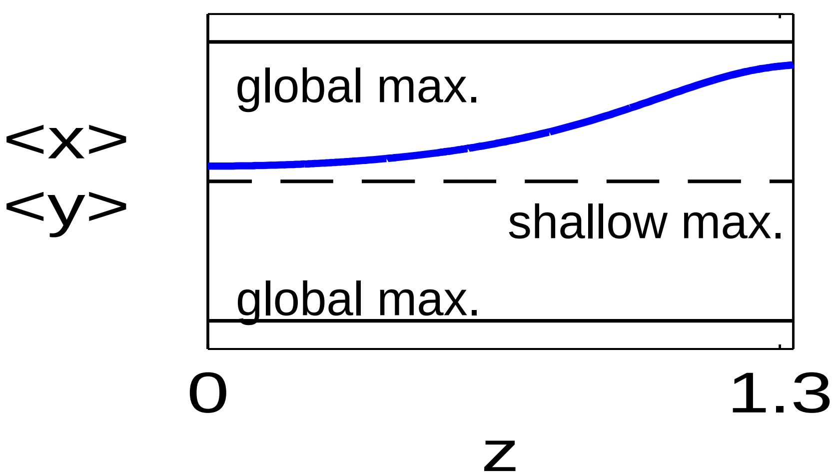
soliton
centered at a
lattice max.



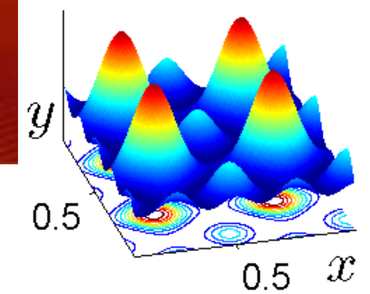
soliton
centered at a
lattice max.



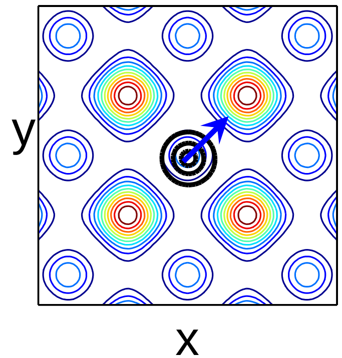
Drift instability



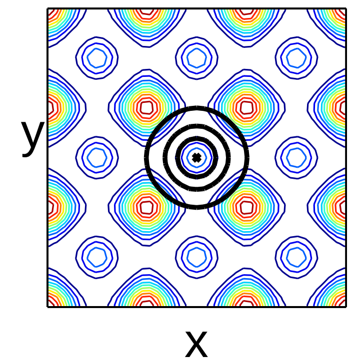
Example



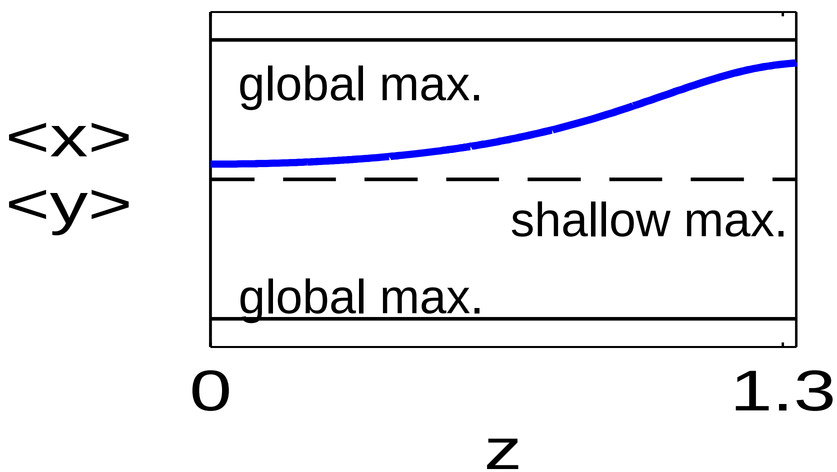
soliton centered at a lattice max.



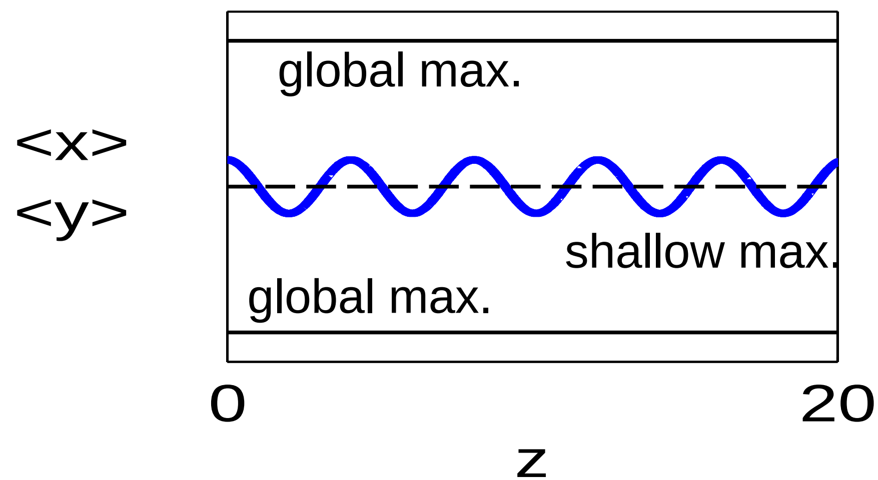
soliton centered at a lattice max.



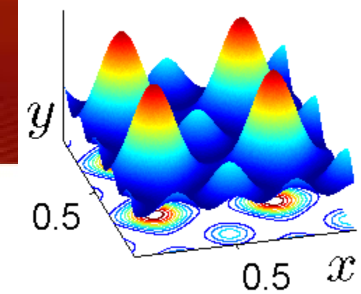
Drift instability



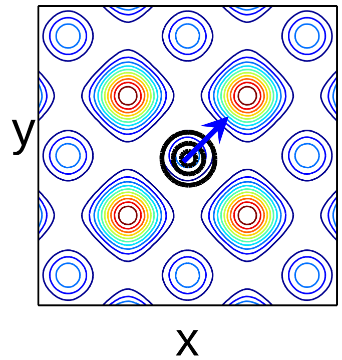
Stability



Example



soliton centered at a lattice max.



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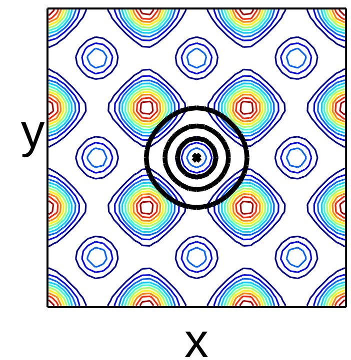
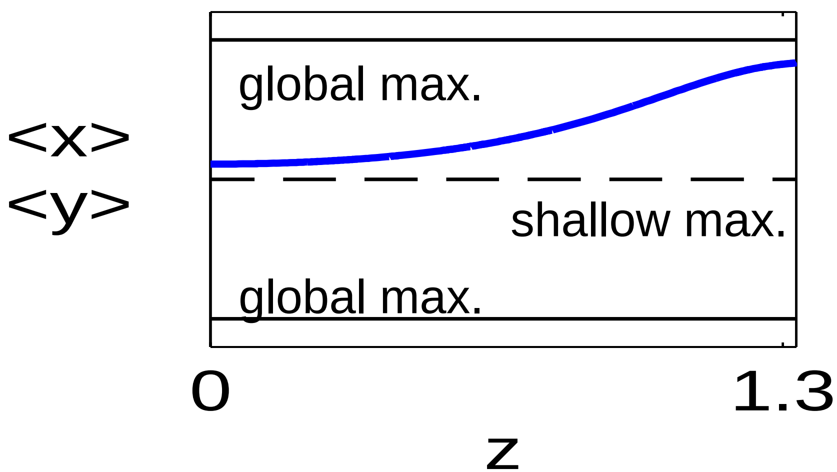
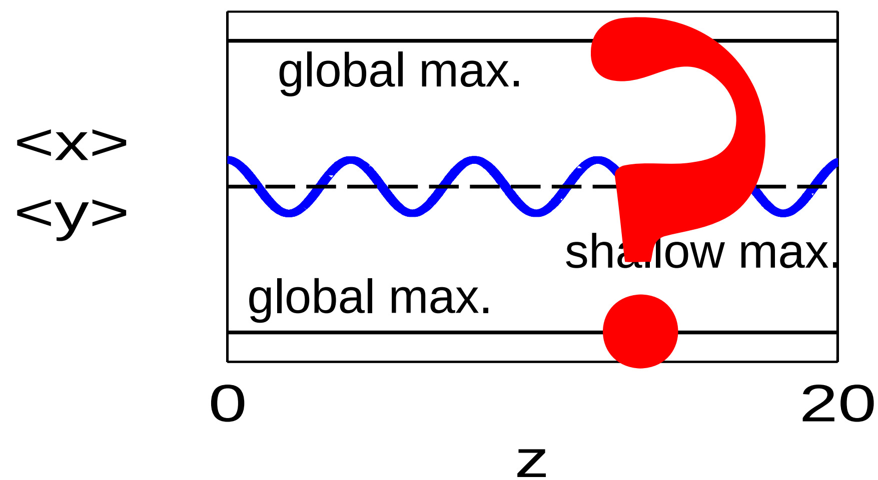


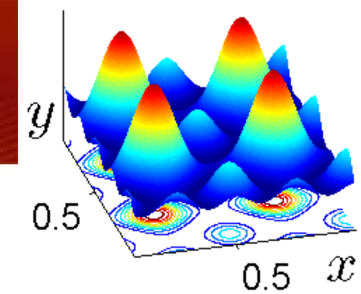
Drift instability



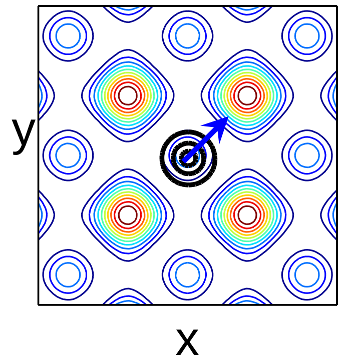
Stability



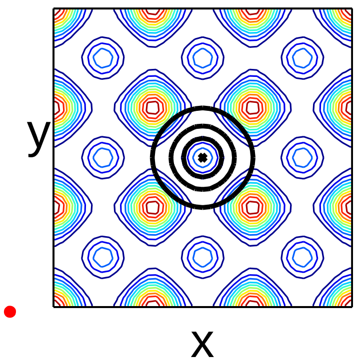
Example



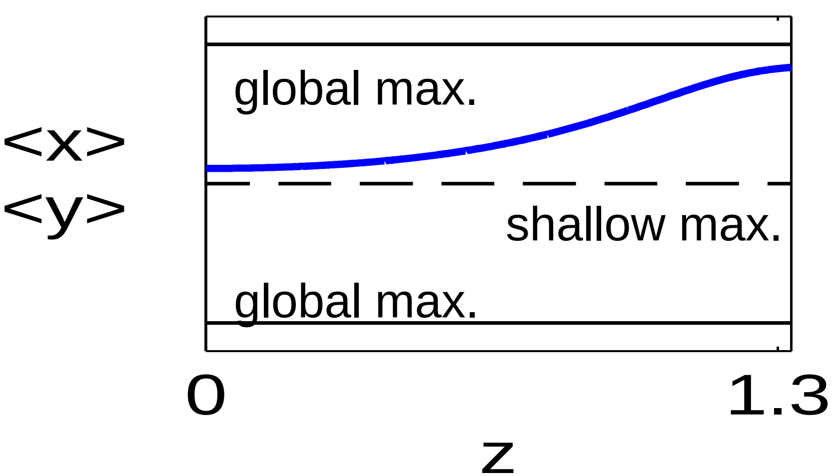
- **Narrow** solitons: centered at a lattice max.



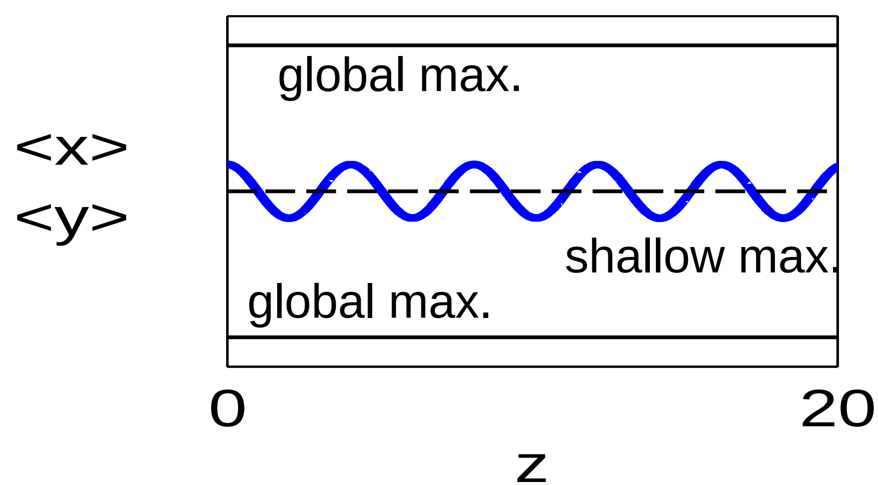
- **Wide** solitons: **effectively** centered at a lattice ~~max.~~ **min.**



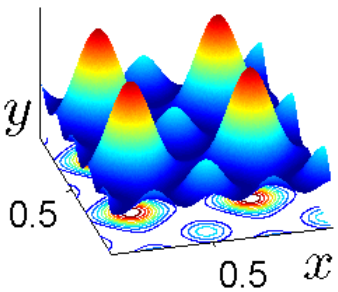
Drift instability



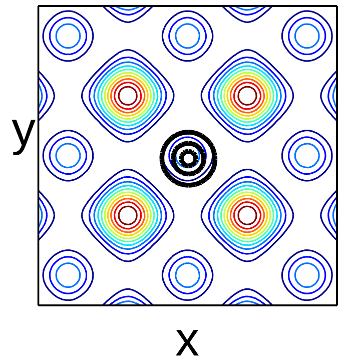
Stability



Example



- **Narrow** solitons: centered at a lattice max.



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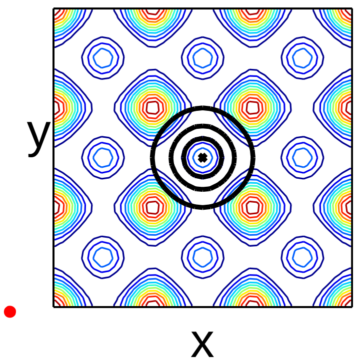
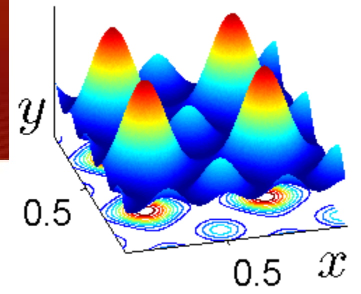
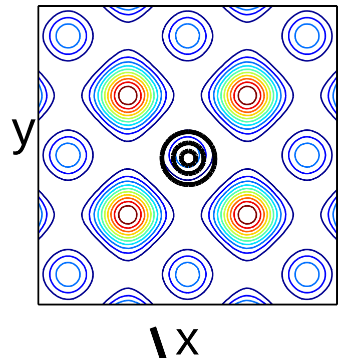


At which width is the transition between drift instability and stability?

Example



- **Narrow** solitons: centered at a lattice max.



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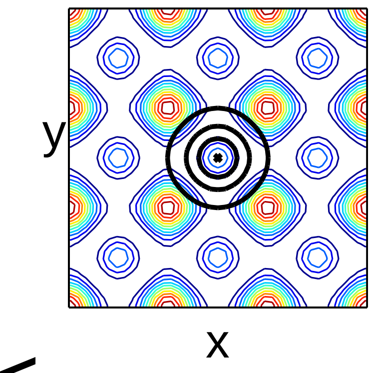
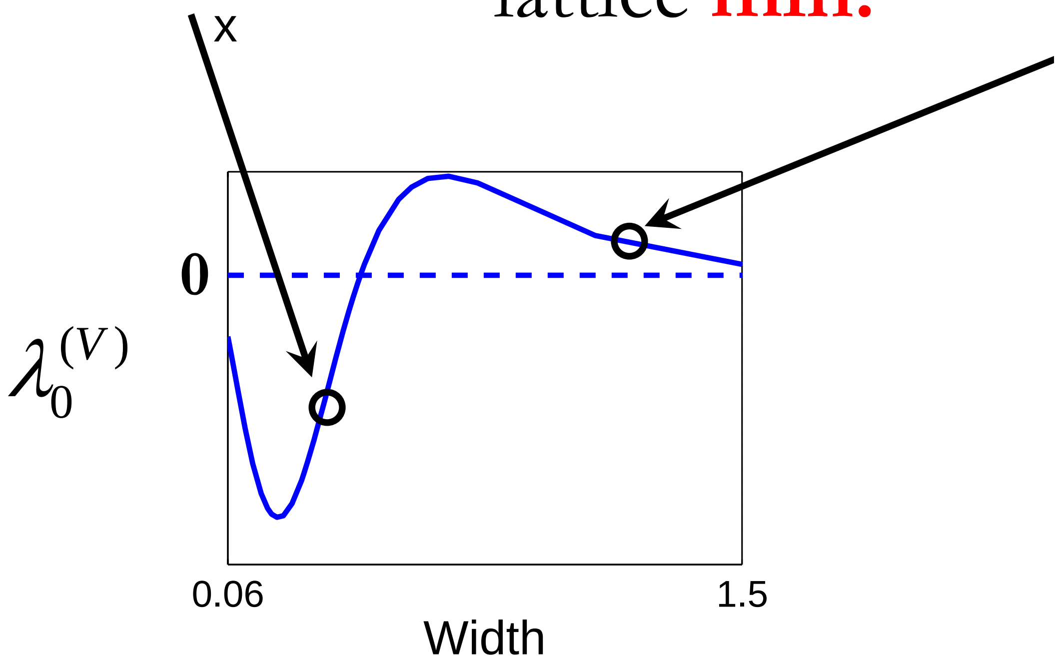
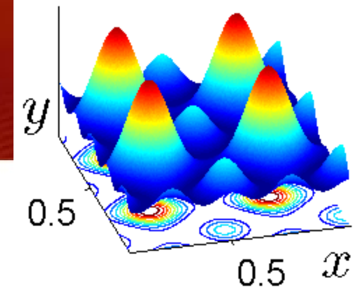
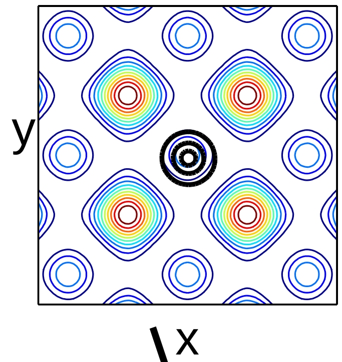




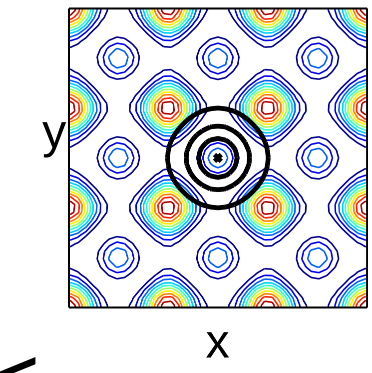
Example

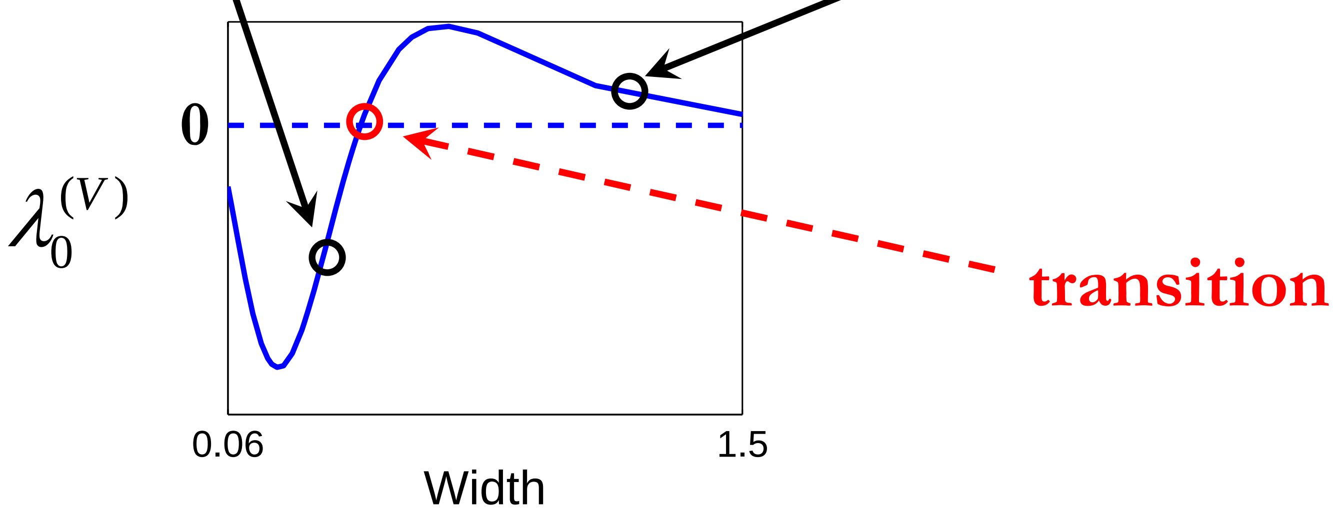


- **Narrow** solitons: centered at a lattice max.



- **Wide** solitons: **effectively** centered at a lattice **min.**





Conclusion

- Dynamics “deciphered” using the qualitative approach
  - “effective centering” determines violation/satisfaction of spectral condition
  - In turn, determines dynamics of “center of mass”
  - Dynamics is determined by slope condition

Outline of the talk

- NLS and solitons - review
- Stability theory
- Qualitative approach
  - Slope condition – width instability
  - Spectral condition – drift instability
- Quantitative approach

Motivation: 2d nonlinear lattices

$$iA_z(z,x,y)+\nabla^2 A+\left(1-V_{nl}(x,y)\right)|A|^2 A=0$$

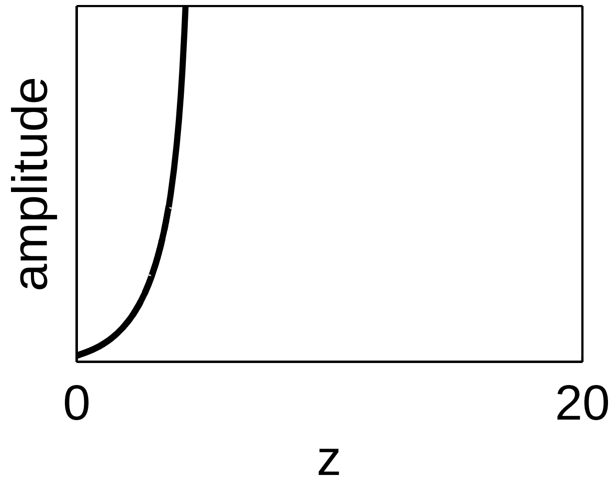
- Can the nonlinear potential stabilize the solitons?
  - spectral condition satisfied only at potential min.
  - slope condition satisfied only for
    - narrow solitons
    - specially designed potential

Narrow solitons in 2d nonlinear lattices

- u(x,y) is a stable narrow soliton
- Test stability of u(x,y) numerically:
  - add extremely small perturbation

$$A(z=0,x,y)=1.0001u(x,y)$$

collapse instability!



Narrow solitons in 2d nonlinear lattices

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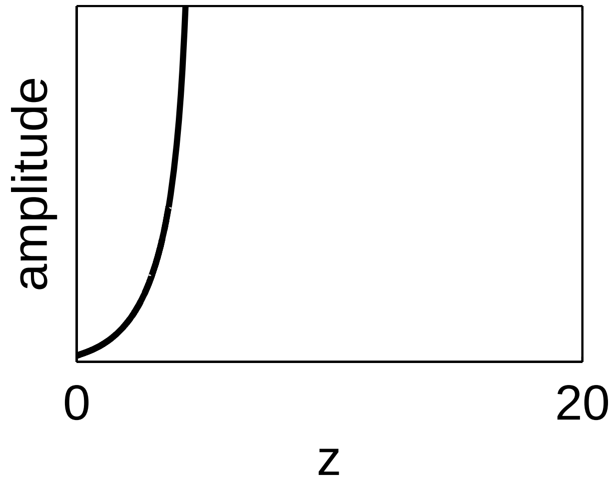


collapse instability!

What is going on?

Strength of stability

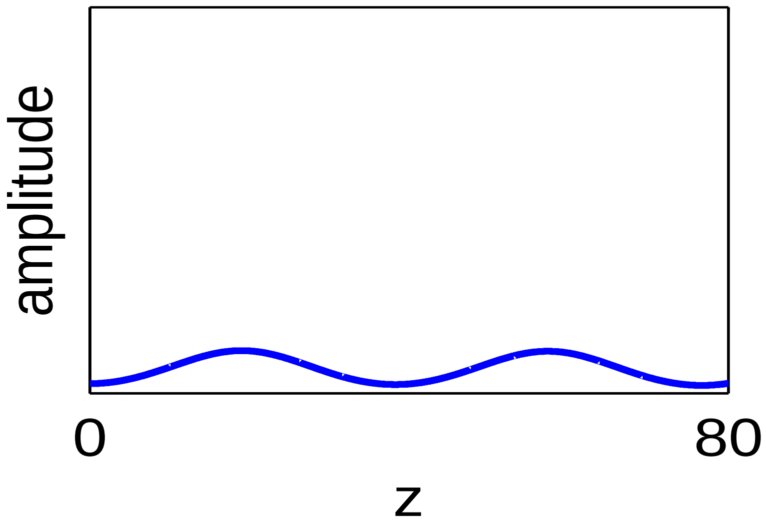
- Slope = 0 $\longrightarrow$ instability
- Slope > 0 $\longrightarrow$ stability
  - What happens for a very small positive slope?
- Strength of stability determined by magnitude of slope
  - small slope leads to weak stabilization
  - stronger stability for larger slope

Strength of stability: nonlinear lattices

- Fixed input beam

  $A(z=0) = 1.0001u(x, y)$

- Change lattice
  - **slope = 0.01 (stable)**

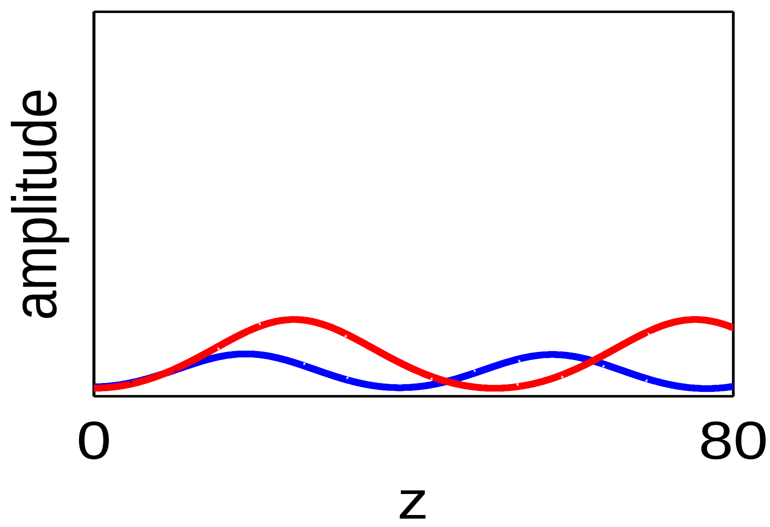


Strength of stability: nonlinear lattices

- Fixed input beam

  $A(z=0) = 1.0001u(x, y)$

- Change lattice
  - **slope = 0.01 (stable)**
  - **slope = 0.006 (less stable)**

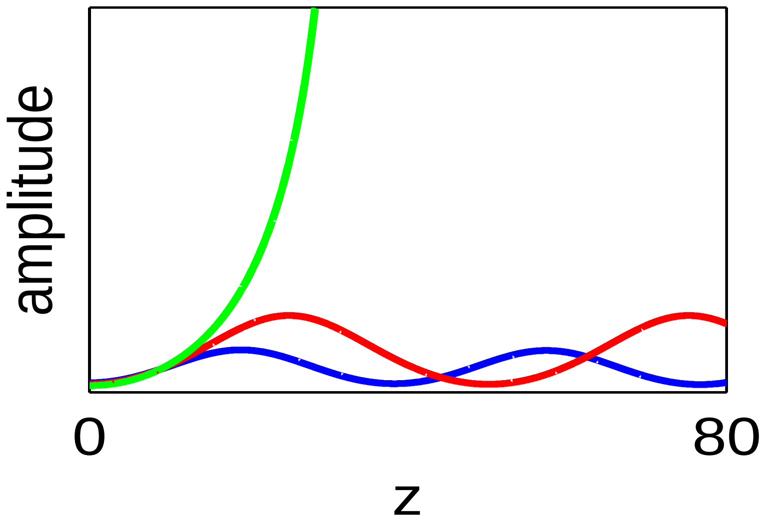


Strength of stability: nonlinear lattices

- Fixed input beam

  $A(z=0) = 1.0001 u(x, y)$

- Change lattice
  - **slope = 0.01 (stable)**
  - **slope = 0.006 (less stable)**
  - **slope = 0.002 (unstable)**

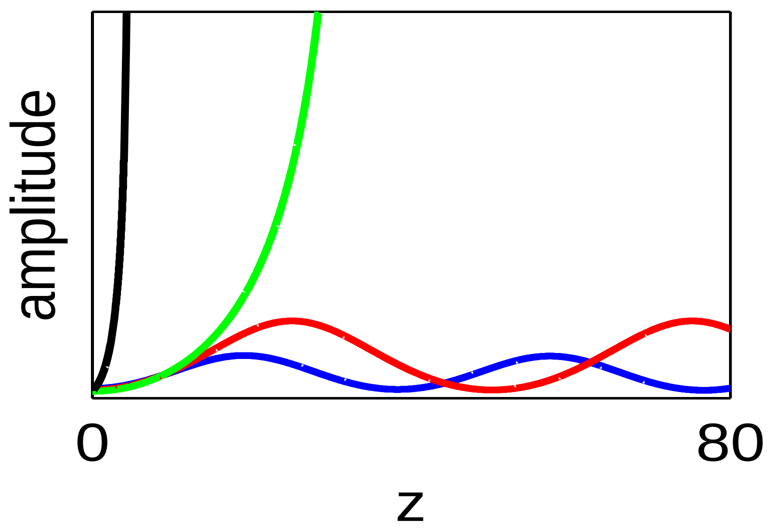


Strength of stability: nonlinear lattices

- Fixed input beam

  $A(z=0) = 1.0001u(x,y)$

- Change lattice
  - **slope = 0.01 (stable)**
  - **slope = 0.006 (less stable)**
  - **slope = 0.002 (unstable)**
  - **slope = 0.0002 (even more unstable)**

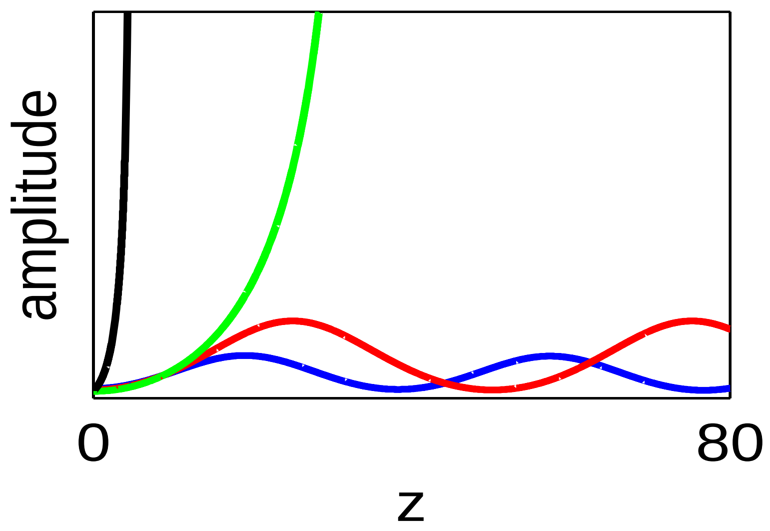


Strength of stability: nonlinear lattices

- Fixed input beam

  $$A(z=0) = 1.0001u(x, y)$$

- Change lattice
  - **slope = 0.01 (stable)**
  - **slope = 0.006 (less stable)**
  - **slope = 0.002 (unstable)**
  - **slope = 0.0002 (even more unstable)**

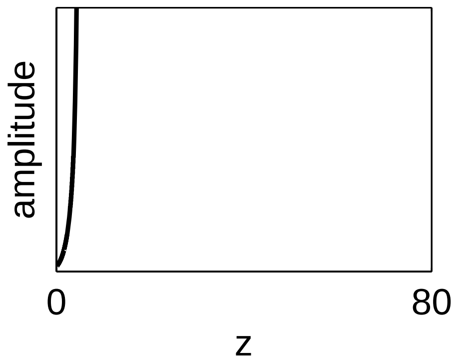


- Strength of stability is determined by magnitude of slope

Narrow solitons in 2d nonlinear lattices

- Original soliton was weakly stable
  - Slope = 0.01

    $$A(z=0) = 1.0001u(x,y)$$



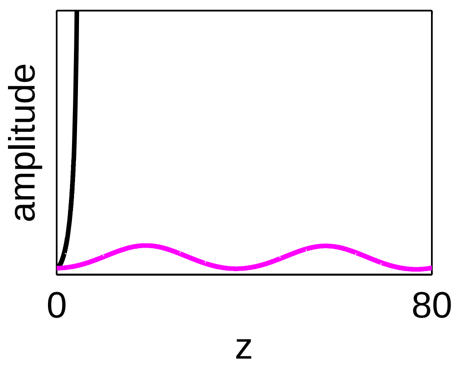
Narrow solitons in 2d nonlinear lattices

- Original soliton was weakly stable
  - Slope = 0.01

$$A(z=0)=1.0001u(x,y)$$

$$A(z=0)=1.00004u(x,y)$$

  - Stability for a smaller perturbation

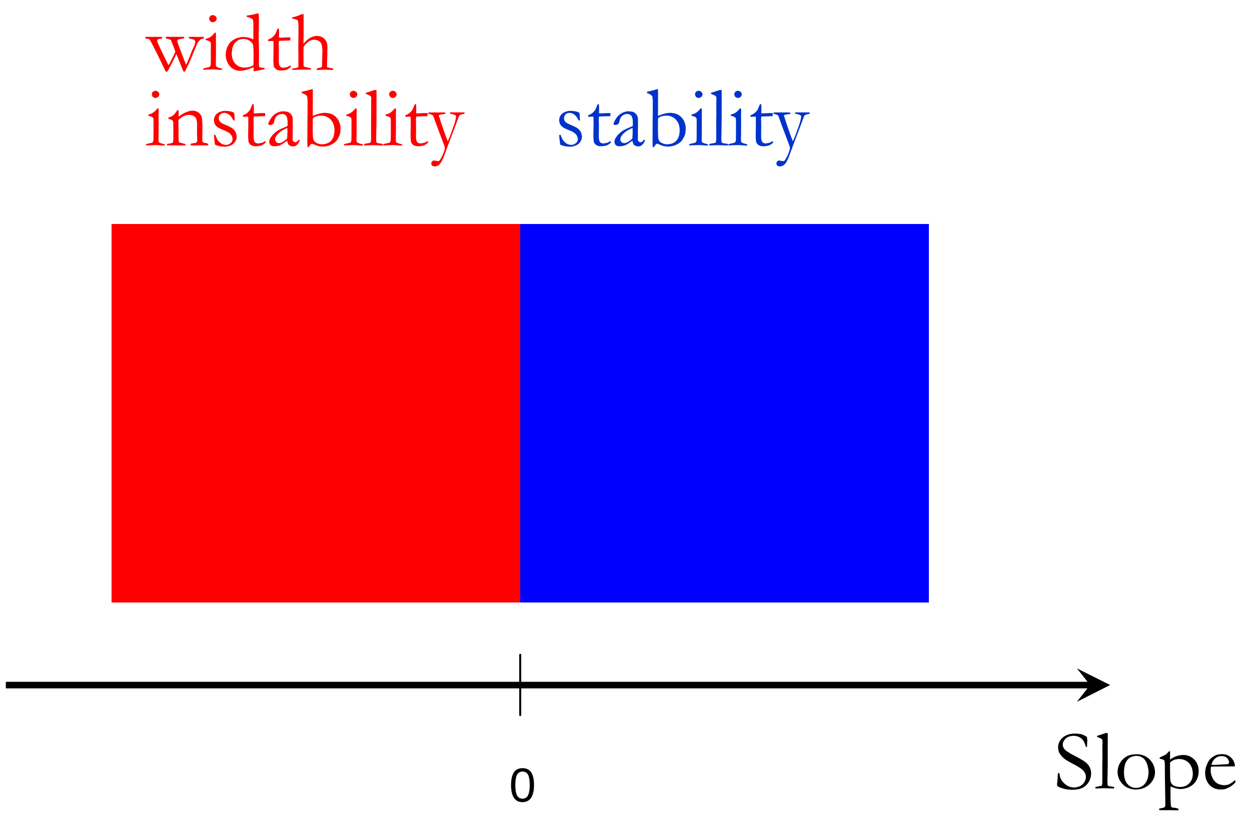


- Soliton is "theoretically" stable, but "practically" unstable

- "Old" approach – check only **sign** of slope
- "New" quantitative approach – check also the **magnitude** of the slope.

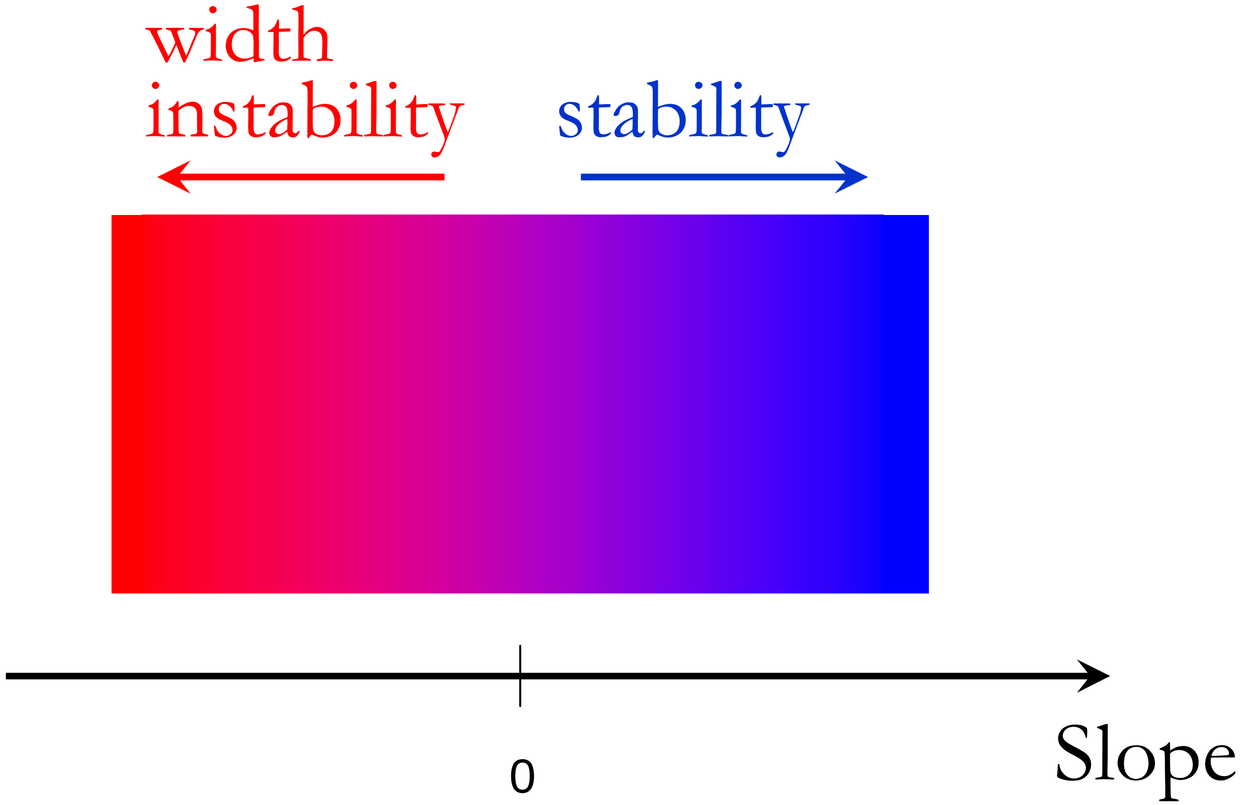
Slope condition: “old” approach

Discontinuous transition between stability and instability



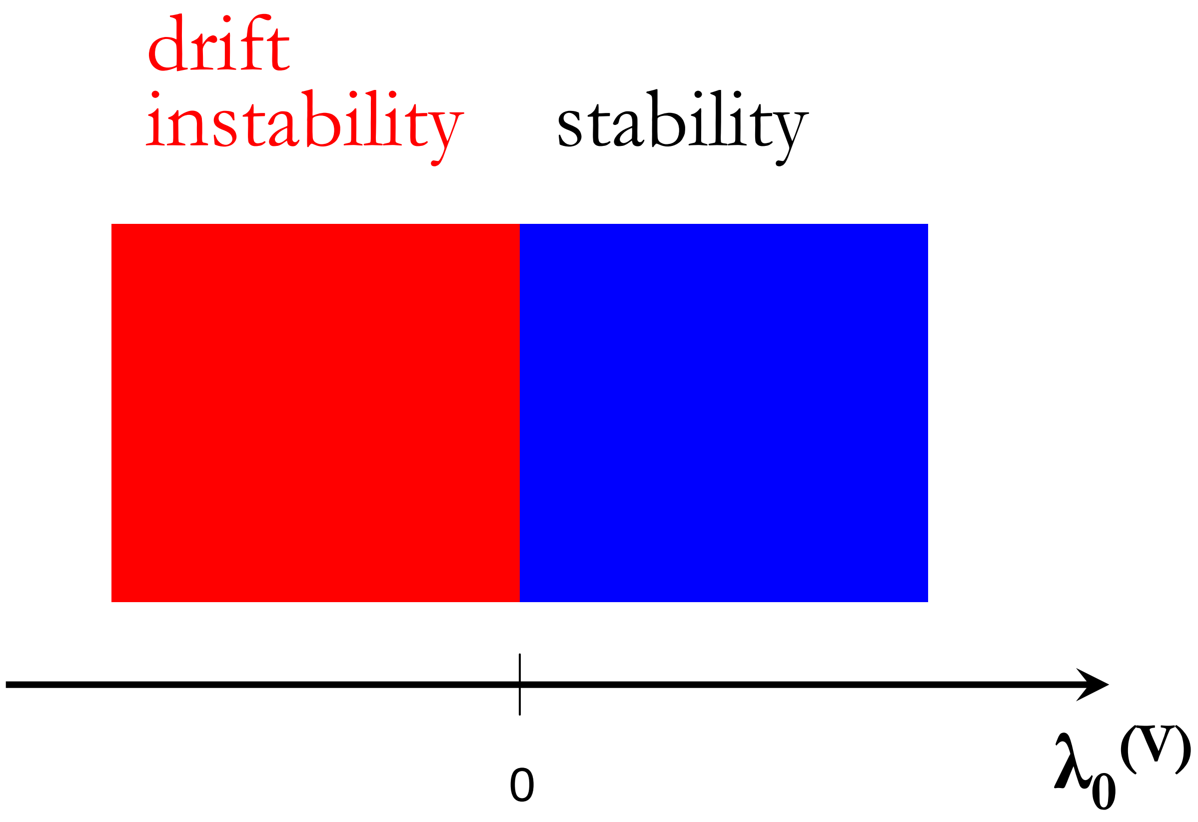
Slope condition: quantitative approach

Continuous transition between stability and instability



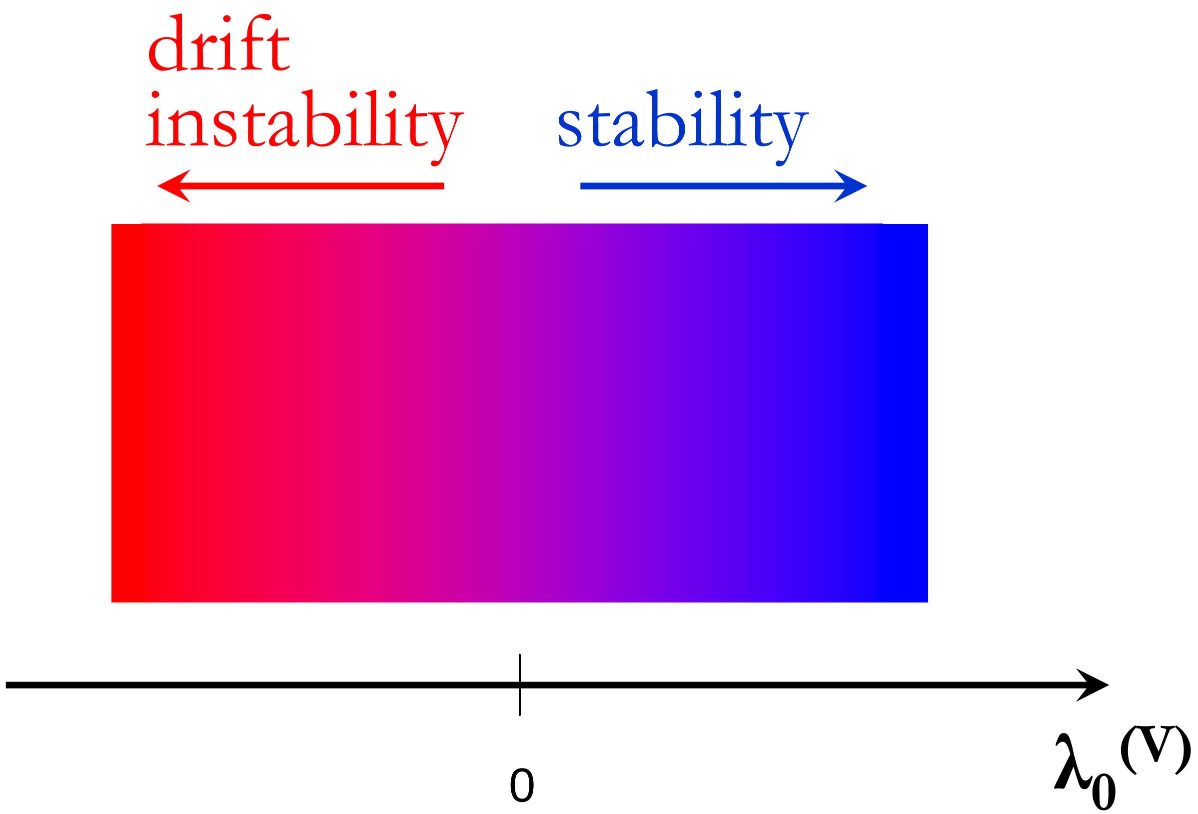
Spectral condition: "old" approach

Discontinuous transition between stability and instability



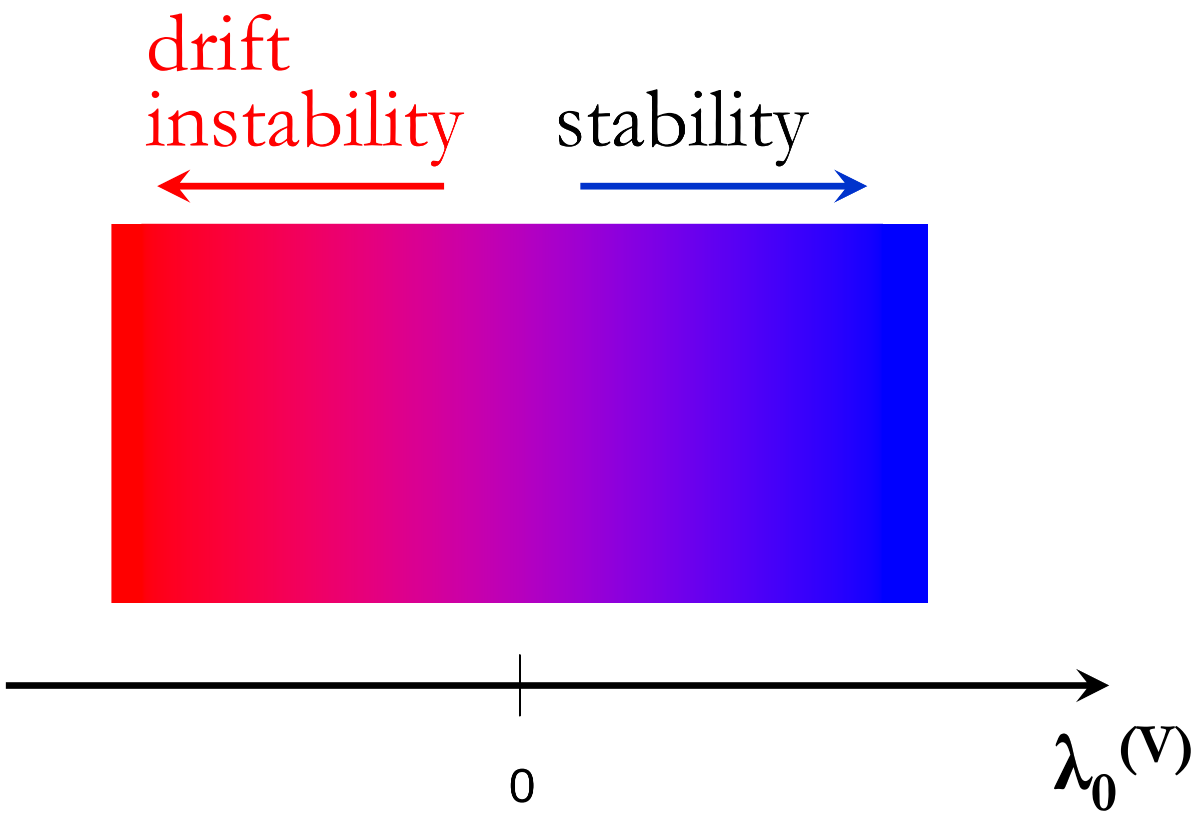
Spectral condition: quantitative approach

Continuous transition between stability and instability



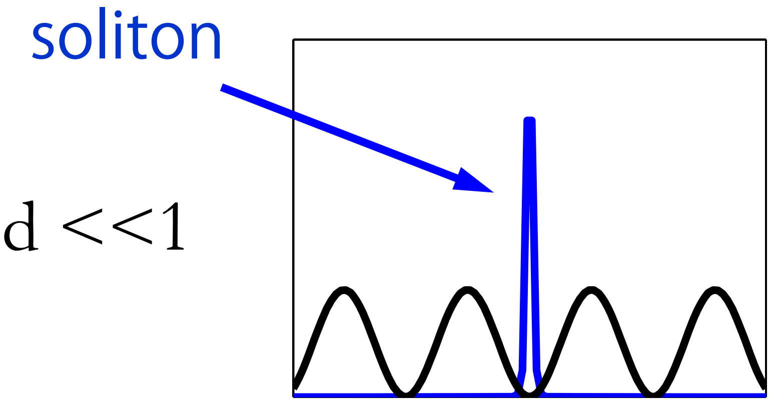
Spectral condition: quantitative approach

Continuous transition between stability and instability



Can we get an analytical prediction for the drift rate as a function of $\lambda_0^{(V)}$?

Narrow solitons



- $\varepsilon =$ soliton width/potential period <<1
- Can use perturbation analysis!
- Expand potential as $V(x) = V(0) + \frac{1}{2}V''(0)(\varepsilon x)^2 + \cdots$
- Solve for $\langle x \rangle = \int x|A|^2$

Narrow solitons – cont. $V(x)=V(0)+\frac{1}{2}V''(0)\varepsilon^2x^2+\cdots$

- Compute effective force

$$\underbrace{\frac{d^2}{dz^2}\langle x\rangle = -2d\int |A|^2\nabla V}_{\text{Ehrenfest law}} \simeq \underbrace{-2dV''(0)\varepsilon^2}_{\Omega^2}\underbrace{\int |A|^2\, x}_{\langle x\rangle},$$

$$\boxed{\frac{d^2}{dz^2}\langle x\rangle = \Omega^2\ \langle x\rangle, \quad \Omega^2 = -2dV''(0)\varepsilon^2}$$

- Center of mass obeys an oscillator equation!

Narrow solitons - cont.

- Use perturbation analysis to compute eigenvalue

$$\lambda_0^{(V)} \approx \frac{V''(0)\varepsilon^2}{\left(\nu + V(0)\right)V(0)}$$

- Combine with $\Omega^2 = -2dV''(0)\varepsilon^2$ and get:

$$\Omega^2 = -C^2 \lambda_0^{(V)}, \quad C^2 = \frac{4d}{\left(\nu + V(0)\right)V(0)}$$

Narrow solitons – cont.

- Conclusion:

$$\frac{d^2}{dz^2}\langle x\rangle = \Omega^2 \langle x\rangle, \qquad \Omega^2 = -C^2\lambda_0^{(V)}$$

$$\begin{cases} \lambda_0^{(V)} > 0 \Rightarrow \Omega^2 < 0 \Rightarrow \text{oscillatory solutions} \Rightarrow \text{stability} \\ \lambda_0^{(V)} < 0 \Rightarrow \Omega^2 > 0 \Rightarrow \text{exponential solutions} \Rightarrow \text{instability} \end{cases}$$

(Lattice min.)

(Lattice max.)

Analytical quantitative relation between spectral condition ($\lambda_0^{(V)}$) and dynamics (Ω)

- But so far only for narrow beams
- Can we compute the drift rate also for wider solitons?

Calculation of drift rate in a general setting

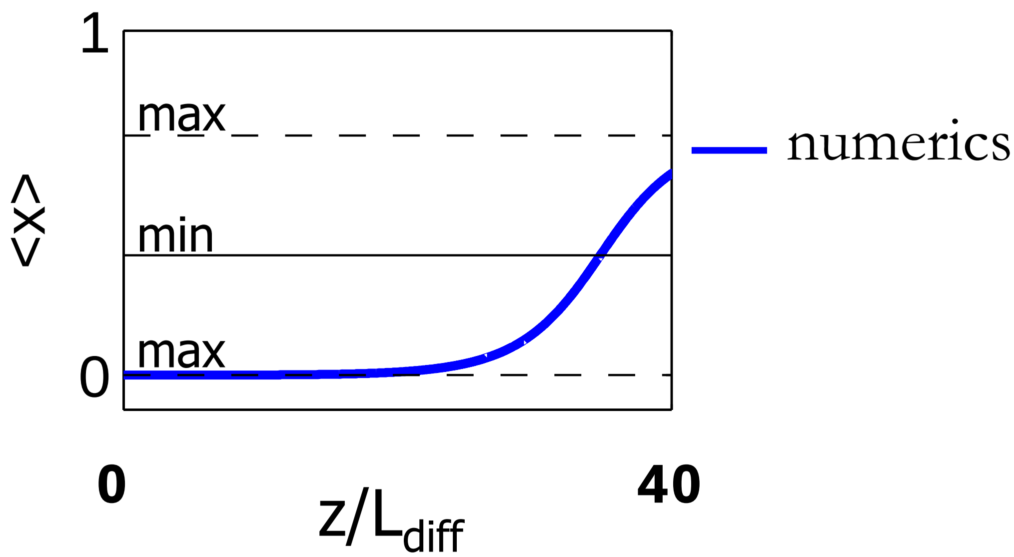
$$\frac{d^2}{dz^2}\langle x\rangle = \Omega^2 \langle x\rangle, \qquad \Omega^2 = -C^2\lambda_0^{(V)}$$

$$C^2 = \frac{\left\langle f^{(V)}, f^{(V)}\right\rangle}{\left\langle f^{(V)}, \left(L_-^{(V)}\right)^{-1} f^{(V)}\right\rangle} > 0, \quad L_+^{(V)} f^{(V)} = \lambda_0^{(V)} f^{(V)}$$

- Valid for
  - any soliton width
  - any potential (periodic/non-periodic, single/multi waveguide, …)
  - any nonlinearity (Kerr, cubic-quintic, saturable, …)
  - any dimension

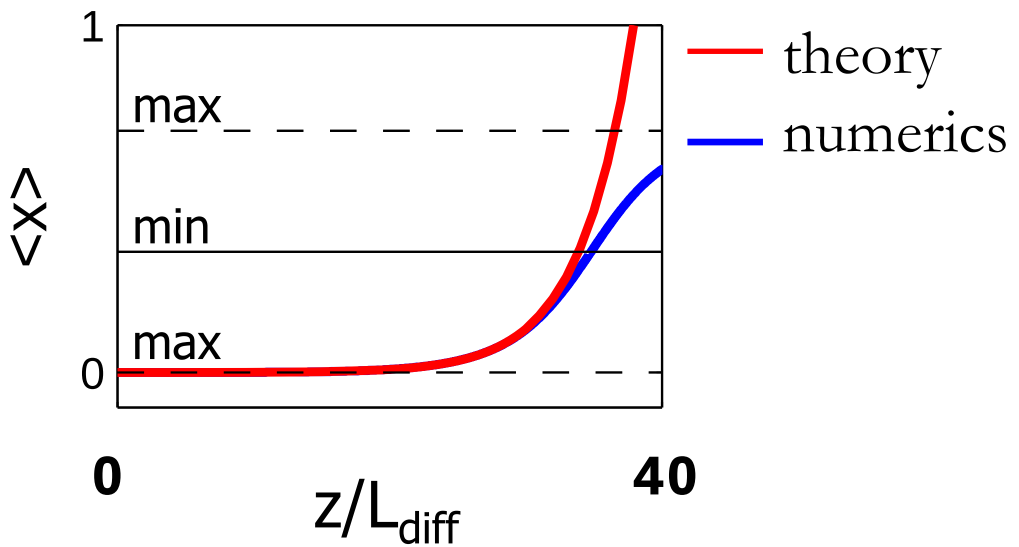
Examples of lateral dynamics (1)

- Soliton slightly shifted from lattice max. $A(0,x) = u(x - \underbrace{\delta}_{shift})$



Examples of lateral dynamics (1)

- Soliton slightly shifted from lattice max. $A(0,x) = u(x - \underbrace{\delta}_{shift})$
- Solution of oscillator equation $\langle x \rangle = \delta \cosh(\Omega z)$
  - Ω is the drift (=instability) rate



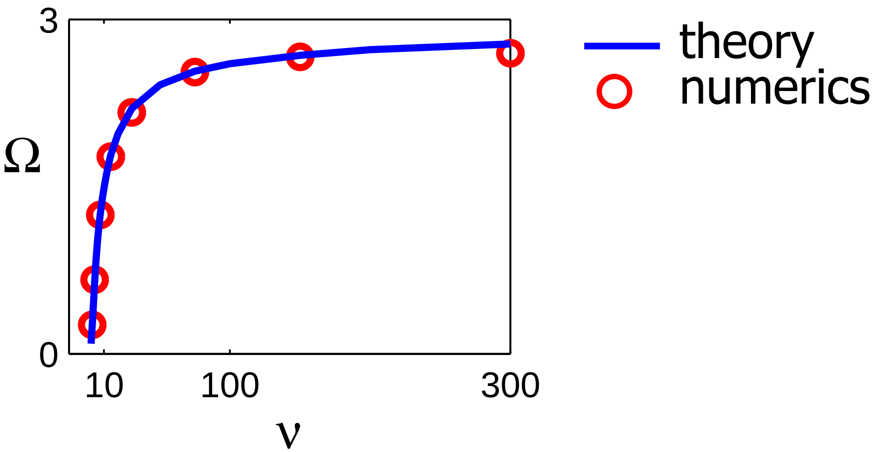
  – agreement up to the lattice min.

  – up to many diffraction lengths

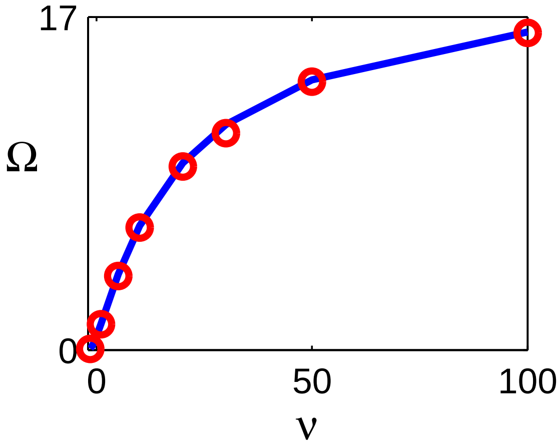
Drift rates

- Excellent agreement between analysis and numerics

d=1, weak lattice

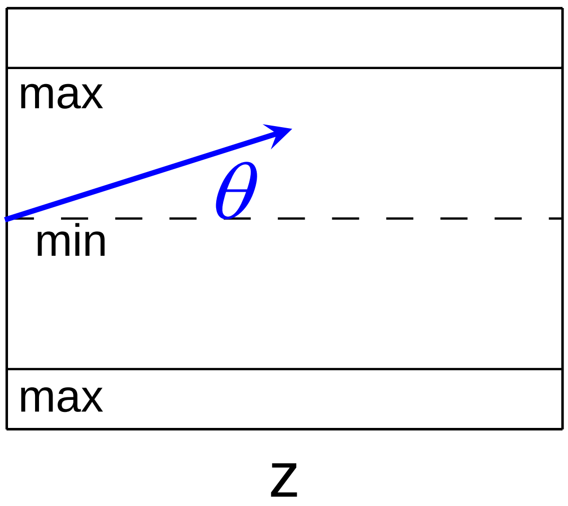


d=2, strong lattice



Examples of lateral dynamics (2)

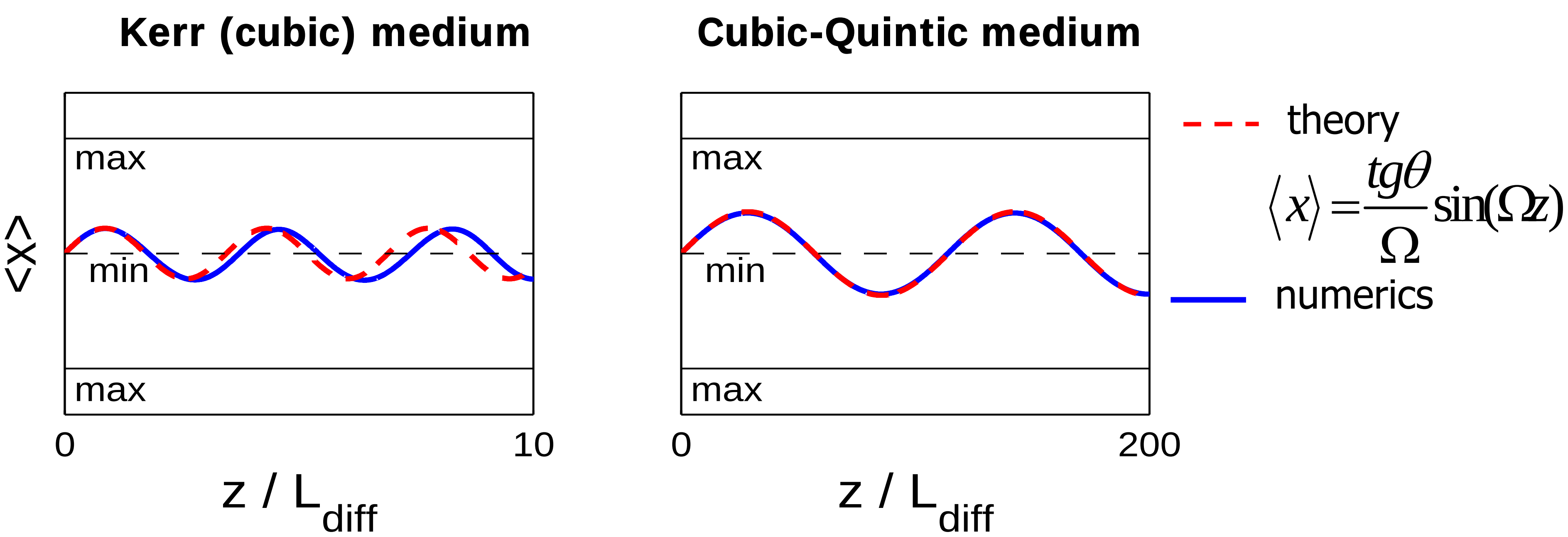
- So far, studied **instability** dynamics
- Oscillator equation good also for **stability** dynamics
  - soliton moving at an angle from a lattice min.



- Solution of oscillator equation is $$\langle x \rangle = \frac{tg\theta}{\Omega}\sin(\Omega z)$$
  - Ω determines the maximal deviation (=strength of stability)

Examples of lateral dynamics (2)

- d=2, periodic lattice



Implications of quantitative study

- Experimental example (Morandotti *et al.* 2000):
  - experiment in a slab waveguide array
  - soliton centered at a lattice max. does not drift to a lattice min. over 18 diffraction lengths
- Explanation: absence of observable drift due to small drift rate
- Theoretical instability but practical stability

Summary

- Qualitative approach
  - Slope condition $\Longrightarrow$ focusing instability
  - Spectral condition $\Longrightarrow$ drift instability
- Quantitative approach
  - continuous transition between stability and instability
  - analytical formula for lateral dynamics
  - still open: find analytical formula for width dynamics
- "Theoretical" vs. "practical" stability/instability
- Results valid for any physical configuration of lattice, dimension, nonlinearity