ALTERNATIVE DERIVATION OF EM CLOAKS AND CONCENTRATORS

Arthur D. Yaghjian

AFRL (Consultant), Hanscom AFB, MA

and

Stefano Maci

University of Siena, Italy

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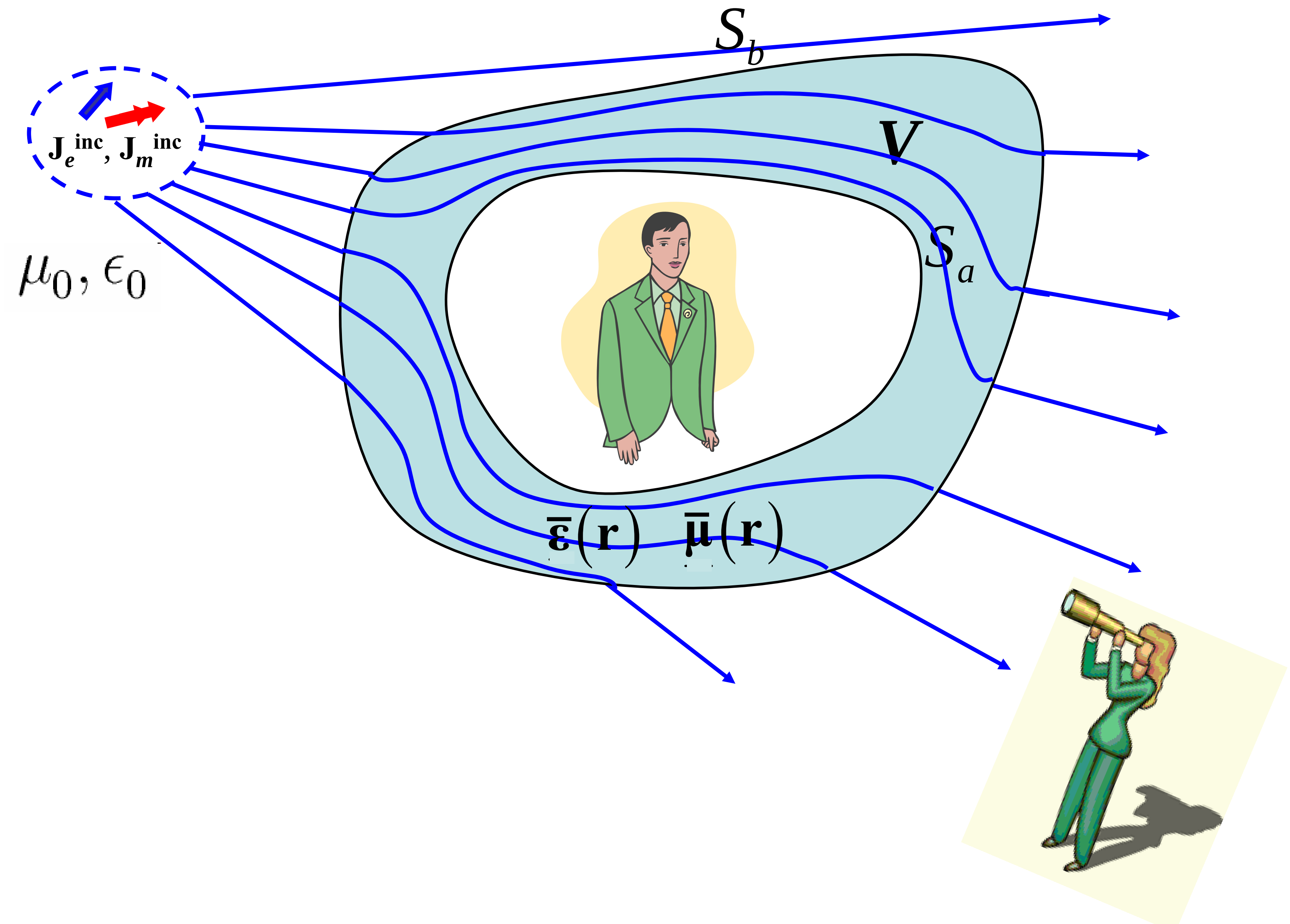
(also at arXiv)

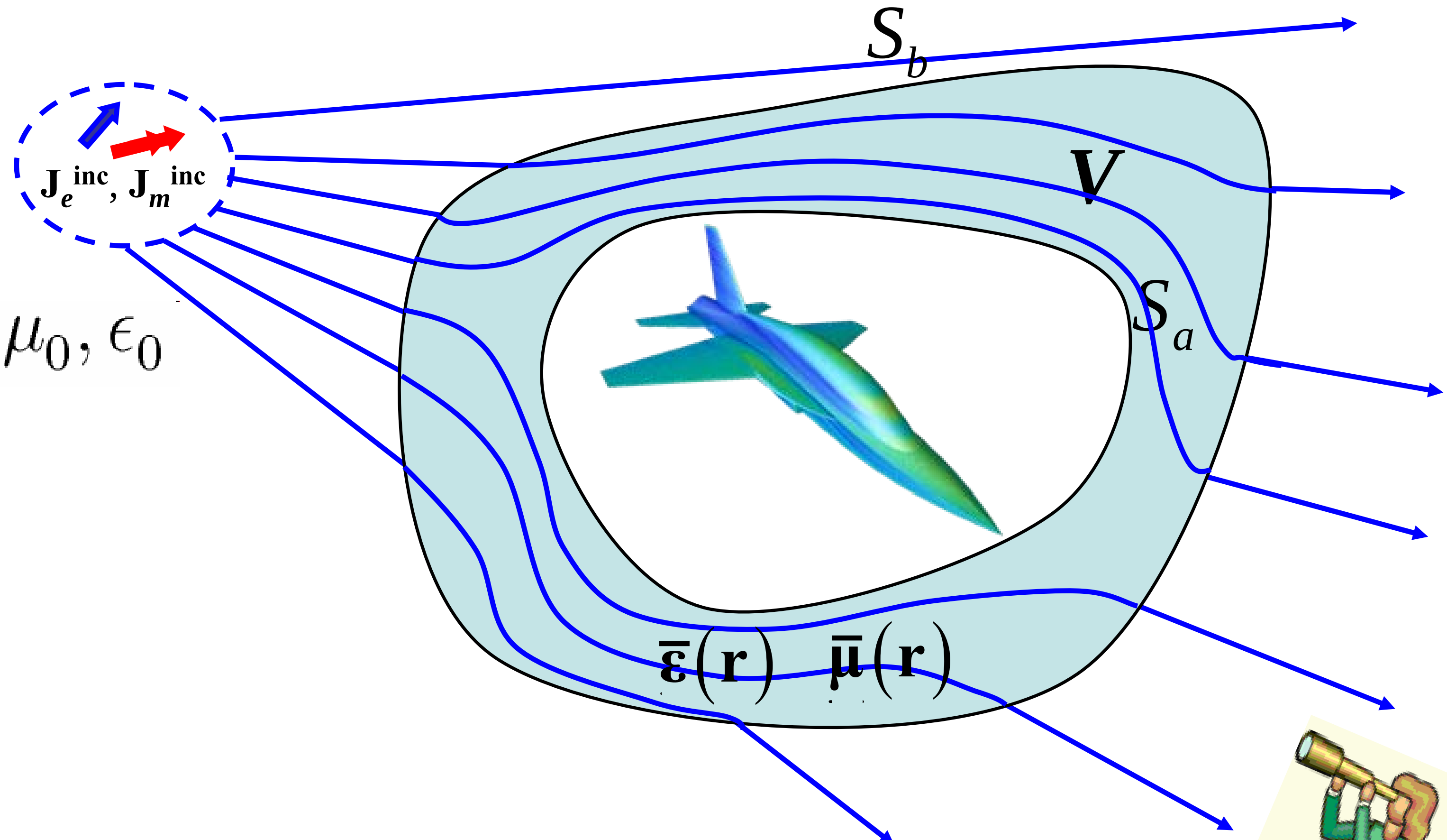
DIFFERENT CLOAKS FOR DIFFERENT FOLKS

- **Cloaking with absorber (LO)**
- **Nonscattering dipole antennas (Dicke 1948, Kahn and Kurss 1965)**
- **Reduced dipolar scattering (Kerker 1975, Alu and Engheta, 2007)**
- **Superlens dipole cloak (Milton et al. 2006)**
- **Dual polarized hard surfaces (Kildal 1996)**
- **EIT (Greenleaf, Lassas, Uhlmann)**
- **Transmission lines (Alitalo et al. 2007)**
- **Active sensors and sources (Miller 2006)**
- **Transformational optics (Pendry 2006)**

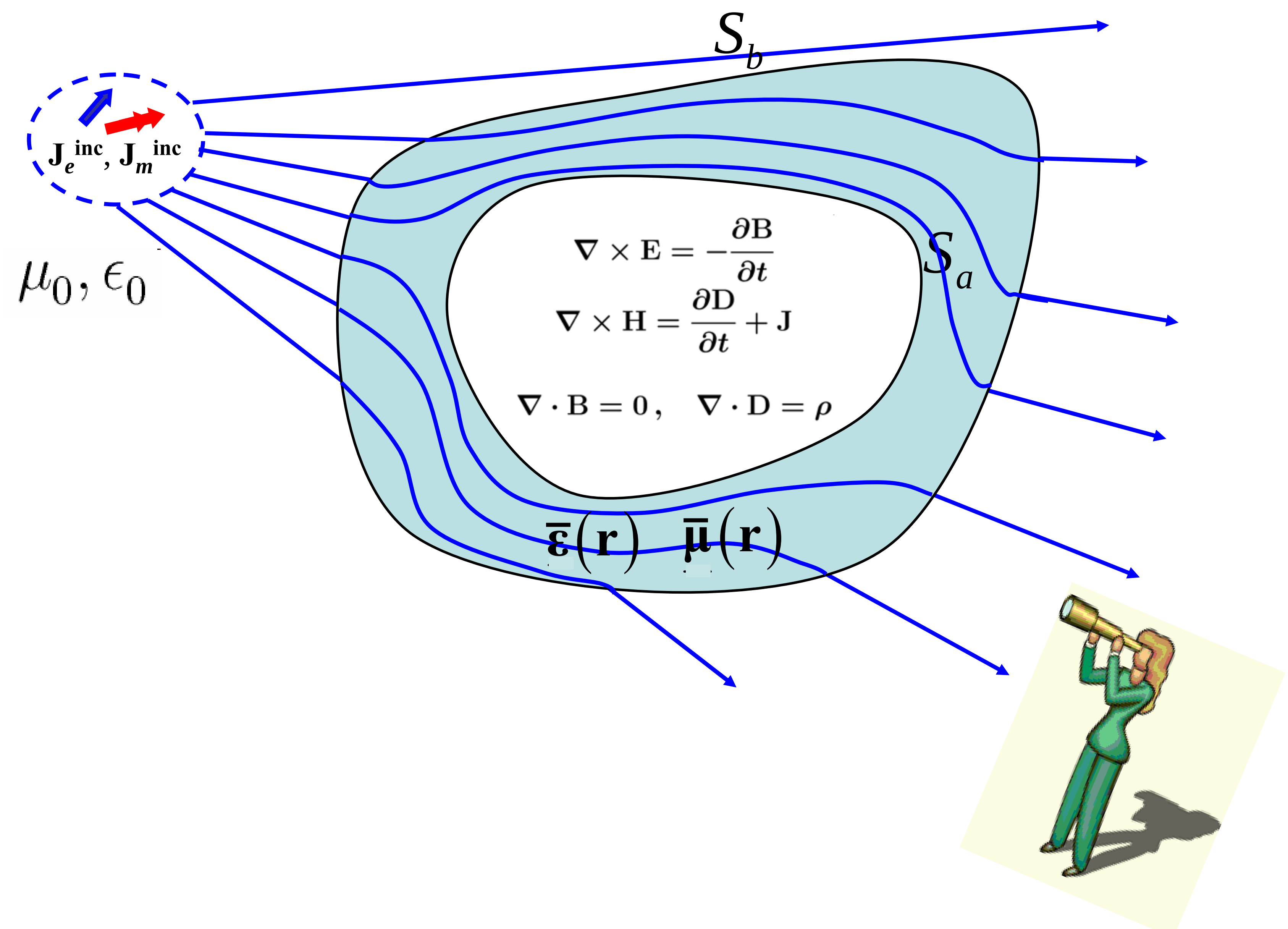
INTRIGUE OF INVISIBILITY



INTRIGUE OF INVISIBILITY



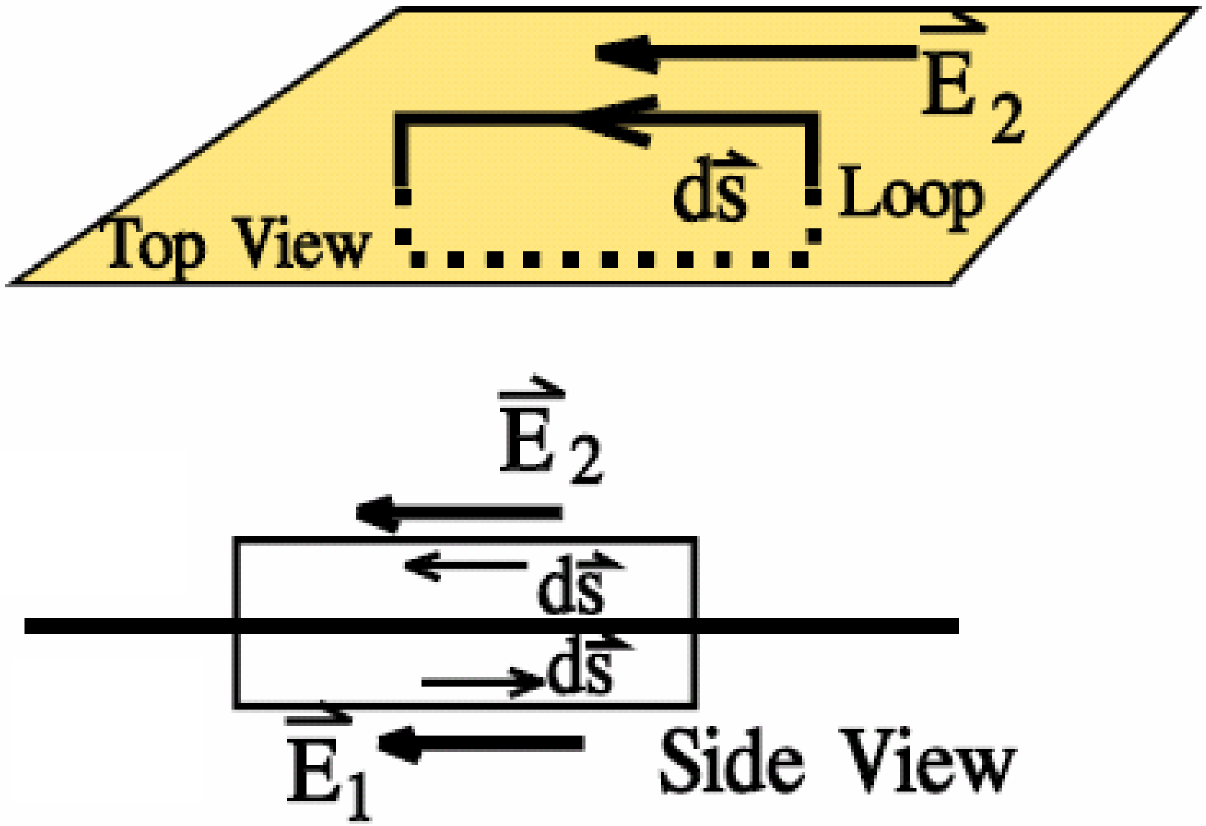
INTRIGUE OF INVISIBILITY



MOTIVATION

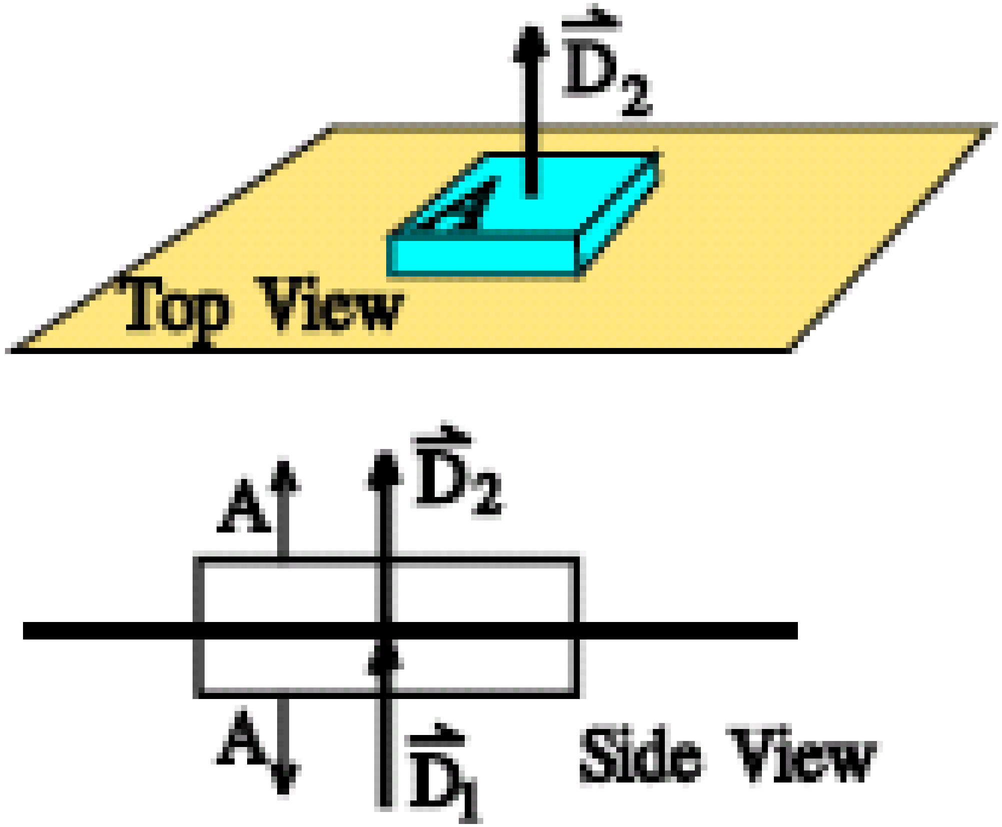
- Cloaking theorem is intriguing, surprising, and potentially useful. Can it be proven by more conventional EM theory?
- Not the usual direct or inverse scattering problem (find $\overline{\mu}$ and $\overline{\varepsilon}$).
- Boundary conditions at inner and outer surfaces of cloaks are not explicit.
- Cloaking of static fields ($\omega = 0$)?
- Reformulate cloaking as an electro-magnetic boundary value problem.

PRELIMINARIES
(Boundary Conditions)



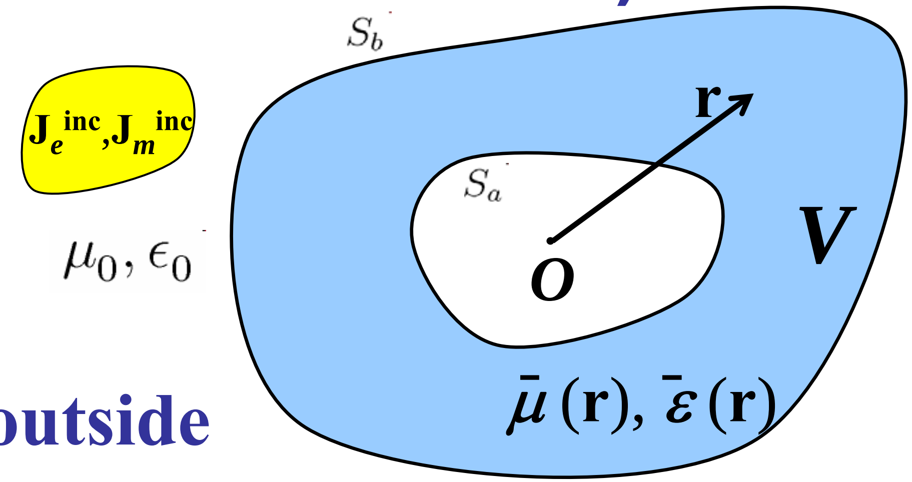
Assumes no delta functions (surface densities) in tangential magnetic induction (B) !

PRELIMINARIES
(Boundary Conditions)



Assumes no delta functions (surface densities) in tangential displacement vector (D) !

PRELIMINARIES
(Zero Scattered and Total Fields)

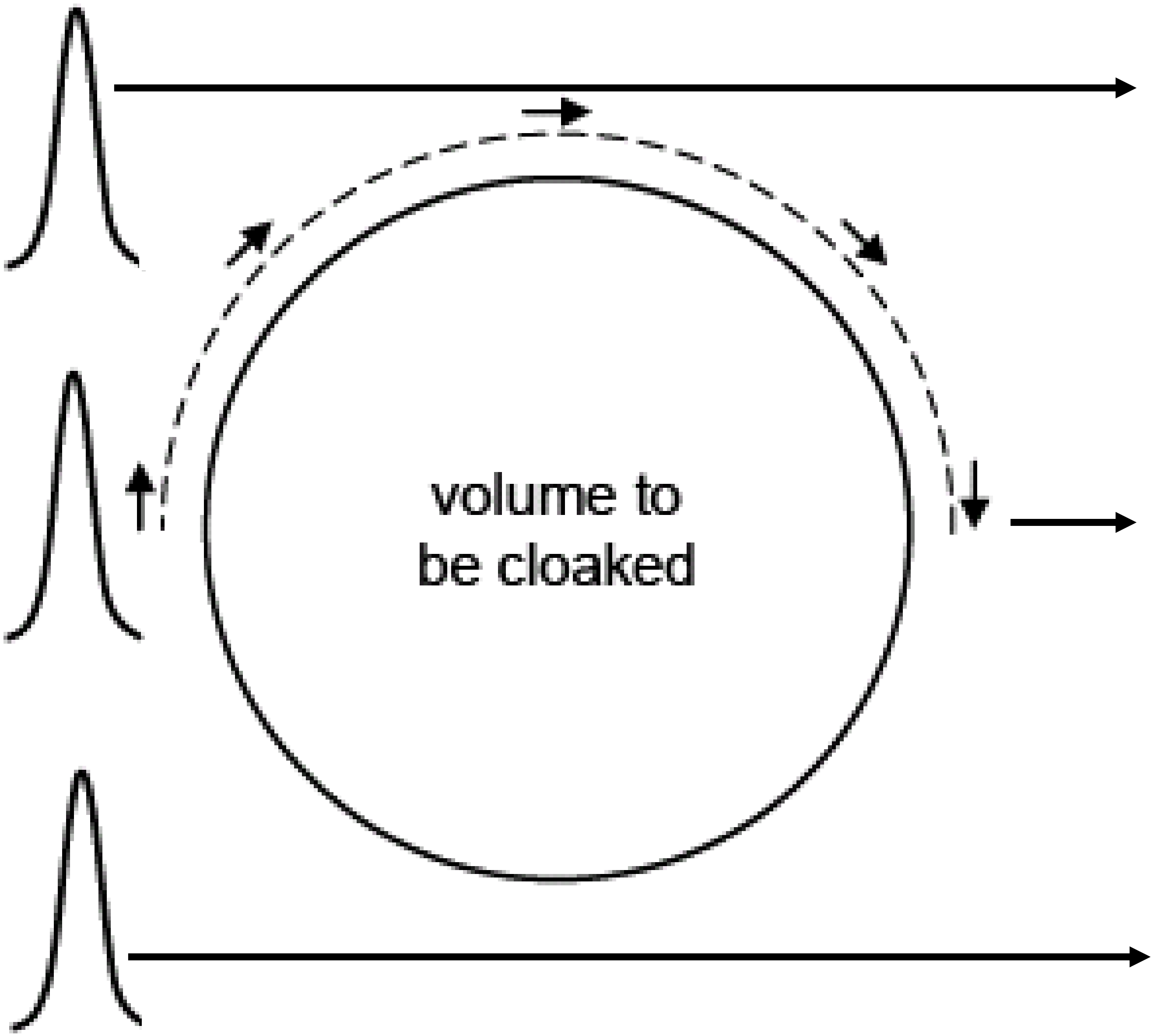


1) Cloaking material must be lossless.

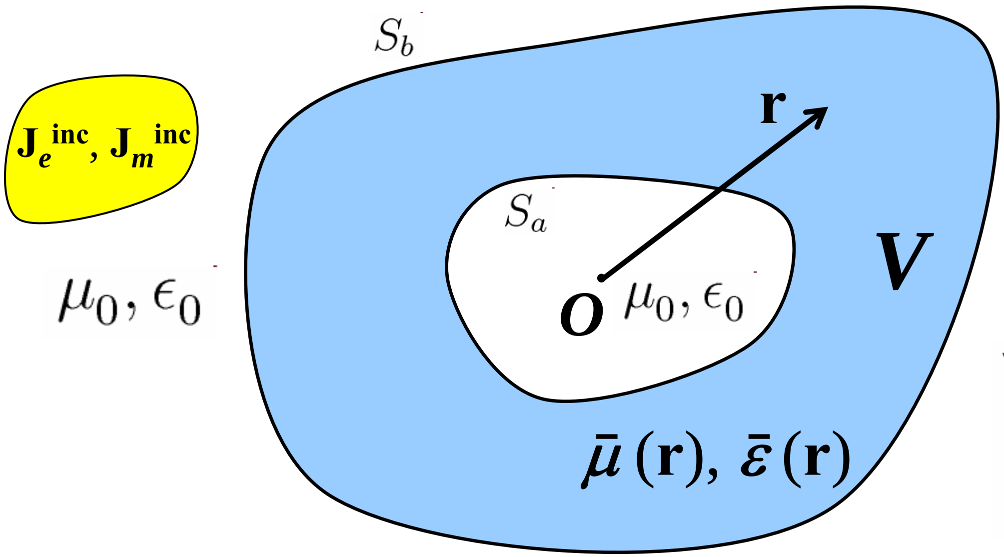
2) If a unique solution exists outside S_a^+ for the determined $[\bar{\mu}, \bar{\varepsilon}]$, which give the b.c.'s across S_b and at S_a^+ $[\mu_n = \varepsilon_n = 0]$, then any homogeneous solution in the cavity is uncoupled from the solution outside S_a^+ (seen by inserting a small loss in the cavity).

3) Then reciprocity implies that sources inside the cavity do not radiate outside the cavity. These interior sources induce surface polarization densities at S_a.

PULSES CANNOT BE CLOAKED



ALTERNATIVE DERIVATION OF EM CLOAKS AND CONCENTRATORS



$$\mathbf{D}(\mathbf{r}) = \overline{\boldsymbol{\epsilon}}(\mathbf{r}) \cdot \mathbf{E}(\mathbf{r})$$

$$\mathbf{B}(\mathbf{r}) = \overline{\boldsymbol{\mu}}(\mathbf{r}) \cdot \mathbf{H}(\mathbf{r})$$

$$\nabla \times \mathbf{E}(\mathbf{r}) - i\omega\overline{\boldsymbol{\mu}}(\mathbf{r}) \cdot \mathbf{H}(\mathbf{r}) = -\mathbf{J}_m^{\mathrm{inc}}(\mathbf{r})$$

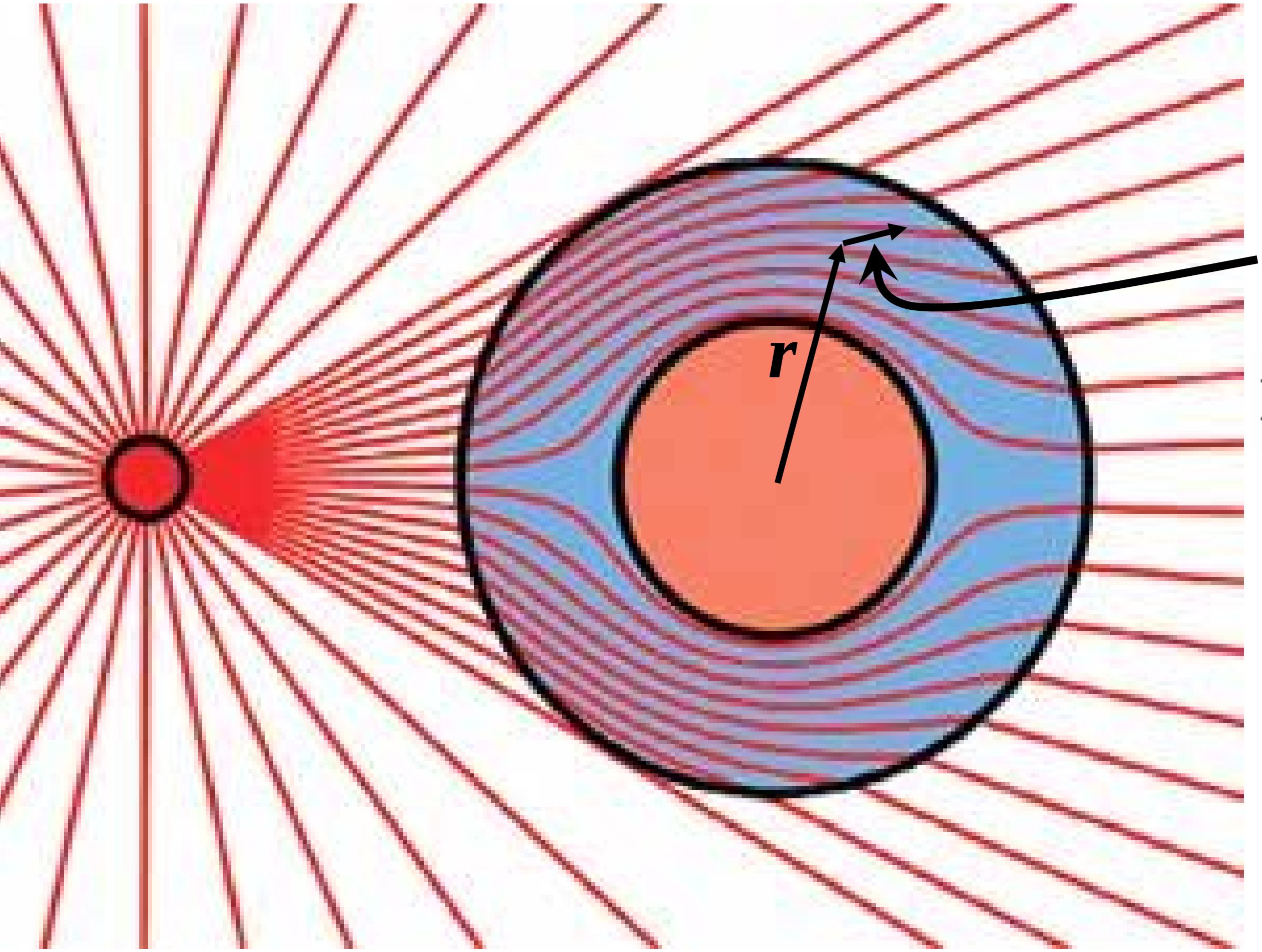
$$\nabla \times \mathbf{H}(\mathbf{r}) + i\omega\overline{\boldsymbol{\epsilon}}(\mathbf{r}) \cdot \mathbf{E}(\mathbf{r}) = \mathbf{J}_e^{\mathrm{inc}}(\mathbf{r})$$

Tangential $\mathbf{E}(\mathbf{r})$ and $\mathbf{H}(\mathbf{r})$ are continuous across S_b

Normal $\mathbf{D}(\mathbf{r})$ and $\mathbf{B}(\mathbf{r})$ are zero at S_a^+

We want to find $\overline{\boldsymbol{\mu}}(\mathbf{r})$ and $\overline{\boldsymbol{\epsilon}}(\mathbf{r})$ such that the total fields inside S_a and the scattered fields outside S_b are zero for all incident fields.

PERFECT CLOAKING FOR ALL INCIDENT FIELDS

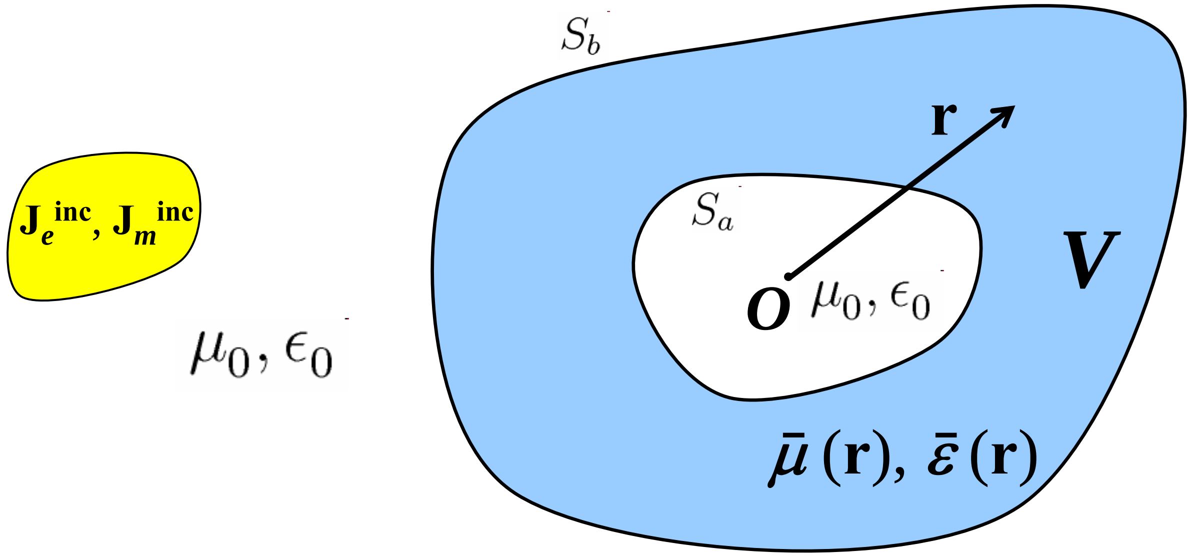


$$\mathbf{E}(\mathbf{r}) = \overline{\mathbf{A}}_e(\mathbf{r}) \cdot \mathbf{E}^{\mathrm{inc}}[\mathbf{f}(\mathbf{r})]$$

$$\mathbf{H}(\mathbf{r}) = \overline{\mathbf{A}}_h(\mathbf{r}) \cdot \mathbf{H}^{\mathrm{inc}}[\mathbf{f}(\mathbf{r})]$$

$$\mathbf{f}(\mathbf{r} \to S_b^-) = \mathbf{r}$$

SAME EQUATION FOR $\mathbf{E}^{inc}$ AND $\mathbf{H}^{inc}$

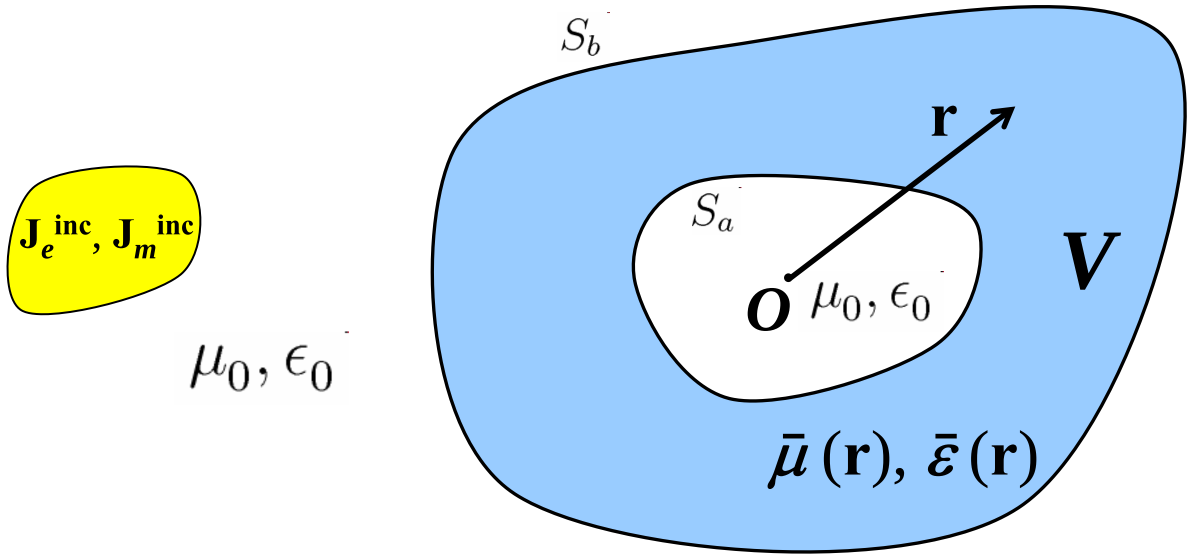


$$\nabla \times \left[\overline{\mathbf{A}}_e(\mathbf{r}) \cdot \mathbf{E}^{\mathrm{inc}}[\mathbf{f}(\mathbf{r})]\right] - \mu_0^{-1}\overline{\boldsymbol{\mu}}(\mathbf{r}) \cdot \overline{\mathbf{A}}_h(\mathbf{r}) \cdot \left[\nabla \times \mathbf{E}^{\mathrm{inc}}(\mathbf{r})\right]_{\mathbf{r}\Rightarrow\mathbf{f}(\mathbf{r})} = 0\,, \qquad \mathbf{r} \in V$$

$$\nabla \times \left[\overline{\mathbf{A}}_h(\mathbf{r}) \cdot \mathbf{H}^{\mathrm{inc}}[\mathbf{f}(\mathbf{r})]\right] - \epsilon_0^{-1}\overline{\boldsymbol{\epsilon}}(\mathbf{r}) \cdot \overline{\mathbf{A}}_e(\mathbf{r}) \cdot \left[\nabla \times \mathbf{H}^{\mathrm{inc}}(\mathbf{r})\right]_{\mathbf{r}\Rightarrow\mathbf{f}(\mathbf{r})} = 0\,, \qquad \mathbf{r} \in V$$

$$\boxed{\begin{gathered}\overline{\mathbf{A}}_e(\mathbf{r}) = \overline{\mathbf{A}}_h(\mathbf{r}) = \overline{\mathbf{A}}(\mathbf{r}) \\ \frac{\overline{\boldsymbol{\mu}}(\mathbf{r})}{\mu_0} = \frac{\overline{\boldsymbol{\epsilon}}(\mathbf{r})}{\epsilon_0} = \overline{\boldsymbol{\alpha}}(\mathbf{r})\end{gathered}}$$

THE BOUNDARY VALUE PROBLEM



$$\nabla \times \left[\overline{\mathbf{A}}(\mathbf{r}) \cdot \mathbf{E}^{\mathrm{inc}}[\mathbf{f}(\mathbf{r})]\right] - \overline{\boldsymbol{\alpha}}(\mathbf{r}) \cdot \overline{\mathbf{A}}(\mathbf{r}) \cdot \left[\nabla \times \mathbf{E}^{\mathrm{inc}}(\mathbf{r})\right]_{\mathbf{r} \Rightarrow \mathbf{f}(\mathbf{r})} = 0\,, \qquad \mathbf{r} \in V$$

$$\mathbf{E}(\mathbf{r}) = \overline{\mathbf{A}}(\mathbf{r}) \cdot \mathbf{E}^{\mathrm{inc}}[\mathbf{f}(\mathbf{r})]$$

$$\mathbf{H}(\mathbf{r}) = \overline{\mathbf{A}}(\mathbf{r}) \cdot \mathbf{H}^{\mathrm{inc}}[\mathbf{f}(\mathbf{r})]$$

$$\mathbf{f}(\mathbf{r}) \overset{\mathbf{r} \to S_b^-}{=} \mathbf{r}$$

$$\hat{\mathbf{n}} \times \overline{\mathbf{A}}(\mathbf{r}) \overset{\mathbf{r} \to S_b^-}{=} \hat{\mathbf{n}} \times \overline{\mathbf{I}}$$

$$\hat{\mathbf{n}} \cdot \left\{\overline{\boldsymbol{\alpha}}(\mathbf{r}) \cdot \overline{\mathbf{A}}(\mathbf{r})\right\} \overset{\mathbf{r} \to S_a^+}{=} 0$$

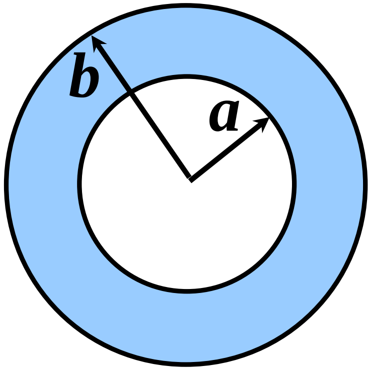
$$\mathbf{D}(\mathbf{r}) = \epsilon_0 \overline{\boldsymbol{\alpha}}(\mathbf{r}) \cdot \mathbf{E}(\mathbf{r})$$

$$\mathbf{B}(\mathbf{r}) = \mu_0 \overline{\boldsymbol{\alpha}}(\mathbf{r}) \cdot \mathbf{H}(\mathbf{r})$$

Note that equations to solve are independent of ω !

SPHERICAL CLOAK

$$\nabla \times \left[\overline{\mathbf{A}}(\mathbf{r}) \cdot \mathbf{E}^{\mathrm{inc}}[\mathbf{f}(\mathbf{r})]\right] - \overline{\boldsymbol{\alpha}}(\mathbf{r}) \cdot \overline{\mathbf{A}}(\mathbf{r}) \cdot \left[\nabla \times \mathbf{E}^{\mathrm{inc}}(\mathbf{r})\right]_{\mathbf{r}\Rightarrow\mathbf{f}(\mathbf{r})} = 0\,, \qquad \mathbf{r} \in V$$



$$\mathbf{f}(\mathbf{r}) \overset{\mathbf{r}\to S_b^-}{=} \mathbf{r}$$

$$\hat{\mathbf{n}} \times \overline{\mathbf{A}}(\mathbf{r}) \overset{\mathbf{r}\to S_b^-}{=} \hat{\mathbf{n}} \times \overline{\mathbf{I}}$$

$$\hat{\mathbf{n}} \cdot \left\{\overline{\boldsymbol{\alpha}}(\mathbf{r}) \cdot \overline{\mathbf{A}}(\mathbf{r})\right\} \overset{\mathbf{r}\to S_a^+}{=} 0$$

$$[\mathbf{f}(\mathbf{r})] = [f(r), g = \theta, h = \phi]$$

$$\overline{\mathbf{A}}(\mathbf{r}) = A_r(r)\hat{\mathbf{r}}\hat{\mathbf{r}} + A_s(r)\left(\hat{\boldsymbol{\theta}}\hat{\boldsymbol{\theta}} + \hat{\boldsymbol{\phi}}\hat{\boldsymbol{\phi}}\right)$$

$$\overline{\boldsymbol{\alpha}}(\mathbf{r}) = \alpha_r(r)\hat{\mathbf{r}}\hat{\mathbf{r}} + \alpha_s(r)\left(\hat{\boldsymbol{\theta}}\hat{\boldsymbol{\theta}} + \hat{\boldsymbol{\phi}}\hat{\boldsymbol{\phi}}\right)$$

$$f(b) = b, \quad A_s(b) = 1, \quad \alpha_r(a)A_r(a) = 0$$

$$\nabla \times \left[\overline{\mathbf{A}}(\mathbf{r}) \cdot \mathbf{E}^{\mathrm{inc}}[\mathbf{f}(\mathbf{r})]\right] = A_s(r)\nabla \times \mathbf{E}^{\mathrm{inc}}[f(r), \theta, \phi] + \frac{dA_s(r)}{dr}\hat{\mathbf{r}} \times \mathbf{E}^{\mathrm{inc}}[f(r), \theta, \phi] + [A_s(r) - A_r(r)]\hat{\mathbf{r}} \times \nabla E_r^{\mathrm{inc}}[f(r), \theta, \phi]$$

$$\left[\nabla \times \mathbf{E}^{\mathrm{inc}}(\mathbf{r})\right]_{\mathbf{r}\Rightarrow\mathbf{f}(\mathbf{r})} = \frac{r}{f(r)}\nabla \times \mathbf{E}^{\mathrm{inc}}[f(r), \theta, \phi] + \left[\frac{1}{f'(r)} - \frac{r}{f(r)}\right]\hat{\mathbf{r}} \times \frac{\partial \mathbf{E}^{\mathrm{inc}}[f(r), \theta, \phi]}{\partial r}$$

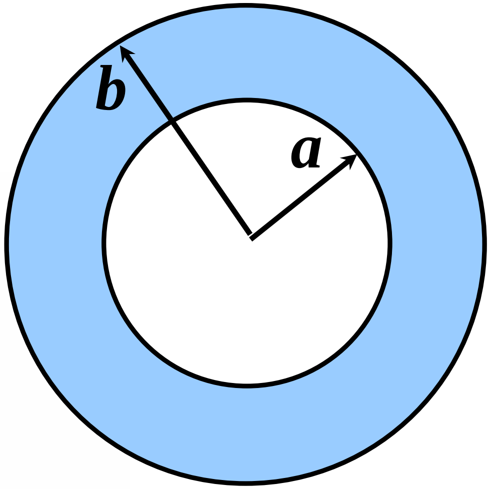
SPHERICAL CLOAK

$$\boxed{\left[\frac{A_r(r)}{A_s(r)} - \frac{f(r)}{r\alpha_r(r)}\right]} \left[\nabla \times \mathbf{E}^{\mathrm{inc}}[f(r),\theta,\phi]\right]_r = 0$$

$$- \left[\frac{A_r(r)}{A_s(r)} - \frac{r\alpha_s(r)}{f(r)}\right] \hat{\mathbf{r}} \times \nabla E_r^{\mathrm{inc}}[f(r),\theta,\phi]$$

$$+ \left[1 - \frac{\alpha_s(r)}{f'(r)}\right] \hat{\mathbf{r}} \times \frac{\partial \mathbf{E}^{\mathrm{inc}}[f(r),\theta,\phi]}{\partial r}$$

$$+ \left[\frac{1}{A_s(r)}\frac{dA_s(r)}{dr} + \frac{1}{r} - \frac{\alpha_s(r)}{f(r)}\right] \hat{\mathbf{r}} \times \mathbf{E}^{\mathrm{inc}}[f(r),\theta,\phi] = 0$$



$$\alpha_s(r) = f'(r)$$

$$\alpha_r(r) = \frac{1}{f'(r)}\left[\frac{f(r)}{r}\right]^2 \qquad f(b) = b$$

$$A_s(r) = \frac{f(r)}{r} \qquad f(a) = 0$$

$$A_r(r) = f'(r)$$

$$\boxed{\omega \neq 0}$$

SPHERICAL CLOAK

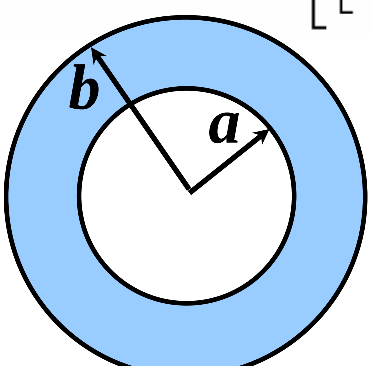
$$\overline{\boldsymbol{\alpha}} = \frac{\overline{\boldsymbol{\epsilon}}}{\epsilon_0} = \frac{\overline{\boldsymbol{\mu}}}{\mu_0} = \frac{1}{f'(r)}\left[\frac{f(r)}{r}\right]^2 \hat{\mathbf{r}}\hat{\mathbf{r}} + f'(r)\left(\hat{\boldsymbol{\theta}}\hat{\boldsymbol{\theta}} + \hat{\boldsymbol{\phi}}\hat{\boldsymbol{\phi}}\right)$$

$$\mathbf{E}(\mathbf{r}) = \left[f'(r) - \frac{f(r)}{r}\right] E_r^{\mathrm{inc}}[f(r),\theta,\phi]\hat{\mathbf{r}} + \frac{f(r)}{r}\mathbf{E}^{\mathrm{inc}}[f(r),\theta,\phi]$$

$$\mathbf{H}(\mathbf{r}) = \left[f'(r) - \frac{f(r)}{r}\right] H_r^{\mathrm{inc}}[f(r),\theta,\phi]\hat{\mathbf{r}} + \frac{f(r)}{r}\mathbf{H}^{\mathrm{inc}}[f(r),\theta,\phi]$$

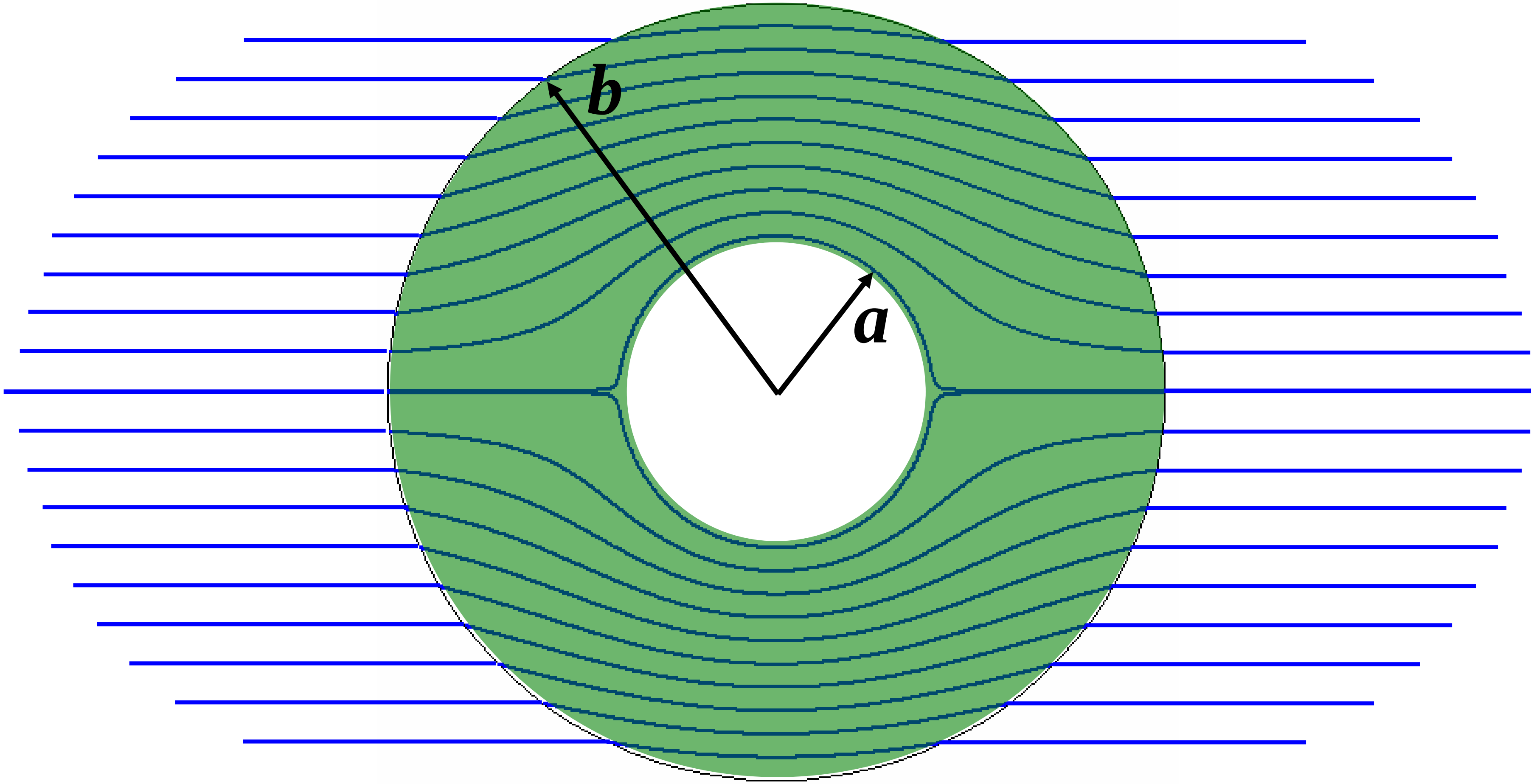
$$\frac{\mathbf{D}(\mathbf{r})}{\epsilon_0} = \overline{\boldsymbol{\alpha}}(\mathbf{r})\cdot\mathbf{E}(\mathbf{r}) = \left[\left[\frac{f(r)}{r}\right]^2 - \frac{f(r)f'(r)}{r}\right] E_r^{\mathrm{inc}}[f(r),\theta,\phi]\hat{\mathbf{r}} + \frac{f(r)f'(r)}{r}\mathbf{E}^{\mathrm{inc}}[f(r),\theta,\phi]$$

$$\frac{\mathbf{B}(\mathbf{r})}{\mu_0} = \overline{\boldsymbol{\alpha}}(\mathbf{r})\cdot\mathbf{H}(\mathbf{r}) = \left[\left[\frac{f(r)}{r}\right]^2 - \frac{f(r)f'(r)}{r}\right] H_r^{\mathrm{inc}}[f(r),\theta,\phi]\hat{\mathbf{r}} + \frac{f(r)f'(r)}{r}\mathbf{H}^{\mathrm{inc}}[f(r),\theta,\phi]$$



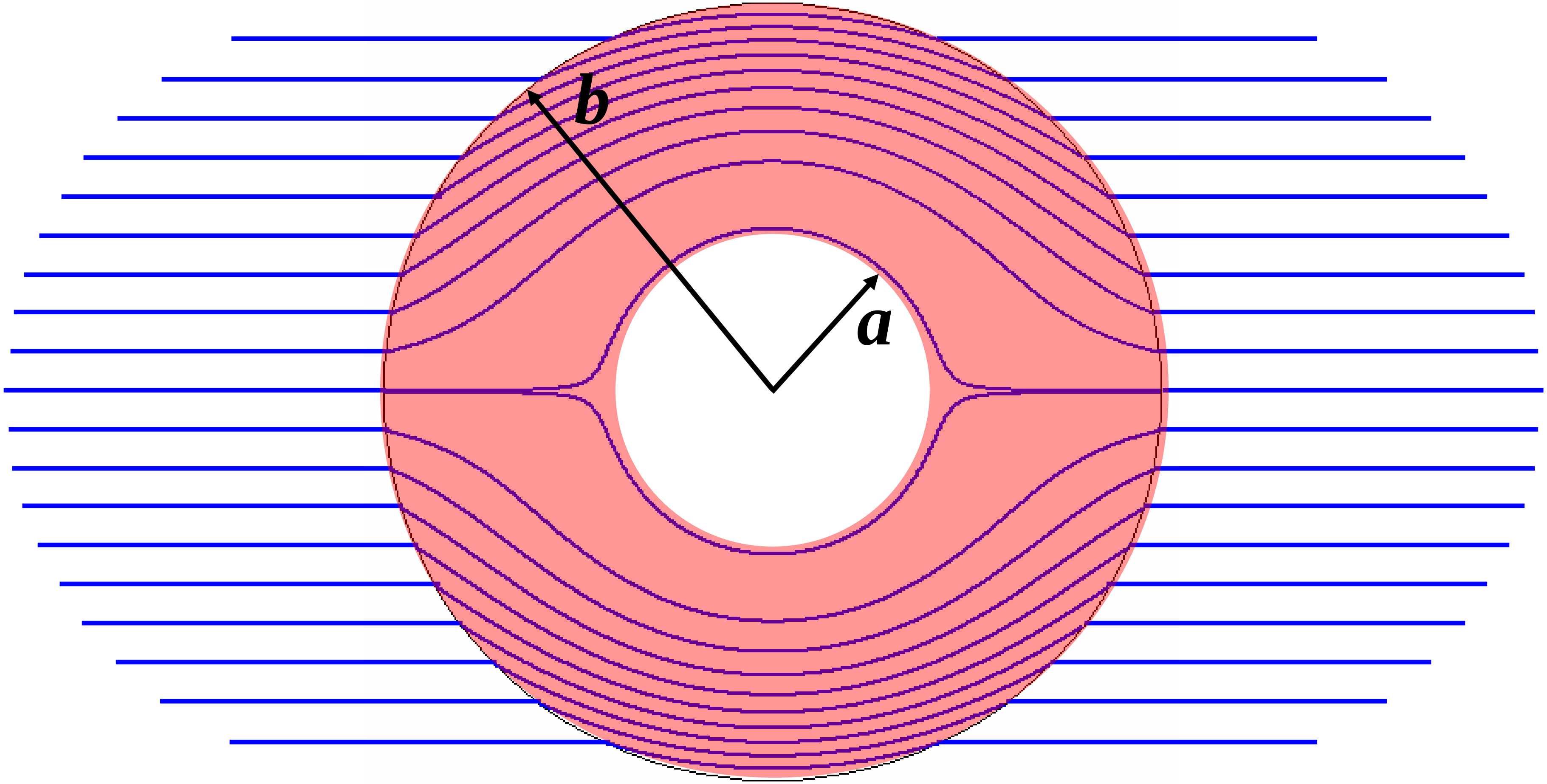
$$\boxed{f(r) = \frac{b(r-a)^p}{(b-a)^p}, \qquad p > 0}$$

RAYS FOR PENDRY SPHERICAL CLOAK (p=1)



$$f(r) = \frac{b(r-a)^p}{(b-a)^p}, \qquad p > 0$$

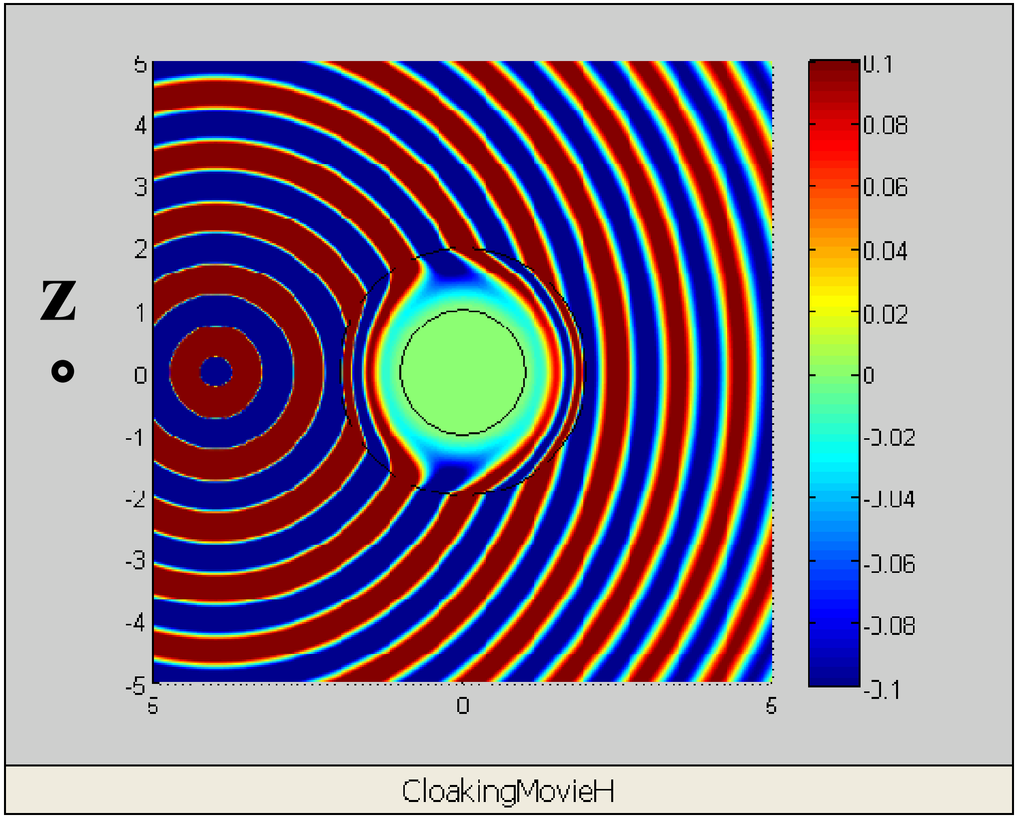
RAYS FOR 2nd ORDER SPHERICAL CLOAK (p=2)



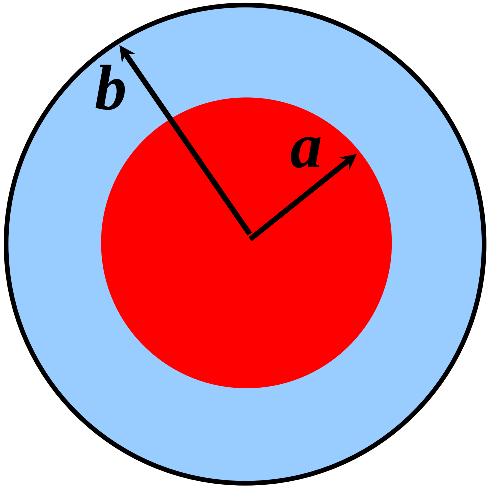
$$f(r) = \frac{b(r-a)^p}{(b-a)^p}, \qquad p > 0$$

PENDRY CLOAK (p=1)
(electric dipole near a spherical annulus)

Re[E_z] in H-Plane



SPHERICAL CONCENTRATOR

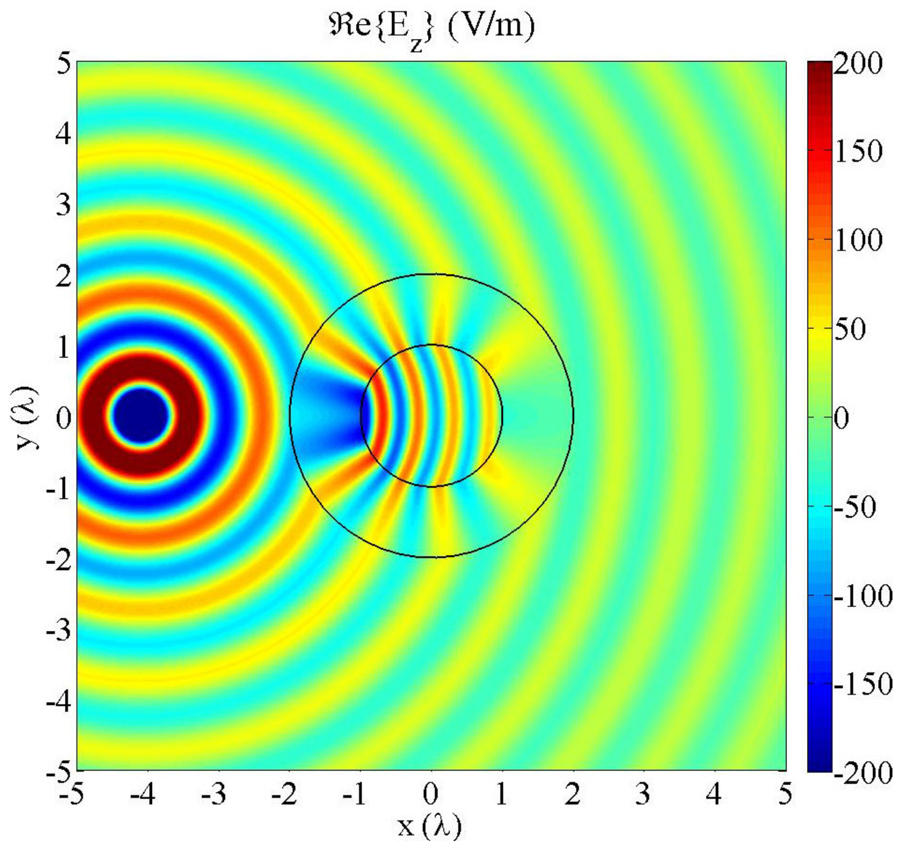


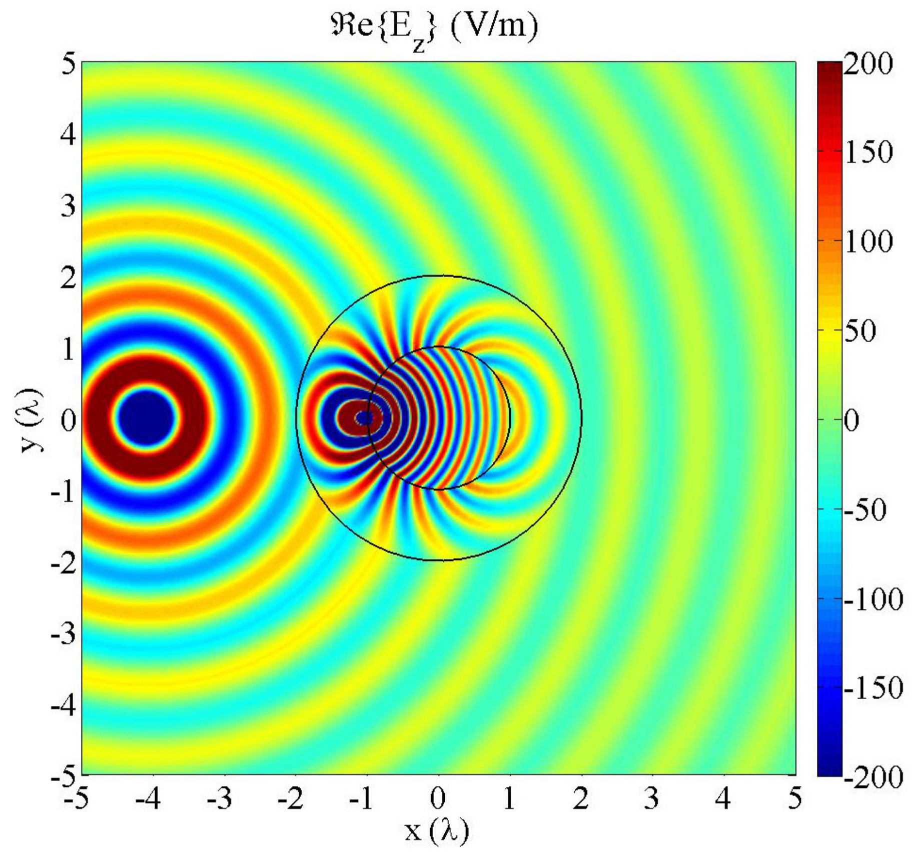
Relax the b.c. $f(a) = 0$ and the cloak becomes a nonscattering sphere for continuous, piecewise continuously differentiable functions $f(r)$. With $f(r)= Mr,\ \ r < a$, the fields are magnified by a factor M.

$$\mathbf{E}(\mathbf{r}) = M\mathbf{E}^{\mathrm{inc}}(Mr,\theta,\phi)\,, \qquad 0 \le r < a$$
$$\mathbf{H}(\mathbf{r}) = M\mathbf{H}^{\mathrm{inc}}(Mr,\theta,\phi)\,, \qquad 0 \le r < a$$

$$\overline{\boldsymbol{\alpha}} = \frac{\overline{\boldsymbol{\epsilon}}}{\epsilon_0} = \frac{\overline{\boldsymbol{\mu}}}{\mu_0} = M\bar{\mathbf{I}}\,, \qquad 0 \le r < a$$

FIELDS OF A SPHERICAL CONCENTRATOR





MAGNETOSTATIC ($\omega = 0$) CLOAKING

$$\nabla \times \mathbf{H}(\mathbf{r}) = 0 \qquad \nabla \cdot \mathbf{B}(\mathbf{r}) = 0 \qquad \mathbf{B}(\mathbf{r}) = \mu_0 \overline{\boldsymbol{\alpha}}(\mathbf{r}) \cdot \mathbf{H}(\mathbf{r})$$

Basic equations

$$\boxed{\begin{gathered} \mathbf{B}(\mathbf{r}) = \overline{\mathbf{A}}(\mathbf{r}) \cdot \mathbf{B}^{\mathrm{inc}}[\mathbf{f}(\mathbf{r})] \\ \nabla \times \{\overline{\boldsymbol{\alpha}}^{-1}(\mathbf{r}) \cdot \overline{\mathbf{A}}(\mathbf{r}) \cdot \mathbf{B}^{\mathrm{inc}}[\mathbf{f}(\mathbf{r})]\} = 0 \\ \nabla \cdot \{\overline{\mathbf{A}}(\mathbf{r}) \cdot \mathbf{B}^{\mathrm{inc}}[\mathbf{f}(\mathbf{r})]\} = 0 \end{gathered}}$$

Boundary conditions

$$\boxed{\begin{gathered} \mathbf{f}(\mathbf{r}) \overset{\mathbf{r} \to S_b^-}{=} \mathbf{r} \\ \hat{\mathbf{n}} \times [\overline{\boldsymbol{\alpha}}^{-1}(\mathbf{r}) \cdot \overline{\mathbf{A}}(\mathbf{r})] \overset{\mathbf{r} \to S_b^-}{=} \hat{\mathbf{n}} \times \overline{\mathbf{I}} \\ \hat{\mathbf{n}} \cdot \overline{\mathbf{A}}(\mathbf{r}) \overset{\mathbf{r} \to S_b^-}{=} \hat{\mathbf{n}} \\ \hat{\mathbf{n}} \times [\overline{\boldsymbol{\alpha}}^{-1}(\mathbf{r}) \cdot \overline{\mathbf{A}}(\mathbf{r})] \overset{\mathbf{r} \to S_a^+}{=} 0 \end{gathered}}$$

SPHERICAL MAGNETOSTATIC CLOAK

$$\alpha_s(r) = \frac{r f'(r)}{f(r)}$$

(approaches ∞ as r → a)

$$\alpha_r(r) = \frac{f(r)}{r f'(r)}$$

However, all components of B and H remain finite.

$$A_s(r) = f'(r)$$

$$A_r(r) = \frac{f(r)}{r}$$

$$\boxed{f(r) = \frac{b(r-a)^p}{(b-a)^p}, \qquad p > 0}$$

$$f(b) = b$$

$$f(a) = 0$$

CAUSALITY-ENERGY CONDITIONS

$$\frac{\mathrm{d}(\omega\alpha_{ll})}{\mathrm{d}\omega} - 1 \geq \frac{\omega}{2}\frac{\mathrm{d}\alpha_{ll}}{\mathrm{d}\omega} \geq 0$$

(Nondiamagnetic material)

$$\epsilon_{ll} - 1 \geq 0\,, \qquad \omega \to 0$$

Therefore, electrostatic cloaking and cloaking at low frequencies is impossible!

CAUSALITY-ENERGY CONDITIONS
(Diamagnetic Material)

$$\frac{\mathrm{d}(\omega\alpha_{ll}^{\mathrm{d}})}{\mathrm{d}\omega} - \left(\alpha_{ll}^{\mathrm{d}}\right)^2 \geq \frac{\omega}{2}\frac{\mathrm{d}\alpha_{ll}^{\mathrm{d}}}{\mathrm{d}\omega} \geq 0$$

$$0 \leq \mu_{ll}^{\mathrm{d}} \leq 1\,, \qquad \omega \rightarrow 0$$

Therefore, magnetostatic cloaks appear to be realizable!

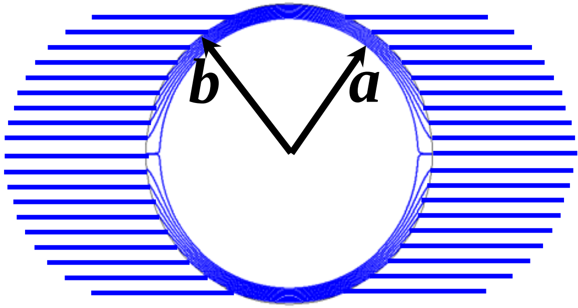
Wood, Pendry, et al. 2007-08

GOOD NEWS

- Perfect cloaks and nonscatterers, such as concentrators, are theoretically possible at any one frequency.
- A host of other new solutions to Max-well's equations are possible based on "transformational electromagnetics" or equivalent boundary value problems.
- Similar techniques may be applicable to other fields of physics, e.g., acoustics.
- Magnetostatic cloaks, unlike electrostatic cloaks and low-frequency cloaking, appear feasible.

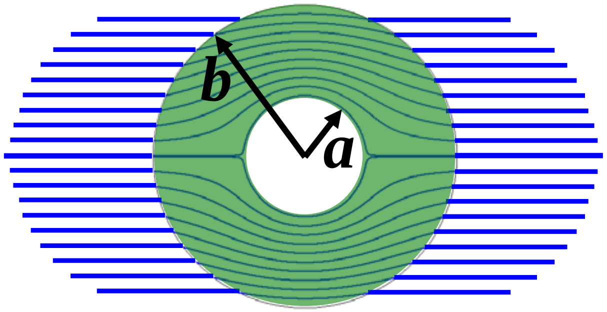
BAD NEWS

- It may be prohibitively difficult to synthesize low-loss anisotropic metamaterials with equal relative $\overline{\mu}$ and $\overline{\varepsilon}$ (unobtainium).
- Ratio of tangential to normal components of $\overline{\mu}$ and $\overline{\varepsilon}$ equals $r^2/(r-a)^2 >> 1$ for thin cloaks.



- Finite bandwidth (pulsed) cloaks violate causality and thus cloaking for $\omega>0$ must always be approximate:

$$\frac{d(\omega\alpha)}{d\omega} - 1 \geq \frac{\omega}{2}\frac{d\alpha}{d\omega} \geq 0\,.$$



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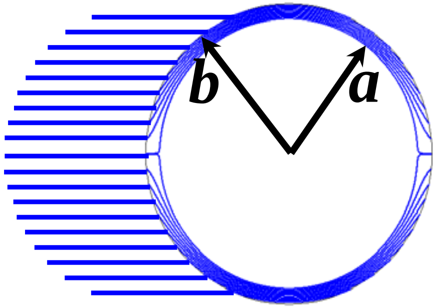

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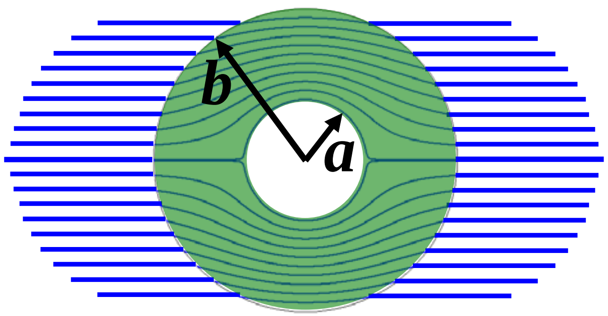

“It is impossible to travel faster than the speed of light, and certainly not desirable, as one's hat keeps blowing off.”

Woody Allen