Lagrangian Data Assimilation (LaDA) for point-vortex systems

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Point-vortex systems: vortex motion

Consider m point vortices in the plane whose the equation of motion of the $k-$th vortex is given by

$$\frac{dx_k}{dt} = -\frac{1}{2\pi}\sum_{\substack{j=1\\ j\neq k}}^{m} \kappa_j \frac{y_k - y_j}{(x_k - x_j)^2 + (y_k - y_j)^2}$$

$$\frac{dy_k}{dt} = \frac{1}{2\pi}\sum_{\substack{j=1\\ j\neq k}}^{m} \kappa_j \frac{x_k - x_j}{(x_k - x_j)^2 + (y_k - y_j)^2},$$

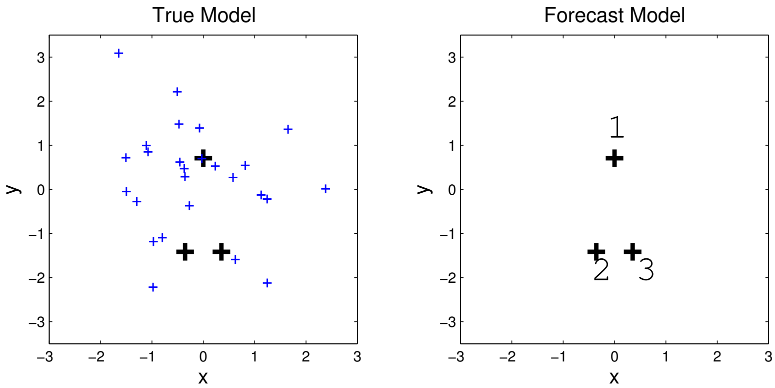
where κ_j denotes the vortex strength.

The motion of passive tracers (denoted by (x, y) without any subscript) depends explicitly on time through the positions of vortices. The velocity filed of the passive tracers are given by

$$\frac{dx}{dt} = -\frac{1}{2\pi}\sum_{j=1}^{m}\kappa_j\frac{y-y_j}{(x-x_j)^2+(y-y_j)^2}$$

$$\frac{dy}{dt} = \frac{1}{2\pi}\sum_{j=1}^{m}\kappa_j\frac{x-x_j}{(x-x_j)^2+(y-y_j)^2}.$$

“True" and Forecast Models



Truth and Model Forecast

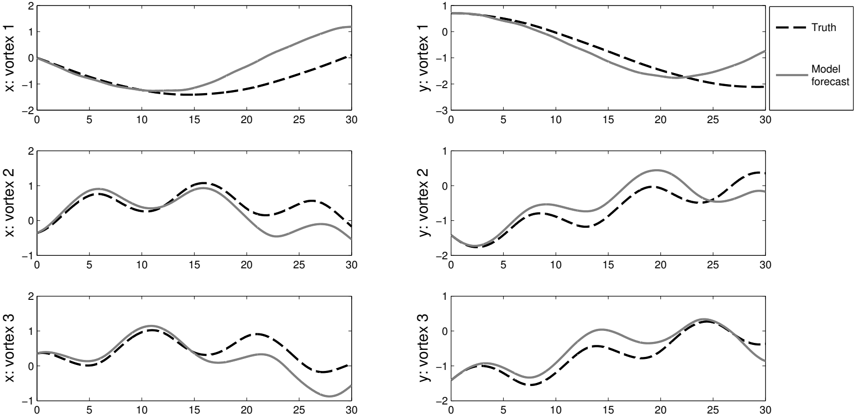


Figure: $\kappa_1 = \kappa_2 = \kappa_3 = 1$, $\kappa_j = 0.03$ for $j = 4, \ldots, 30$. The RMSEs for the three vortices are 0.64, 0.43 and 0.44, respectively.

- We do not want to rely only on the model forecasts to track the large-scale vortices (i.e. estimate the "true" vortex trajectories)
- Suppose that we observe only (noisy) trajectories of passive tracers (or drifters/floats)
- Deployment strategy: Where should we initialize the tracers?
- Can we use some dynamical structures (e.g. LCS or finite-time coherent sets) to aid the design of the launching strategy?

Sequential Data Assimilation

- Given the data $y_{1:t} := (y_1, \ldots, y_t)$, inference about x_t is carried out by

$$P(x_t|y_{1:t}) \propto P(y_t|x_t, y_{1:t-1})P(x_t|y_{1:t-1})$$

- The normalization term is omitted here
- Prior distribution: (deterministic/stochastic) model containing uncertainties in model itself or initial conditions or both
- Likelihood: uncertainties in predicting y_t from x_t (e.g. $y_t = Hx_t +$"noise")

Ensemble KF (ENKF) with perturbed obsevation

- Use the updating dynamics of classical KF, but for a prior non-Gaussian distribution.
- ENKF: use sample statistics to approximate $\mathbf{P}^f$ and $\mathbf{P}^a$
- Anomalies: $\mathbf{X^f} = [x_1^f - \bar{x}^f | \dots | x_N^f - \bar{x}^f]/\sqrt{N-1}$

$$x_i^a(t_k) = x_i^f(t_k) + \mathbf{K}_e(t_k)(\underbrace{y^o(t_k) + \epsilon_i(t_k)}_{y_i} - \underbrace{\mathbf{H}x_i^f(t_k)}_{y_i^f(t_k)})$$

$$\mathbf{X^a} = [x_1^a - \bar{x}^a | \dots | x_N^a - \bar{x}^a]/\sqrt{N-1}$$

$$\mathbf{P}_e^f(t_k) = \mathbf{X^f}{\mathbf{X^f}}^T, \quad \mathbf{P}_e^a(t_k) = \mathbf{X^a}{\mathbf{X^a}}^T$$

$$\mathbf{K}_e(t_k) = \mathbf{X}^f(\mathbf{Y}^f)^T(\mathbf{Y}^f(\mathbf{Y}^f)^T + \mathbf{Y}\mathbf{Y}^T)^{-1}$$

- $\mathbf{P}_e^a(t_k)$ provides a good estimate of the desired form when N large and x^f and ϵ_i uncorrelated
- Use the ensemble mean $\langle x_i^a \rangle$ as the state estimate

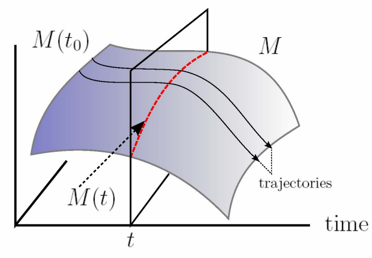
Particle Filtering (PF)

- Based on Sequential Important Sampling (SIS) with Resampling
- Use empirical distributions (weighted ensemble) to estimate a target density
- Does not make the Linear+Gaussian assumption
- ENKF: Use the Kalman update equation to “move" the prior samples by linear regression, but no weight on the samples
- PF: Update the weights of prior samples based on the Likelihood, but no “move", no linear regression
- PF suffers from the curse of dimensionality. So, it’s not available for high-dimensional problems

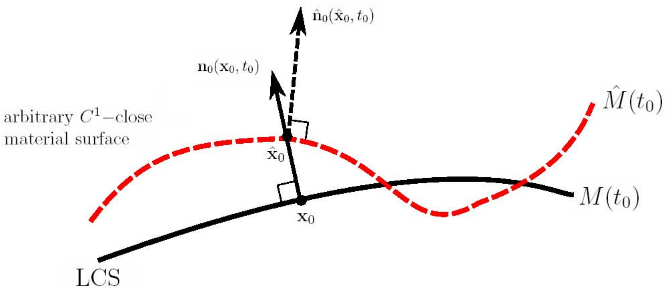
Truth and Model Forecast

- Vortices: $[x_1, y_1, \ldots, x_3, y_3] \equiv \mathbf{x}_F$
- Tracers: $[x_1^d, y_1^d, \ldots, x_3^d, x_3^d] \equiv \mathbf{x}_D$
- State variable: $(\mathbf{x}_F, \mathbf{x}_D)$
- Model uncertainty: SDE with $N(0, 0.05\mathbf{I})$ for a model forecast
- Observation: $(\mathbf{x}_D)+$“noise"
- Observation noise: Gaussian with zero mean and covariance $0.05\mathbf{I}$

What is LCS?

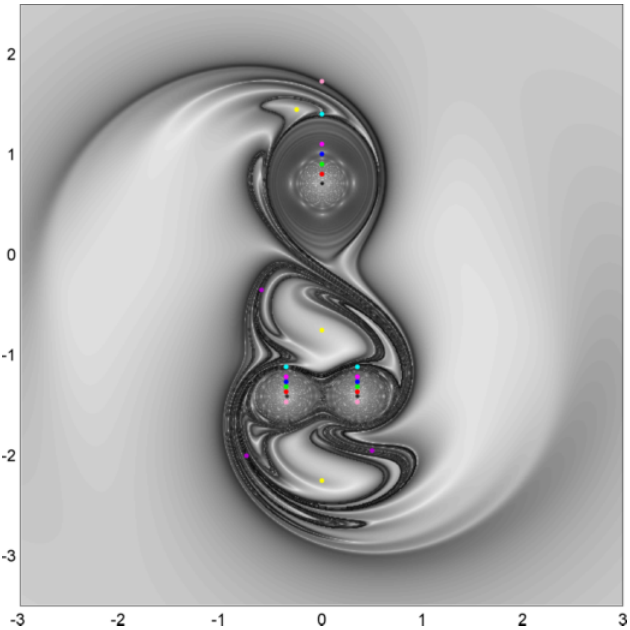


$M(t_0)$
M
trajectories
$M(t)$
time
t

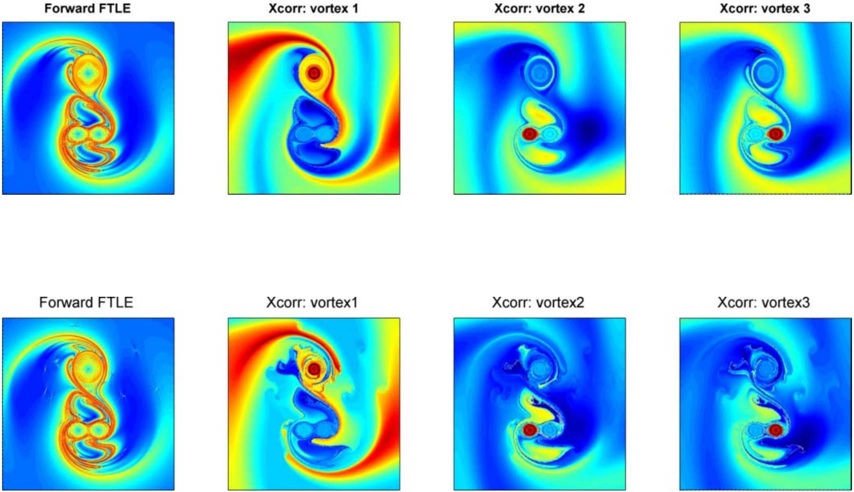


$\hat{\mathbf{n}}_0(\hat{\mathbf{x}}_0, t_0)$
$\mathbf{n}_0(\mathbf{x}_0, t_0)$
arbitrary C^1−close
material surface
$\hat{M}(t_0)$
$\hat{\mathbf{x}}_0$
$\mathbf{x}_0$
$M(t_0)$
LCS

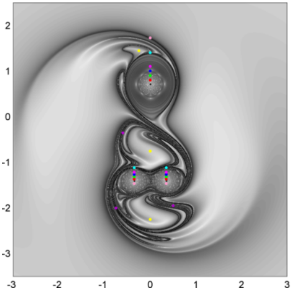
Initial tracers

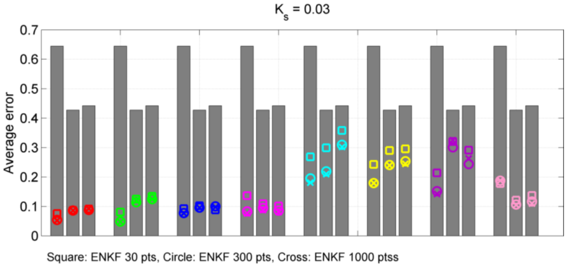


Truth and Model Forecast

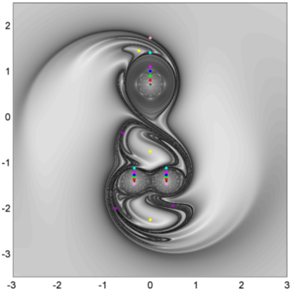


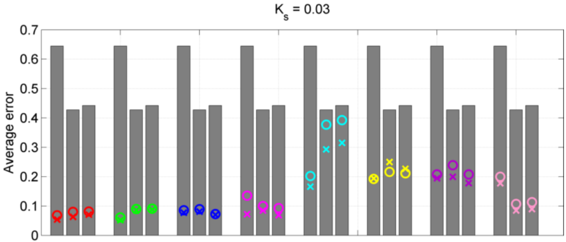
Results: ENKF for three-point vortex system





Results: PF for three-point vortex system





Circle: PF 400 pts, Cross: PF 2000 pts

Summary: Three-point vortex systems

- There is a “vortex core" where vortex and tracer trajectories are highly correlated
- In these regions, data assimilation produces reliable estimate of the true states
- LCS may be used as a proxy to determine these vortex core regions

Parameter Estimation and vortex tracking

- Consider a system of two Rossby wave on the $\beta-$plane

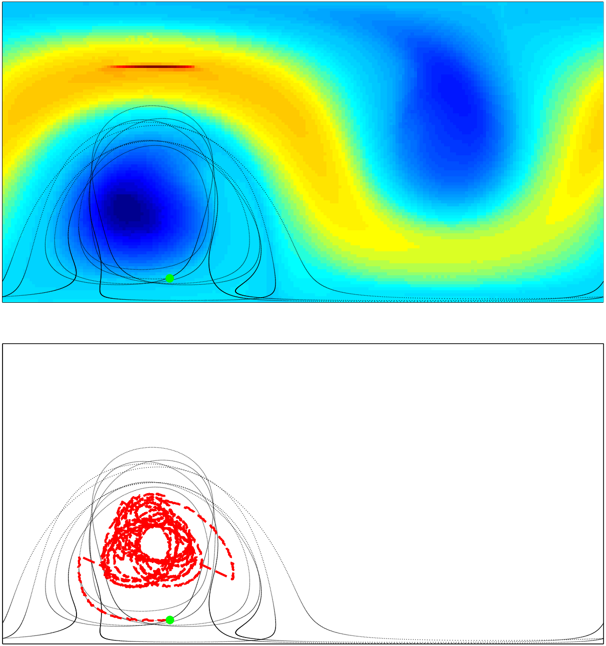
$$\psi = A\sin(k_1(x - c_1 t)\sin(l_1 y) + \epsilon\sin(k_2(x - c_2 t)\sin(l_2 y)$$

- Phase speeds are $c_j = \beta/(k_j^2 + l_j^2)$
- Using the barotropic Co-moving frame
- **Problem**: Estimating the unknown parameters $A, \epsilon, k_1, l_1, k_2, l_2$ and retaining observations within the circulation region at all times
- **Approach**: Use particle filtering and coherent set method to adapt the tracer position
- Twin experiments: assume that the true parameters are $A = 1, \epsilon = 0.2, k_1 = 1, l_1 = 1, k_2 = 1, l_2 = 2$.

True coherent set

(true coherent set)

Observation with/without control



Posterior

(posterior)

Coherent set Estimates