Superparameterization and Dynamic Stochastic Superresolution (DSS) for Filtering Sparse Geophysical Flows

Presented by Nan Chen, Michal Branicki and Chenyue Wu

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Outline

1. Filtering
   - Fourier Domain Kalman Filter (FDKF) with regularly spaced sparse observations.
2. Filtering with Superparameterization
   - linear, analytically solvable model,
   - model error coming from finite discrete approximations.
3. Filtering with Dynamic Stochastic Superresolution (DSS)
   - nonlinear model,
   - using cheap stochastic models to forecast the true nonlinear dynamics.

- **Test Models for Filtering with Superparameterization**, John Harlim and A. J. Majda, submitted, SIAM J. Multiscale Modeling and Simulation, September 9, 2012.
- **Dynamic Stochastic Superresolution of sparseley observed turbulent systems**, M. Branicki and A. J. Majda, submitted, Journal of Computational Physics, May 17, 2012.
- **New methods for estimating poleward eddy heat transport using satellite altimetry**, S. Keating, A. J. Majda and K. S. Smith, Monthly Weather Review, February 9, 2012.

Basic Notions of Filtering and Test Models for Filtering with Superparameterization

I. Filtering the Turbulent Signal

1.1. Kalman Filter

- True signal $\vec{u}_{m+1} \in \mathbb{R}^N$, which is generated from

$$\boxed{\vec{u}_{m+1} = F\vec{u}_m + \vec{\sigma}_{m+1}},$$

- Observation $\vec{v}_{m+1} \in \mathbb{R}^M$:

$$\boxed{\vec{v}_{m+1} = G\vec{u}_{m+1} + \vec{\sigma}^o_{m+1}},$$

where matrix $G \in \mathbb{R}^{M\times N}$ and $\vec{\sigma}^o_m = \{\sigma^o_{j,m}\}$ is an M-dimensional Gaussian while noise vector with zero mean and covariance

$$R^o = \langle \vec{\sigma}^o_m \otimes (\vec{\sigma}^o_m)^T \rangle = \{\langle \vec{\sigma}^o_{i,m} (\vec{\sigma}^o_{j,m})^T \rangle\} = \{\delta(i-j) r^o\}$$

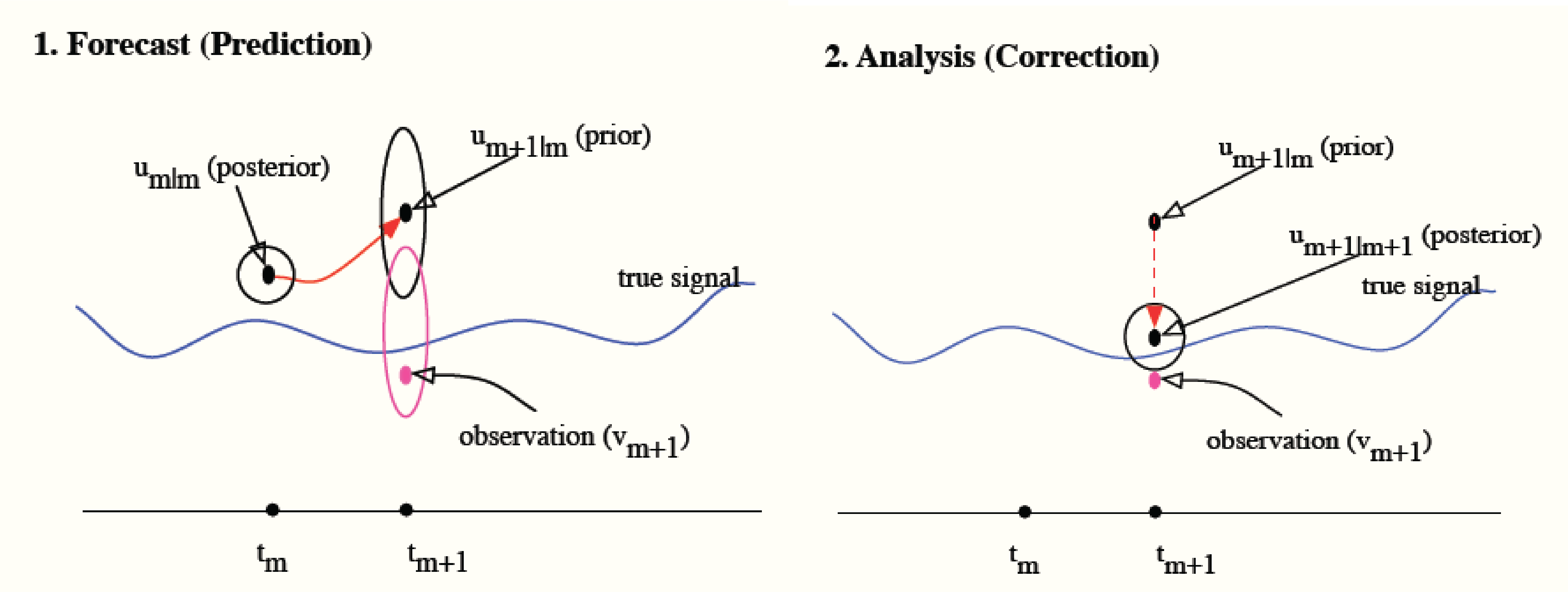
- Forecast model:

$$\boxed{\vec{u}^M_{m+1} = F^M \vec{u}^M_m + \vec{\sigma}^M_{m+1}},$$

where $F^M \in \mathbb{R}^{N\times N}$ and $\vec{\sigma}^M_m$ is an M-dimensional Gaussian while noise vector with zero mean and covariance

$$R^M = \langle \vec{\sigma}^M_m \otimes (\vec{\sigma}^M_m)^T \rangle.$$

Goal: Estimate the true state: $\vec{u}_{m+1} \in \mathbb{R}^N$ from the imperfect prediction model and the observations of the true signal.



Step 1. Forecast:
Run the forecast model from step m to $m+1$,

$$\vec{u}^M_{m+1|m} = F\vec{u}^M_{m|m} + \vec{\sigma}^M_{m+1}.$$

Compute the **prior** mean and covariance

$$\vec{u}^M_{m+1|m} = F^M \vec{u}^M_{m|m},$$
$$R^M_{m+1|m} = F^M R^M_{m|m} (F^M)^T + R^M.$$

Step 2. Analysis:
Compute **posterior** mean and variance

$$\vec{u}^M_{m+1|m+1} = \vec{u}^M_{m+1|m} + K_{m+1}(\vec{v}_{m+1} - G\vec{u}^M_{m+1|m}),$$
$$R^M_{m+1|m+1} = (\mathcal{I} - K_{m+1}G)R^M_{m+1|m},$$

where K_{m+1} is the Kalman gain matrix

$$K_{m+1} = \frac{R^M_{m+1|m}G^T}{GR^M_{m+1|m}G^T + R^o}.$$

1.2. Fourier Domain Kalman Filter (FDKF).

Canonical Filtering Problem: Plentiful Observations

$$\frac{\partial}{\partial t}\vec{u}(x,t) = \mathcal{L}\left(\frac{\partial}{\partial x}\right)\vec{u}(x,t) + \sigma(x)\dot{\vec{W}}(t), \qquad \vec{u} \in \mathbb{R}^s,$$
$$\vec{v}(x_j, t_m) = G\vec{u}(x_j, t_m) + \sigma^o_{j,m}.$$

The dynamics is realized at $2N+1$ discrete points $\{x_j = jh, j = 0, 1, \ldots, 2N\}$ such that $(2N+1)h = 2\pi$. The observations are attainable at all the $2N+1$ grid points. The observation noise $\sigma^o_m = \{\sigma^o_{j,m}\}$ are assumed to be zero mean Gaussian variables and are spatial and temporal independent.

Finite Fourier expansion of $\vec{u}(x,t)$:

$$\vec{u}(x_j, t_m) = \sum_{|k|\le N} \vec{\hat{u}}(t_m)e^{ikx_j}, \qquad \hat{u}_{-k} = \hat{u}^*_k,$$

$$\vec{\hat{u}}(t_m) = \frac{h}{2\pi}\sum_{j=0}^{2N} \vec{u}(x_j, t_m)e^{-ikx_j}.$$

Fourier Analogue of the Canonical Filtering Problem:

$$\vec{\hat{u}}_k(t_{m+1}) = F_k\vec{\hat{u}}_k(t_m) + \vec{\sigma}_{k,m+1},$$
$$\vec{\hat{v}}_k(t_m) = G\vec{\hat{u}}_k(t_m) + \vec{\sigma}^o_{k,m}.$$

Then the original $(2N+1)s \times (2N+1)s$ filtering problem reduces to study $2N+1$ independent $s \times s$ matrix Kalman filtering problems.

1.3. FDKF with regularly spaced sparse observations.

Assume there are N model grid points. We consider the observations at every p model grid points such that the total number of observation is M with $M = N/p$.

Sparse Regularly Spaced Observations in Fourier Space is expressed as follows:

$$\vec{\hat{u}}_k(t_{m+1}) = F_k\vec{\hat{u}}_k(t_m) + \vec{\sigma}_{k,m+1}, \qquad |k| \leq N/2,$$

$$\vec{\hat{v}}_l(t_m) = G\sum_{k\in\mathcal{A}(l)} \vec{\hat{u}}_k(t_m) + \vec{\sigma}^o_{l,m}, \qquad |l| \leq M/2,$$

where the aliasing set of wavenumber l is defined as

$$\mathcal{A}(l) = \{k : k = l + Mq | q \in \mathbb{Z}, |k| \leq N/2\}$$

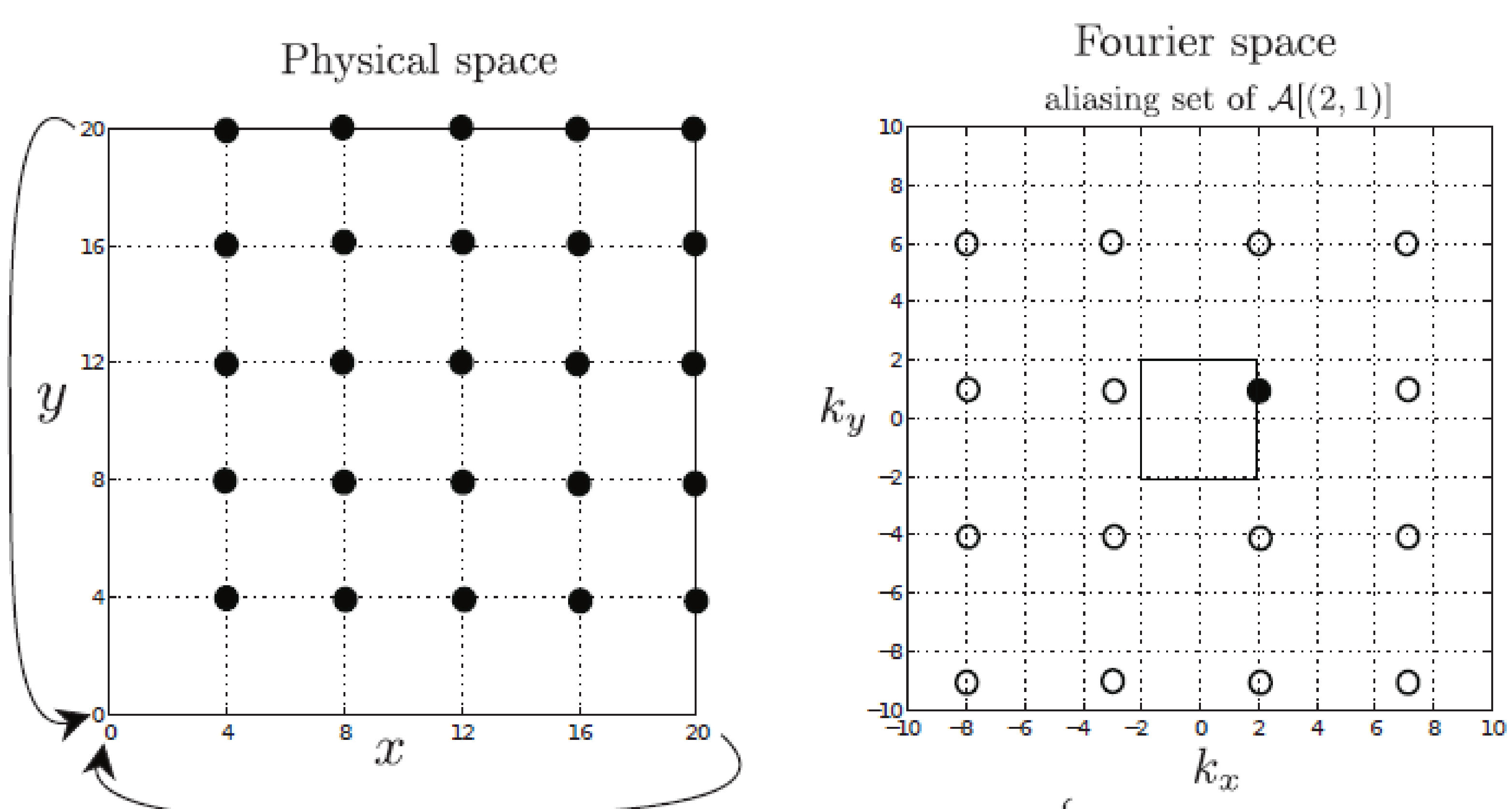


Figure 1: 5×5 sparse observation grid is a regular subset of the 20×20 model mesh so that every $P = 4$ model mesh node is observed. Here $N = 20$ and $M = 5$. There are 25 aliasing sets in all: $A(i,j)$ with $i,j \in \mathbb{Z}$ and $-2 \leq i,j \leq 2$. All primary modes lie inside the region $-2 \leq k_x, k_y \leq 2$.

Covariance matrix in physical domain → Covariance matrix in Fourier domain

$$\begin{pmatrix} R_{11} & R_{12} & R_{13} & R_{14} & R_{15} & R_{16} & R_{17} & R_{18} & R_{19} \\ R_{21} & R_{22} & R_{23} & R_{24} & R_{25} & R_{26} & R_{27} & R_{28} & R_{29} \\ R_{31} & R_{32} & R_{33} & R_{34} & R_{35} & R_{36} & R_{37} & R_{38} & R_{39} \\ R_{41} & R_{42} & R_{43} & R_{44} & R_{45} & R_{46} & R_{47} & R_{48} & R_{49} \\ R_{51} & R_{52} & R_{53} & R_{54} & R_{55} & R_{56} & R_{57} & R_{58} & R_{59} \\ R_{61} & R_{62} & R_{63} & R_{64} & R_{65} & R_{66} & R_{67} & R_{68} & R_{69} \\ R_{71} & R_{72} & R_{73} & R_{74} & R_{75} & R_{76} & R_{77} & R_{78} & R_{79} \\ R_{81} & R_{82} & R_{83} & R_{84} & R_{85} & R_{86} & R_{87} & R_{88} & R_{89} \\ R_{91} & R_{92} & R_{93} & R_{94} & R_{95} & R_{96} & R_{97} & R_{98} & R_{99} \end{pmatrix} \longrightarrow \begin{pmatrix} \begin{matrix} R^a_{11} & R^a_{12} & R^a_{13} \\ R^a_{21} & R^a_{22} & R^a_{23} \\ R^a_{31} & R^a_{32} & R^a_{33} \end{matrix} & 0 & 0 \\ 0 & \begin{matrix} R^b_{11} & R^b_{12} & R^b_{13} \\ R^b_{21} & R^b_{22} & R^b_{23} \\ R^b_{31} & R^b_{32} & R^b_{33} \end{matrix} & 0 \\ 0 & 0 & \begin{matrix} R^c_{11} & R^c_{12} & R^c_{13} \\ R^c_{21} & R^c_{22} & R^c_{23} \\ R^c_{31} & R^c_{32} & R^c_{33} \end{matrix} \end{pmatrix}$$

The aliased Fourier modes in geophysical systems with quadratic, advection-type nonlinearity are expected to be relatively weakly correlated.

In such systems the quadratic nonlinearities *do not directly couple the Fourier modes contained in the same aliasing set*; that is, if mode $\mathbf{k}$ is in the aliasing set $\mathcal{A}$, the quadratic couplings in the dynamics of $u_{\mathbf{k}}$ have the form

$$\frac{du_{\mathbf{k}}}{dt} \propto \sum u_{\mathbf{l}} u_{\mathbf{m}}, \qquad \mathbf{k} \in \mathcal{A}, \mathbf{l}, \mathbf{m} \notin \mathcal{A}$$

Dynamic Stochastic Superresolution of sparsely observed turbulent systems

References:

Branicki & Majda, "Dynamic Stochastic Superresolution of sparsely observed turbulent systems", *J. Comp. Phys.* 2012

Keating, Majda & Smith, "New methods for estimating poleward eddy heat transport using satellite altimetry", *Mon. Wea. Rev.* 2012

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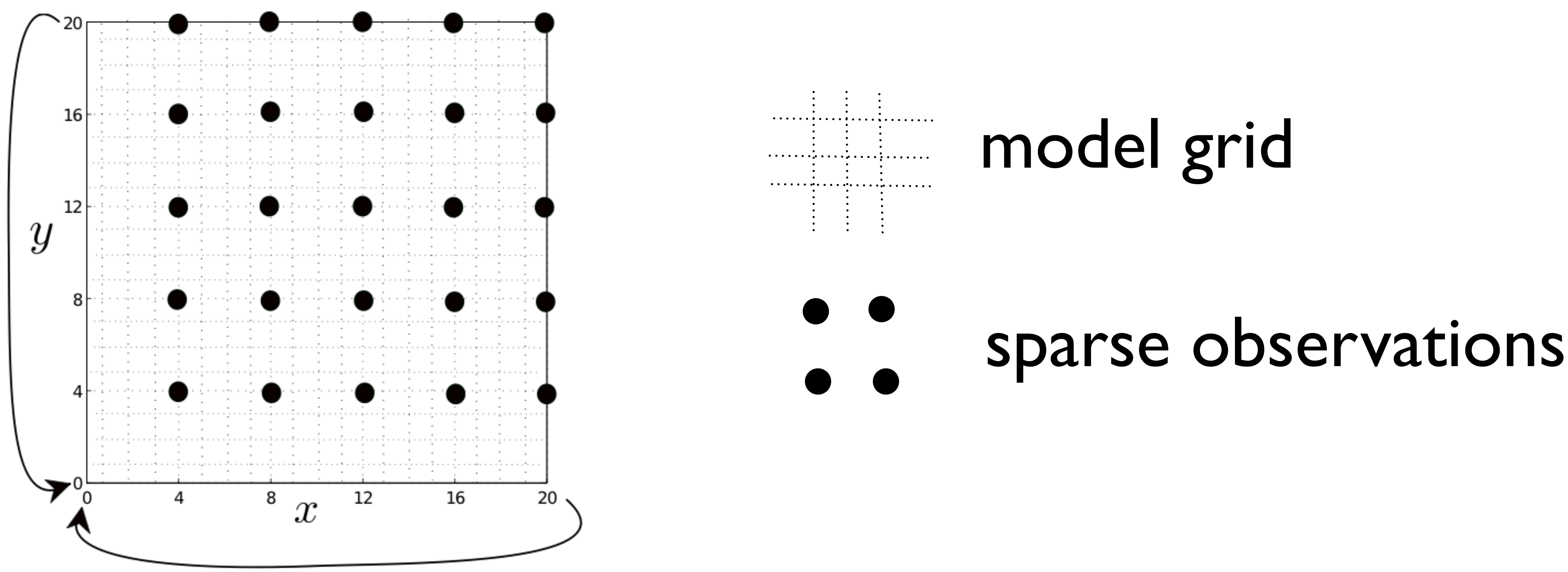
Harlim & Majda, "Mathematical Strategies for Filtering Complex Systems: Regularly Spaced Sparse Observations", *J. Comp. Phys.* 2008

Filtering sparsely observed spatially extended systems

Estimate the evolution of the true solution $u(t,x)$ of

$$\partial_t u = \mathcal{F}(u)$$

given sparse-in-space, discrete-in-time observations



and the model

$$\partial_t u^M = \mathcal{L}\left(\frac{\partial}{\partial x}\right) u^M + \sigma(s)\dot{W}(t)$$

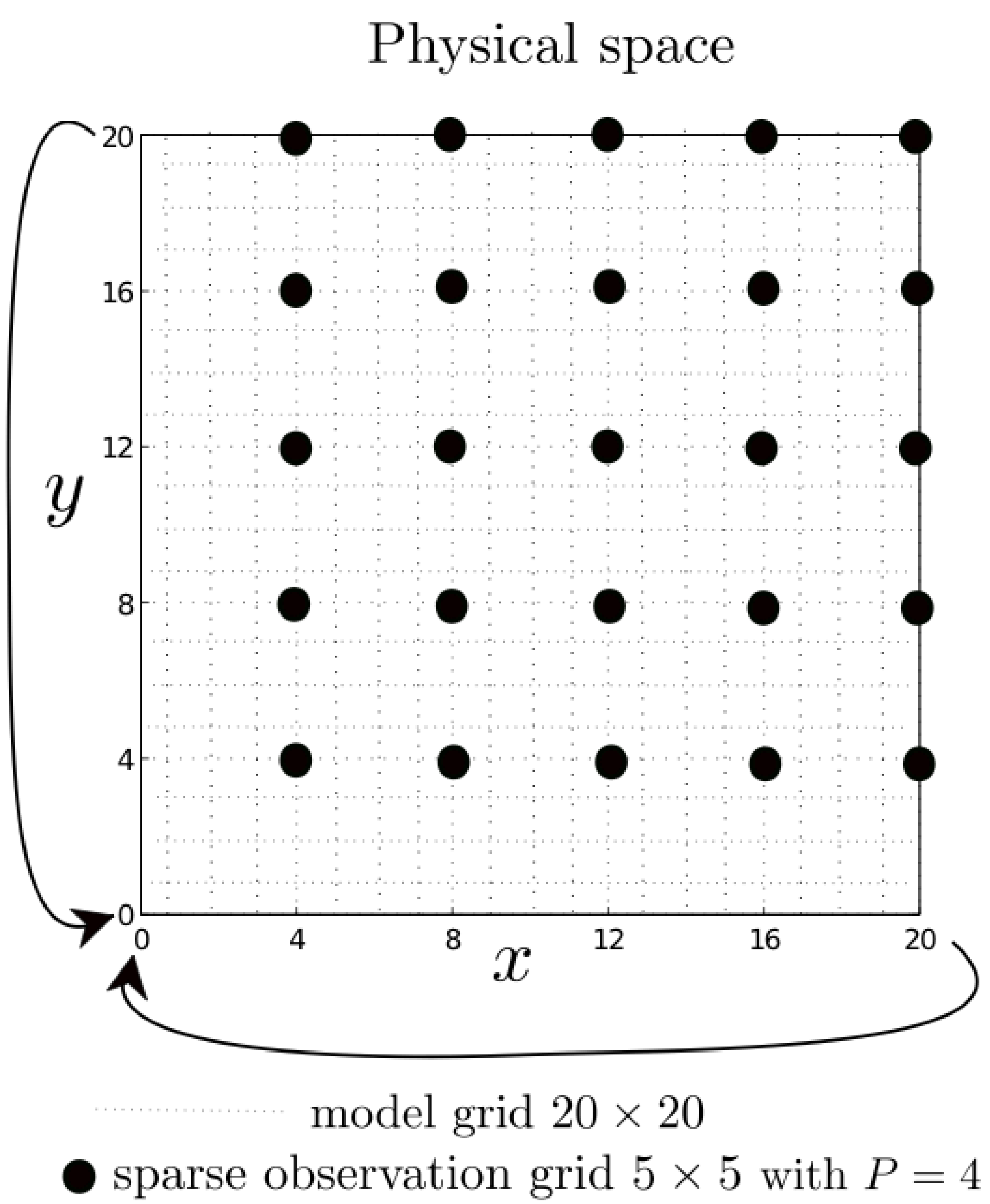
discretized on a regular grid.

Outline:

- DSS via Kalman Filtering in Fourier domain
  - DSS framework for sparsely observed turbulent systems
  - Aliasing - ally or enemy?
  - Examples of DSS skill in spatially extended systems
    - Black swan detection in synthetic examples
    - DSS in 1D turbulent spatially extended systems
    - DSS in 2D turbulent spatially extended systems

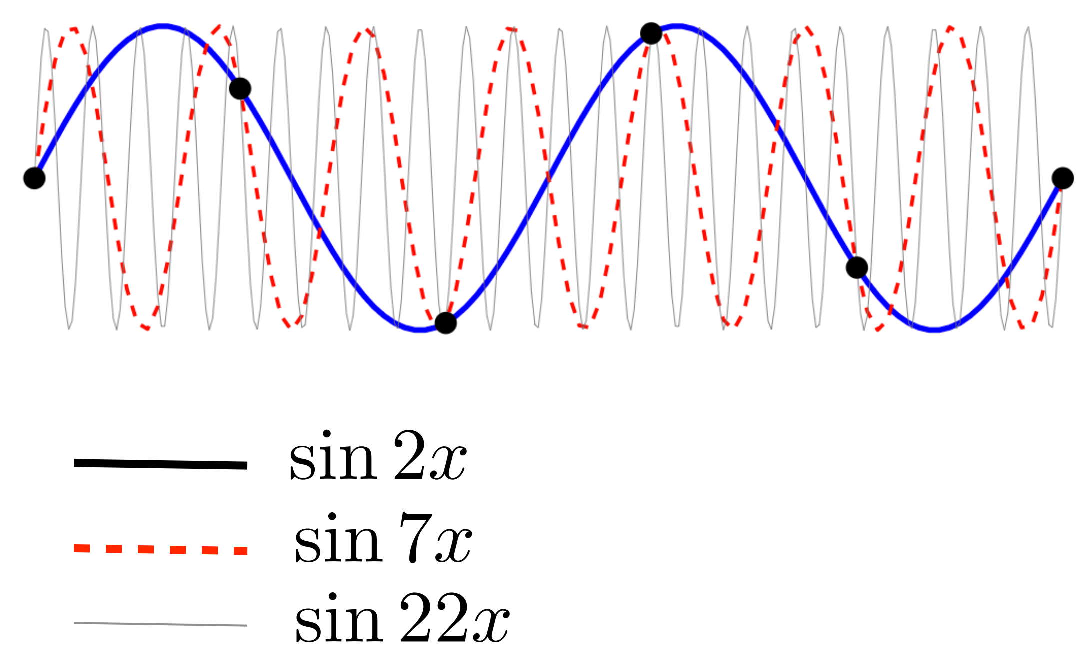
      (eddy heat flux estimation from sparse satellite altimetry)

Consequences of assimilating sparse observations



Aliasing

sparse regular observations alias higher wavenumber information into the resolved wavenumber band



Aliased observations can be used to estimate the unresolved modes

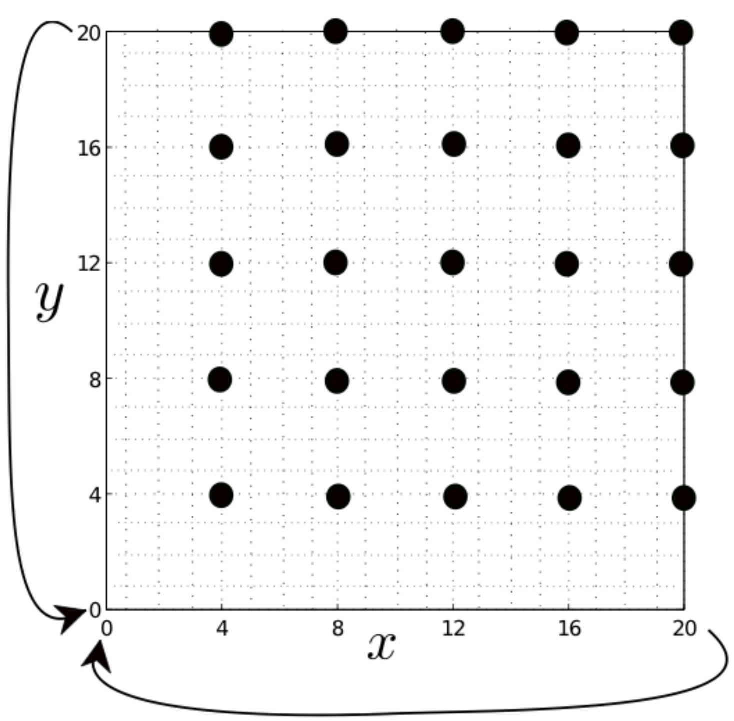
Consequences of assimilating sparse observations

Sparse obs in physical space

$$v_{(i,j)m} = G\, u\big(x_i, y_j, m\Delta t\big) + \sigma_m^{obs}$$

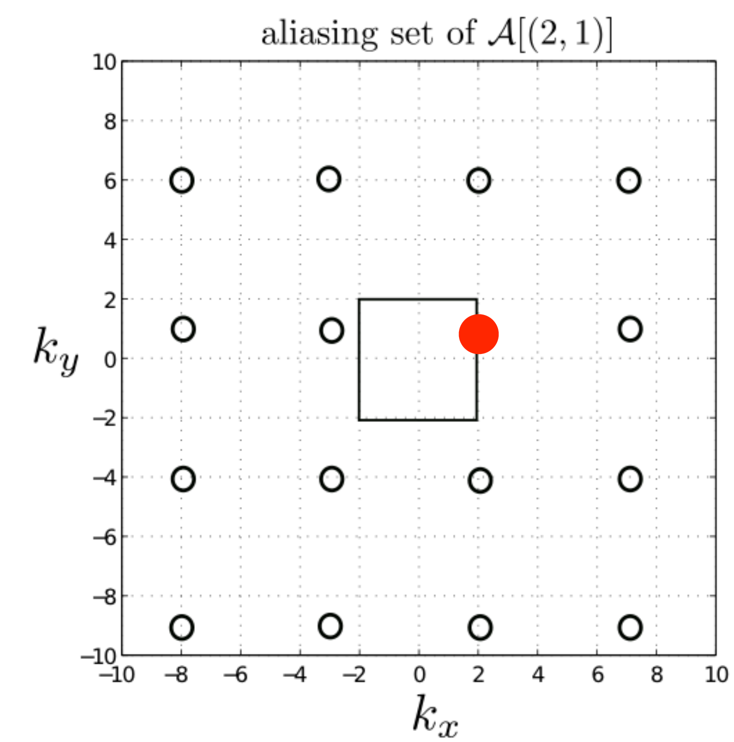
model grid
$\tilde{N} \times \tilde{N}$

observation grid
$\tilde{M} \times \tilde{M}$



Fourier space

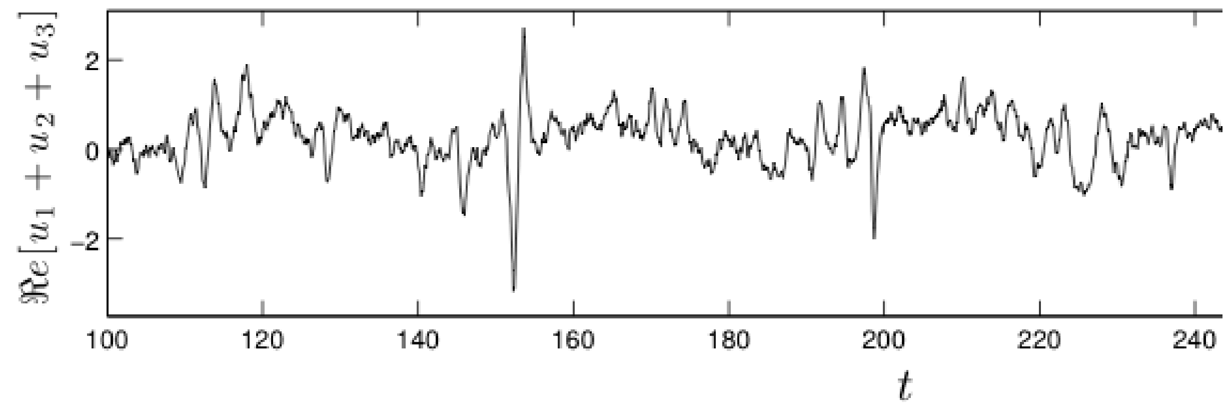
$$\hat{v}_{\{l\}m} = G \sum_{k \in \mathcal{A}(l)} \hat{u}_{k,m} + \sigma_{\{l\}m}^{obs} / \tilde{M}^2$$



Aliasing set of wavenumber $\boldsymbol{l} = (l_x, l_y), \mathbb{Z} \ni |l_x|, |l_y| \leqslant \tilde{M}/2$

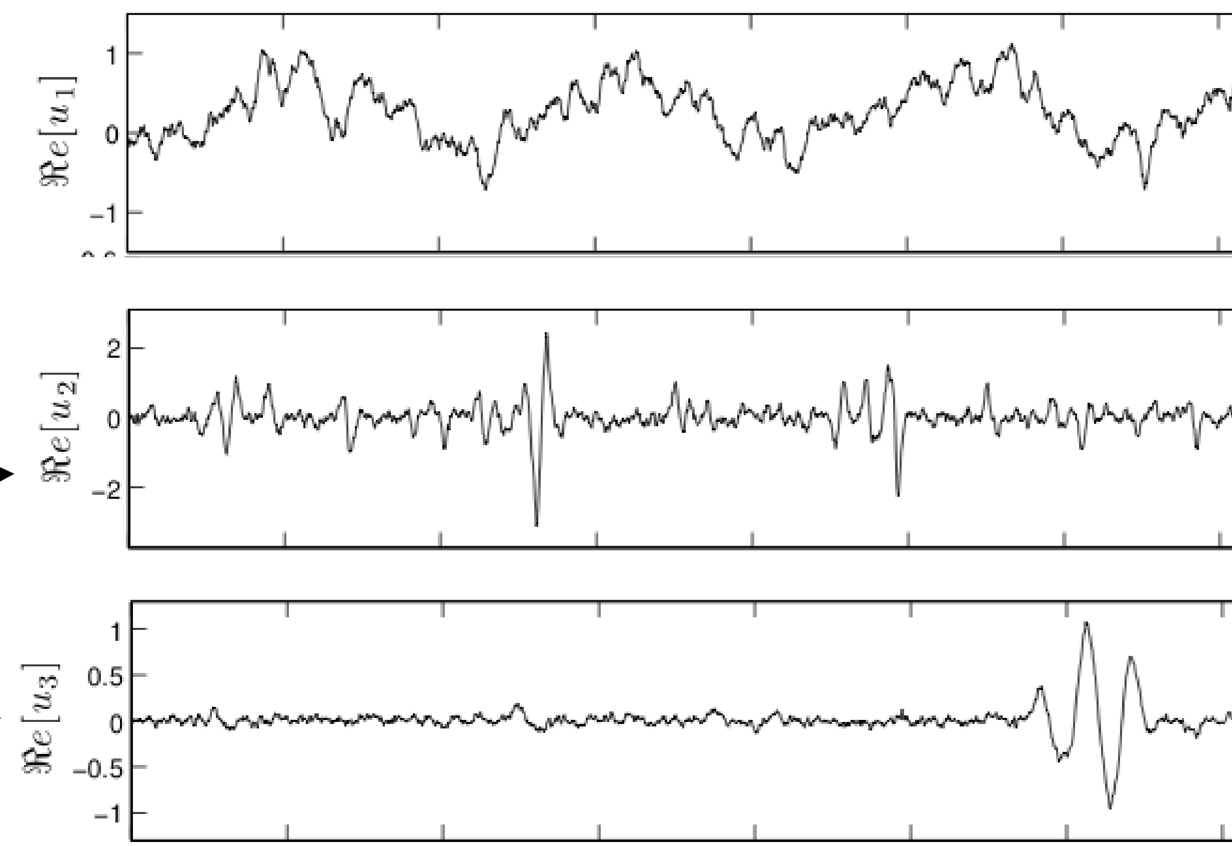
$$\mathcal{A}(\boldsymbol{l}) = \Big\{ \boldsymbol{k} = (k_x, k_y) : \;\; \mathbb{Z} \ni |k_x|, |k_y| \leqslant \tilde{N}/2, \quad k_x = l_x + \tilde{M} q_x, \quad k_y = l_y + \tilde{M} q_y, \quad q_x, q_y \in \mathbb{Z} \Big\}$$

Observed aliased signal



Use aliasing to superresolve the signal "on the fly"

Aliasing modes



Stochastic model with judicious error + Kalman filter

Do we gain from estimating the aliased modes?

YES, when $\langle |u_1|^2 \rangle \sim \langle |u_2|^2 \rangle \sim \langle |u_3|^2 \rangle$

NO, when $\langle |u_1|^2 \rangle \gg \max(\langle |u_2|^2 \rangle, \langle |u_3|^2 \rangle)$

The stochastic forecast model in DSS

Number of judicious simplifications introduced in the model dynamics and its statistics

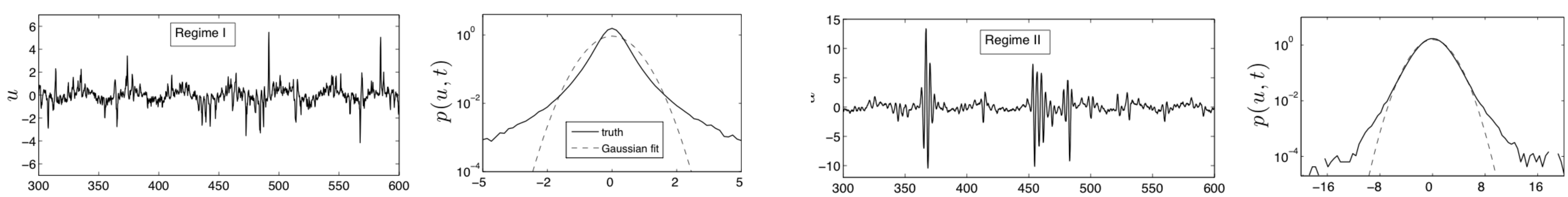
- SPEKF forecast model (Gershgorin et al. 2010)

$$\mathrm{d}\hat{u}_k(t) = \Big[\big(-\gamma_k(t) + i\omega_k\big)\hat{u}_k(t) + b_k(t) + F_k(t)\Big]\mathrm{d}t + \sigma_{u_k}\mathrm{d}W_{u_k}(t),$$

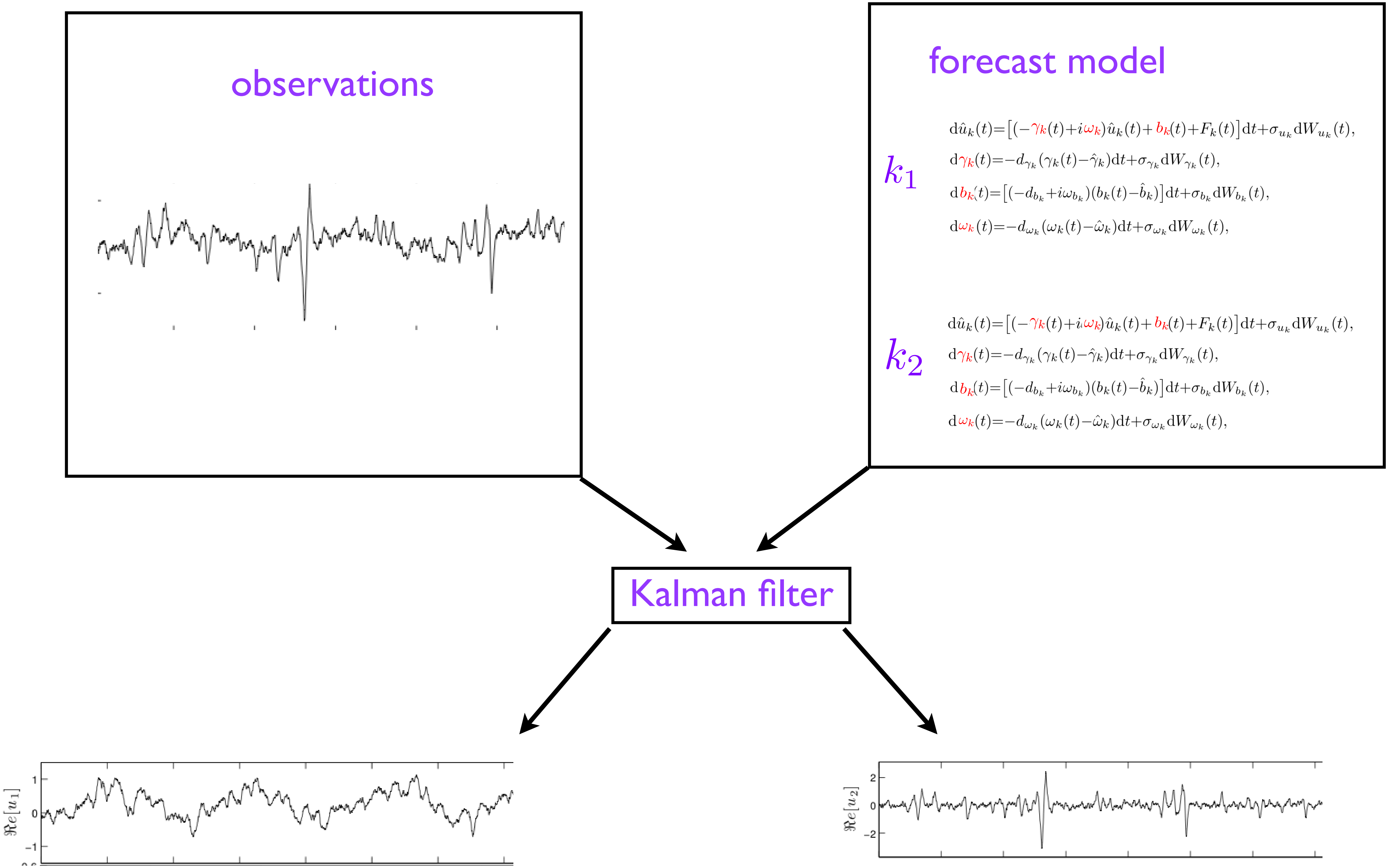
$$\mathrm{d}\gamma_k(t) = -d_{\gamma_k}\big(\gamma_k(t) - \hat{\gamma}_k\big)\mathrm{d}t + \sigma_{\gamma_k}\mathrm{d}W_{\gamma_k}(t),$$

$$\mathrm{d}b_k(t) = \Big[\big(-d_{b_k} + i\omega_{b_k}\big)\big(b_k(t) - \hat{b}_k\big)\Big]\mathrm{d}t + \sigma_{b_k}\mathrm{d}W_{b_k}(t),$$

$$\mathrm{d}\omega_k(t) = -d_{\omega_k}\big(\omega_k(t) - \hat{\omega}_k\big)\mathrm{d}t + \sigma_{\omega_k}\mathrm{d}W_{\omega_k}(t),$$



DSS through Kalman filtering

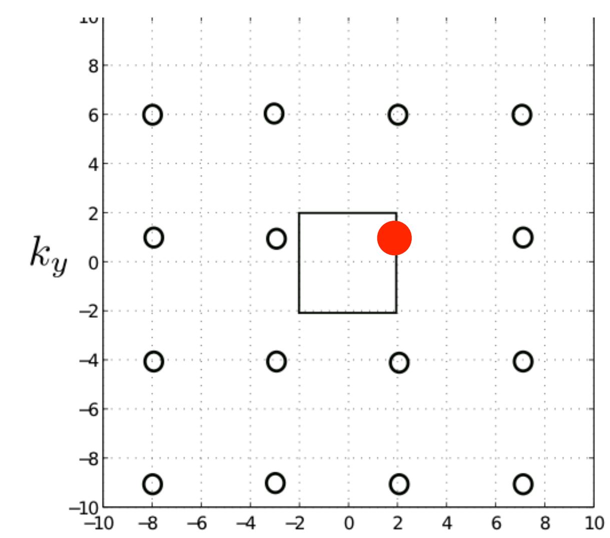


DSS via continuous-discrete Kalman filtering in Fourier space

Estimate the aliasing modes ○ ● using
cheap exactly solvable **non-Gaussian forecast** and **Gaussian** approximation for the **posterior.**

Exactly solvable forecast in each aliasing set $\mathcal{A}(l)$

Harlim & Majda 2008



$$\bar{\boldsymbol{u}}_{m+1|m} = \mathcal{F}_{m+1}[\bar{\boldsymbol{u}}_{m|m}, R_{m|m}]$$

$$R_{m+1|m} = \mathcal{C}_{m+1}[\bar{\boldsymbol{u}}_{m|m}, R_{m|m}]$$

Observations:

$$\boldsymbol{v}_{m+1} = \sum_{k \in \mathcal{A}(l)} u_{k,m+1} + \hat{\sigma}_{m+1}$$

Kalman update:

$$\bar{\boldsymbol{u}}_{m+1|m+1} = \bar{\boldsymbol{u}}_{m+1|m} + K_{m+1}\big(v_{m+1} - \sum_{k \in \mathcal{A}(l)} \bar{u}_{k,m+1|m}\big)$$

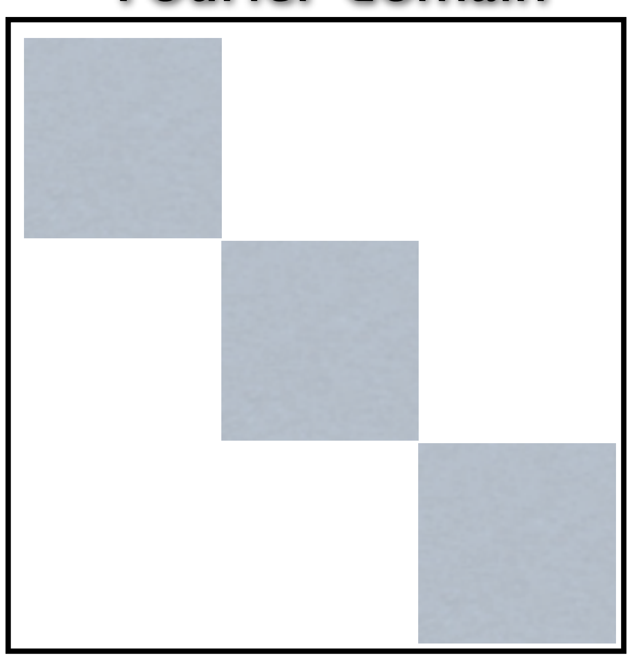
$$R_{m+1|m+1} = R_{m+1|m} - \Lambda(R_{m+1|m}, r^{obs})R_{m+1|m}(G_P G_P^T)R_{m+1|m}$$

Dynamic Stochastic Superresolution (DSS)

Algorithms:

- Diagonal Multi-SPEKF (**m-SPEKF**)
- Exact mean SPEKF+Monte Carlo covariances (**MC SPEKF**) Keating, Majda, Smith, *MWR* 2012
- Multi-SPEKF with Gaussian closure (**GCSSF**) (Branicki&Majda, *JCP* 2012)

Covariance in Fourier domain



GCSSF & MC SPEKF

$$\begin{matrix} R_{11} & R_{12} & R_{13} & R_{14} \\ R_{21} & R_{22} & R_{23} & R_{24} \\ R_{31} & R_{32} & R_{33} & R_{34} \\ R_{41} & R_{42} & R_{43} & R_{44} \end{matrix}$$

m-SPEKF

$$\begin{matrix} R_{11} & & & \\ & R_{22} & & \\ & & R_{33} & \\ & & & R_{44} \end{matrix}$$

GCSSF is superior robust algorithm but both m-SPEKF and MC SPEKF can have significant filtering skill in appropriate settings

Approximate second-order statistics in DSS algorithms

$$d\hat{u}_k(t) = \left[(-\gamma_k(t) + i\omega_k)\hat{u}_k(t) + b_k(t) + F_k(t)\right]dt + \sigma_{u_k} dW_{u_k}(t),$$

$$db_k(t) = \left[(-d_{b_k} + i\omega_{b_k})(b_k(t) - \hat{b}_k)\right]dt + \sigma_{b_k} dW_{b_k}(t),$$

$$d\gamma_k(t) = -d_{\gamma_k}(\gamma_k(t) - \hat{\gamma}_k)dt + \sigma_{\gamma_k} dW_{\gamma_k}(t),$$

$$d\omega_k(t) = -d_{\omega_k}(\omega_k(t) - \hat{\omega}_k)dt + \sigma_{\omega_k} dW_{\omega_k}(t),$$

$$\boldsymbol{v} = \{\hat{u}_k, b_k, \gamma_k, \omega_k\} \quad k \in \mathcal{A}(l)$$

$$a) \quad \dot{\bar{\boldsymbol{v}}} = \boldsymbol{f}(\bar{\boldsymbol{v}}) + \overline{\mathbf{B}(\boldsymbol{v}', \boldsymbol{v}')},$$

$$b) \quad \dot{R} = R\,A^T(\bar{\boldsymbol{v}}) + A(\bar{\boldsymbol{v}})R + Q + \overline{\boldsymbol{v}'\mathbf{B}^T(\boldsymbol{v}', \boldsymbol{v}') + \mathbf{B}(\boldsymbol{v}', \boldsymbol{v}')\boldsymbol{v}'^T},$$

- Gaussian Closure $\overline{\boldsymbol{v}'_M \mathbf{B}^T(\boldsymbol{v}'_M, \boldsymbol{v}'_M) + \mathbf{B}(\boldsymbol{v}'_M, \boldsymbol{v}'_M)\boldsymbol{v}'^T_M} = 0$

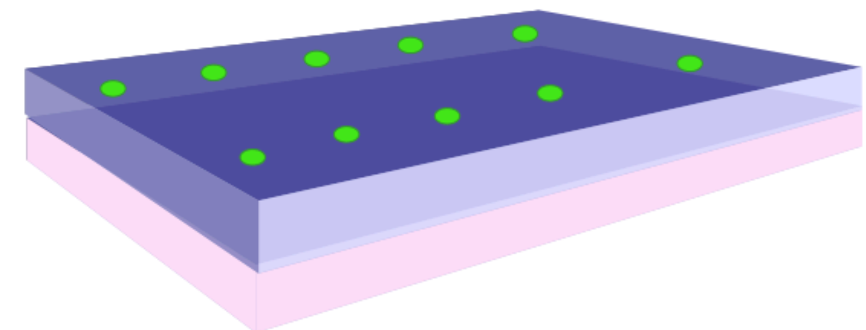
$$a) \quad \dot{\bar{\boldsymbol{v}}}_M = \boldsymbol{f}(\bar{\boldsymbol{v}}_M) + \overline{\mathbf{B}(\boldsymbol{v}'_M, \boldsymbol{v}'_M)},$$

$$b) \quad \dot{R}_M = R_M\,A^T(\bar{\boldsymbol{v}}_M) + A(\bar{\boldsymbol{v}}_M)\,R_M + Q_M,$$

- assume odd moments zero
- keep turbulent fluxes in the mean

DSS for estimating poleward eddy heat transport from sparse observations

Task: estimate a quadratic, sign indefinite quantity from sparse observations of the ocean surface



(Two layer) Philips model

$$\partial_t q_i + J(\psi_i, q_i) + U_i \partial_x q_i + \Pi_i \partial_x \psi_i = -\delta_{i2} r \nabla^2 \psi_i, \qquad i = 1, 2,$$

$$q_i = \nabla^2 \psi_i + (R^2 d_i)^{-1} (\psi_{3-i} - \psi_i), \qquad R = \sqrt{g' H_0} / f_0$$

Heat flux

$$\langle v_1 \tau \rangle = -\left(\frac{d_1}{d_2}\right)^{1/2} k_D^{-2} \langle v_1 q_1 \rangle$$

$$\tau = \sqrt{d_1 d_2} (\psi_1 - \psi_2)$$

DSS for estimating poleward eddy heat transport from sparse observations

EOF   stream

$$\begin{pmatrix} \tilde{\chi}^{+}_{kl} \\ \tilde{\chi}^{-}_{kl} \end{pmatrix} = \mathsf{V}(k,l) \begin{pmatrix} \tilde{\psi}^{1}_{kl} \\ \tilde{\psi}^{2}_{kl} \end{pmatrix}$$

Partial observations

$$\tilde{\psi}^{\mathrm{obs}}_{kl} = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{pmatrix} \tilde{\psi}^{1}_{kl} \\ \tilde{\psi}^{2}_{kl} \end{pmatrix} + \tilde{\sigma}^{\mathrm{obs}}_{kl}, \qquad \tilde{\psi}^{1,2}_{k_i,l_j} = \sum_{i,j=1}^{P} \hat{\psi}^{1,2}_{k_i,l_j}$$

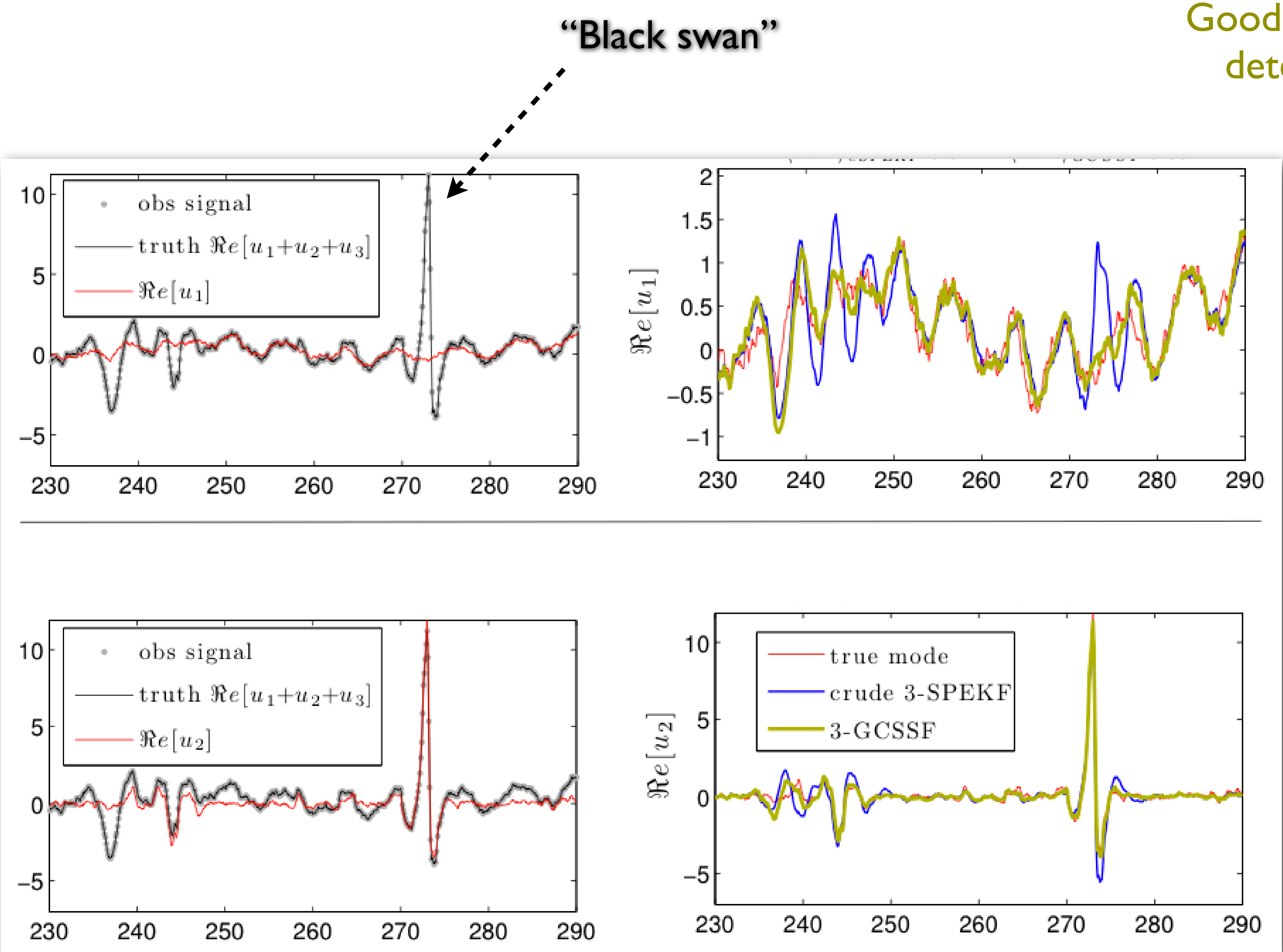
SPEKF   SPEKF   SPEKF   SPEKF

↑   ↑   ↑   ↑

$$\boldsymbol{z}_{kl} = \left[\hat{\chi}^{+}_{k_1 l_1}, \; \hat{\chi}^{-}_{k_1 l_1}, \; \cdots \; \hat{\chi}^{+}_{k_P l_P}, \; \hat{\chi}^{-}_{k_P l_P}\right]^T$$

DSS for uncorrelated modes in the aliasing set

Good “Black swan” detection with GCSSF



References:

Branicki & Majda, Dynamic Stochastic Superresolution of sparsely observed turbulent systems, *J. Comp. Phys.* 2012

Keating, Majda & Smith, New methods for estimating poleward eddy heat transport using satellite altimetry, *Mon. Wea. Rev.* 2012

Majda & Harlim, *Filtering Complex Turbulent Systems*, Cambridge Univ. Press 2012

Harlim & Majda, Filtering Turbulent Sparsely Observed Geophysical Flows, *Mon. Wea. Rev.* 2010

Harlim & Majda, Mathematical Strategies for Filtering Complex Systems: Regularly Spaced Sparse Observations, *J. Comp. Phys.* 2008

DSS in nonlinear turbulent spatially extended systems

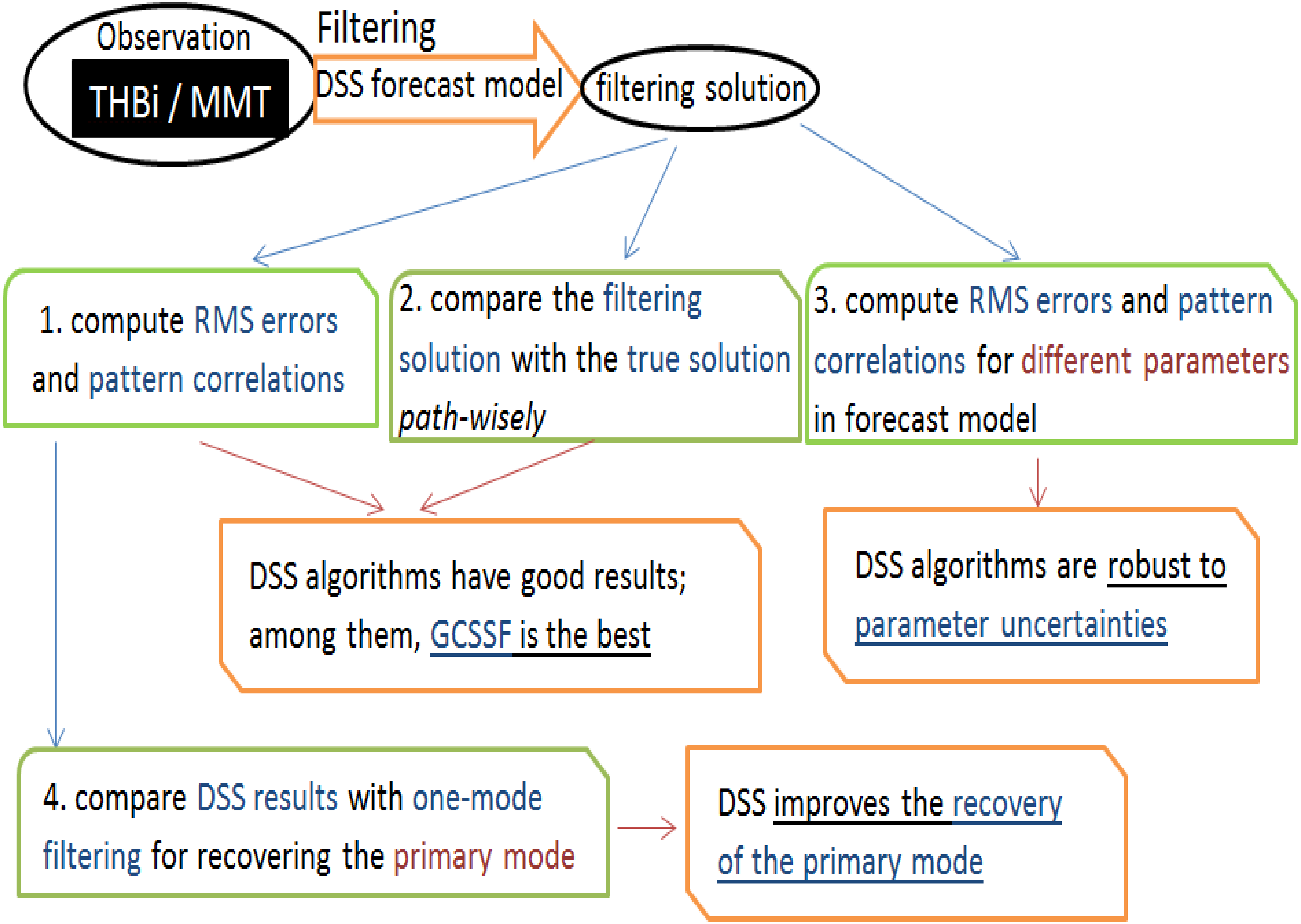
Performance of DSS algorithms on TBHi model and MMT model

Lecturer: Chenyue WU

Nov 15, 2012

From M. Branicki and A. J. Majda, “Dynamic Stochastic Superresolution of sparsely observed turbulent systems”

Overview



Observation
THBi / MMT
Filtering
DSS forecast model
filtering solution
1. compute RMS errors and pattern correlations
2. compare the filtering solution with the true solution path-wisely
3. compute RMS errors and pattern correlations for different parameters in forecast model
DSS algorithms have good results; among them, GCSSF is the best
DSS algorithms are robust to parameter uncertainties
4. compare DSS results with one-mode filtering for recovering the primary mode
DSS improves the recovery of the primary mode

Truncated Burgers-Hilbert (TBHi) equation

inviscid Burgers-Hopf equation with dispersive terms added via Hilbert transform

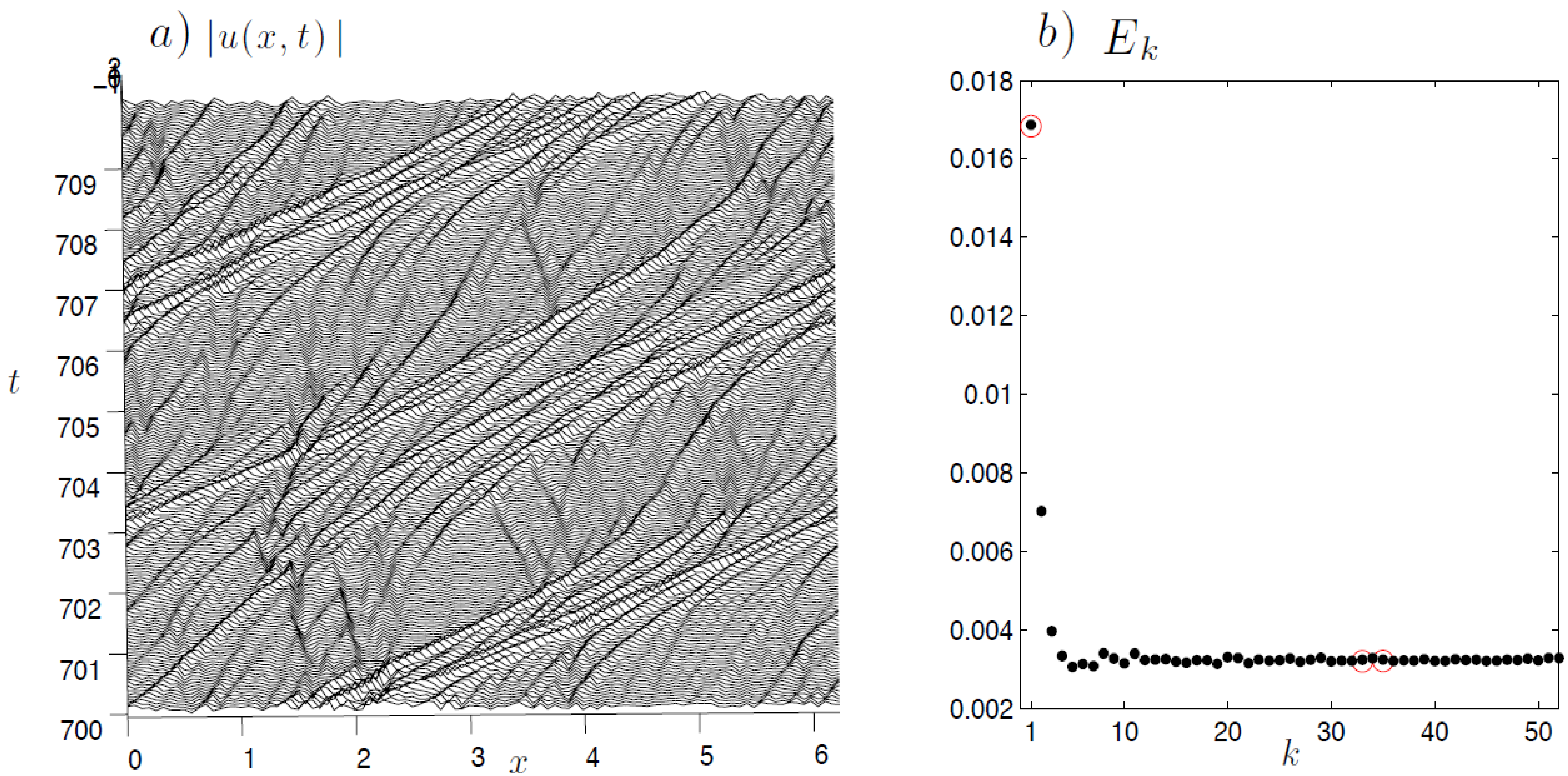
$$u_t + \frac{1}{2}(u^2)_x = \alpha H[u]$$

H is defined by: $\hat{H}[f](k) = -i \operatorname{sgn}(k)\hat{f}(k)$. in Fourier space.

Truncated form

$$(u_\Lambda)_t + \frac{1}{2}P_\Lambda(u_\Lambda^2)_x = \alpha H[u_\Lambda]$$

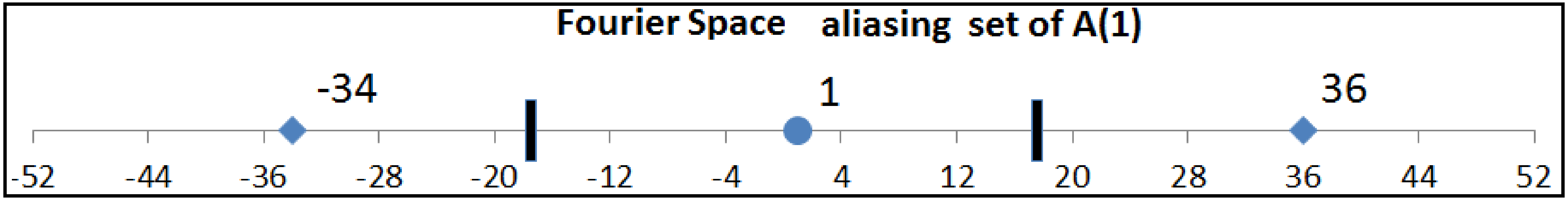
TBHi $(u_\Lambda)_t + \frac{1}{2}P_\Lambda(u_\Lambda^2)_x = H[u_\Lambda]$



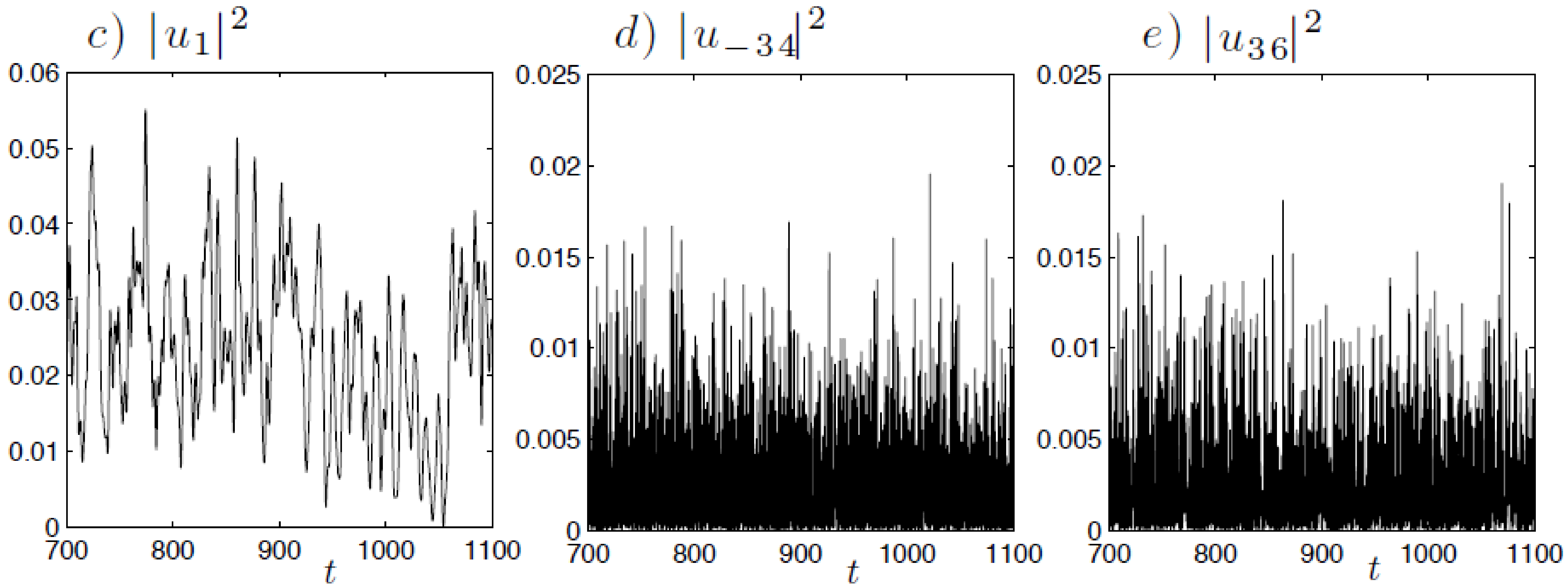
- energy-conserving
- near-equipartition of energy

TBHi $(u_\Lambda)_t + \frac{1}{2} P_\Lambda (u_\Lambda^2)_x = H[u_\Lambda]$

Sparse observation network: $P = 3$, $\tilde{N} = 105$, $\tilde{M} = 35$.



$\mathcal{A}(1) = \{1, -34, 36\}$. energy $E = (0.017, 0.003, 0.003)$.
decorrelation time $\tau_{corr} = (3.5, 0.1, 0.1)$.



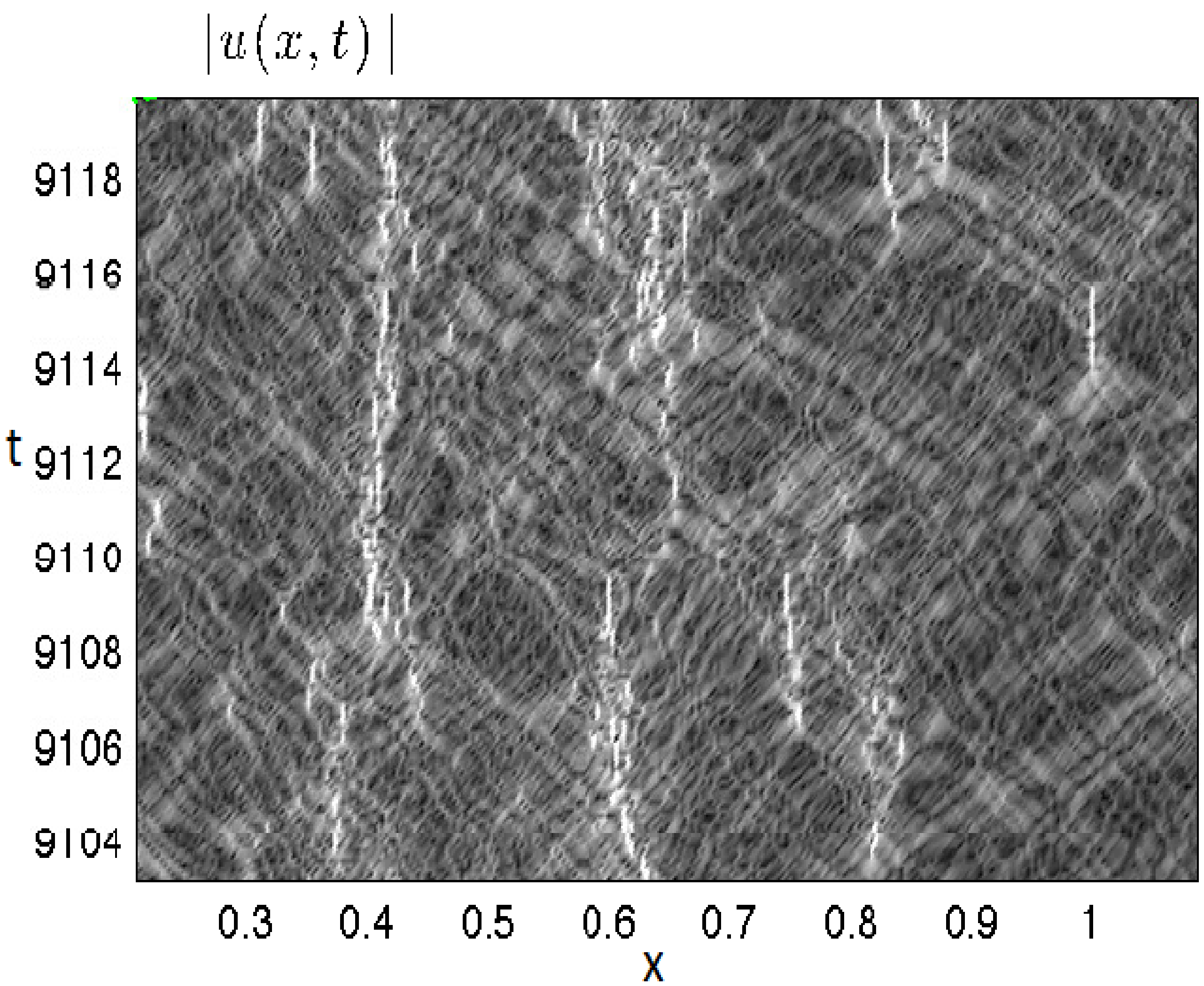
MMT equation

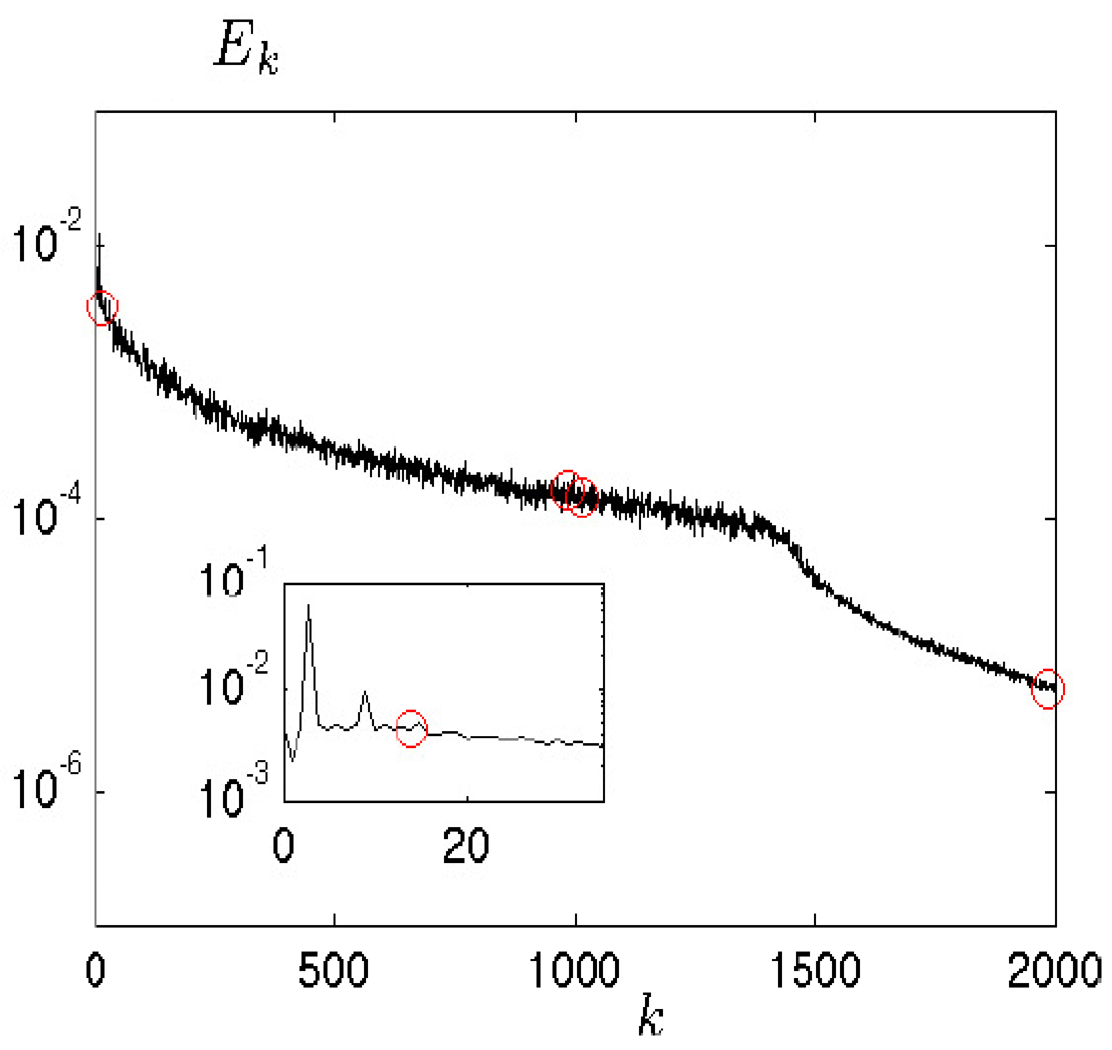
MMT equation

$$iu_t = |\partial_x|^{\frac{1}{2}} u + \lambda |u|^2 u - iAu + F$$

Here, we consider the case with the focusing nonlinearity, $\lambda = -1$, which induces spatially coherent 'solitonic' excitations at random spatial locations.

MMT $iu_t = |\partial_x|^{\frac{1}{2}} u - |u|^2 u - iAu + F$

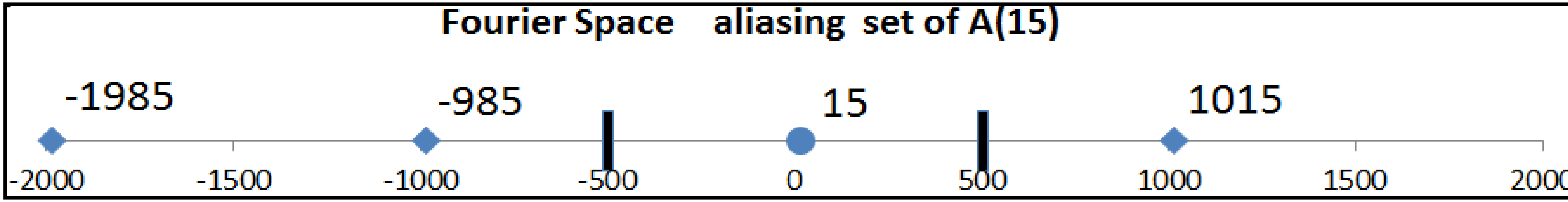




- forced and dissipative
- dispersive
- decaying energy spectrum

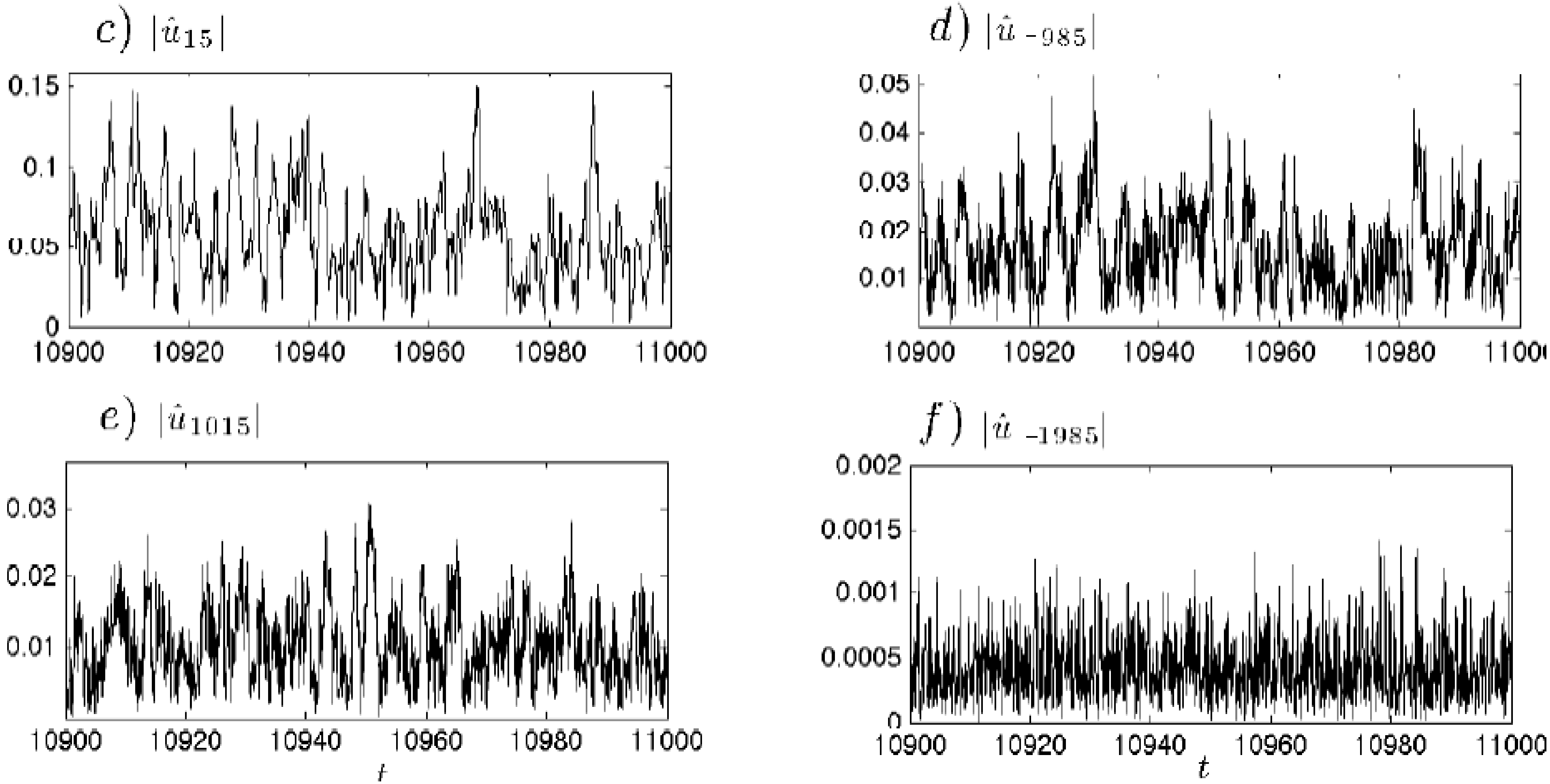
MMT $iu_t = |\partial_x|^{\frac{1}{2}} u - |u|^2 u - iAu + F$

Sparse observation network: $P = 4$, $\tilde{N} = 4000$, $\tilde{M} = 1000$.



$\mathcal{A}(15) = \{15, -985, 1015, -1985\}$.

$E = (3.5, 0.3, 0.15, 0.005) \times 10^{-3}$. $\tau_{corr} = (1, 0.015, 0.01)$



parameter estimate for DSS

How do we choose the parameter for DDS algorithm?

$$(a) \quad \mathrm{d}\hat{u}_k(t) = \big[(-\gamma_k(t) + i\omega_k)\hat{u}_k(t) + b_k(t) + F_k(t)\big]\mathrm{d}t + \sigma_{u_k}\mathrm{d}W_{u_k}(t),$$

$$(b) \quad \mathrm{d}b_k(t) = \big[(-d_{b_k} + i\omega_{b_k})(b_k(t) - \hat{b}_k)\big]\mathrm{d}t + \sigma_{b_k}\mathrm{d}W_{b_k}(t),$$

$$(c) \quad \mathrm{d}\gamma_k(t) = -d_{\gamma_k}(\gamma_k(t) - \hat{\gamma}_k)\mathrm{d}t + \sigma_{\gamma_k}\mathrm{d}W_{\gamma_k}(t),$$

$$(d) \quad \mathrm{d}\omega_k(t) = -d_{\omega_k}(\omega_k(t) - \hat{\omega}_k)\mathrm{d}t + \sigma_{\omega_k}\mathrm{d}W_{\omega_k}(t),$$

Actually, only rough estimates for these parameters are needed to achieve nearly optimal DSS skill.

parameter estimate for DSS

The parameters in SPEKF are estimated through the linear regression to the climetology.

We have the Mean Stochastic Model (MSM)

$$du_k^{\mathrm{MSM}} = -(\gamma_k^{\mathrm{MSM}} + i\omega_k^{\mathrm{MSM}})u_k^{\mathrm{MSM}}dt + \sigma_k^{\mathrm{MSM}}dW_k^{\mathrm{MSM}},$$

and set:

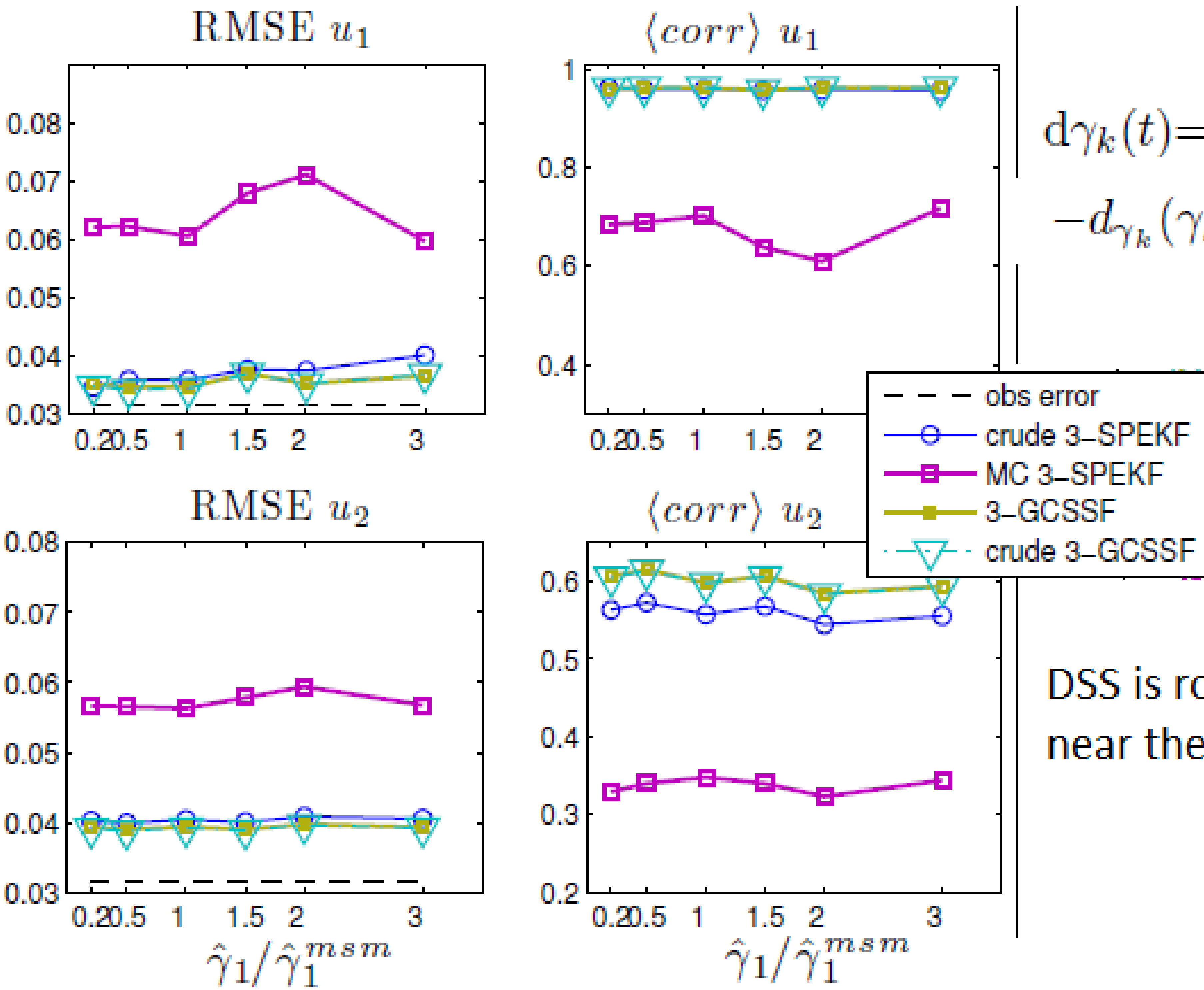
$$\hat{\gamma}_k = \gamma_k^{MSM}, \quad \hat{\omega}_k = \omega_k^{MSM}, \quad \hat{b}_k = 0$$

We set:

$$d_{\gamma_k} = d_{b_k} = 0.1\hat{\gamma}_k, \quad \omega_{b_k} = \omega_k, \quad \sigma_{\gamma_k} = \sigma_{b_k} = 4\sigma_k$$

Robustness of DSS algorithms to parameter uncertainties

RMS error and pattern correlation for the TBHi system as a function of the mean damping parameters in the forecast models.



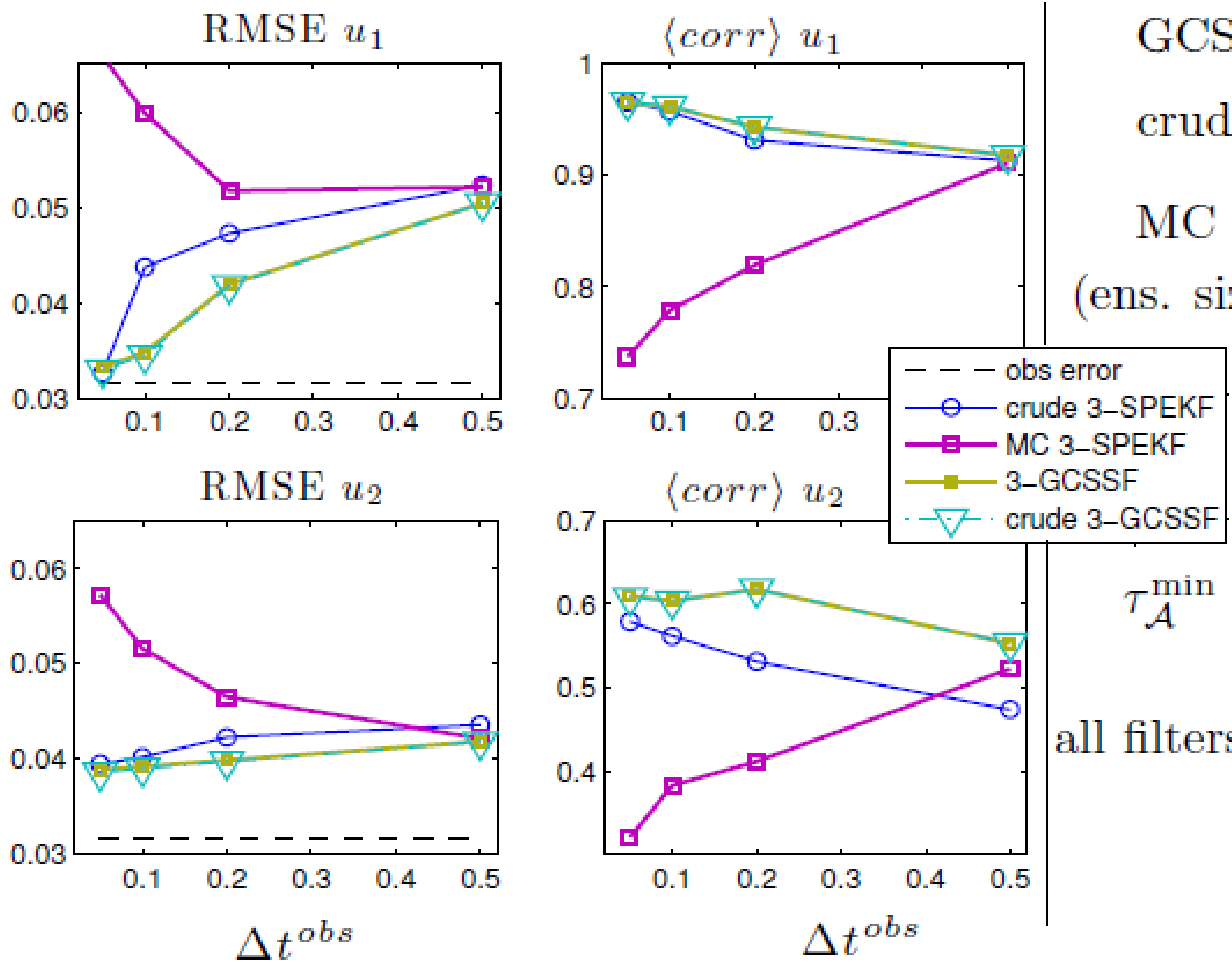
$$d\gamma_k(t) = -d_{\gamma_k}(\gamma_k(t) - \hat{\gamma}_k)dt + \sigma_{\gamma_k} dW_{\gamma_k}(t)$$

DSS is robust and near-optimal near the MSM parameter values

RMS error and pattern correlation for the TBHi system as a function of observation time

$r^o = 0.003 \quad E_1 = (0.017, 0.003, 0.003)$

$\tau_{corr} = (3.5, 0.1, 0.1)$



$\Delta t^{obs} \lesssim \tau_{\mathcal{A}}^{\min}$:

GCSSF >

crude mSPEKF >

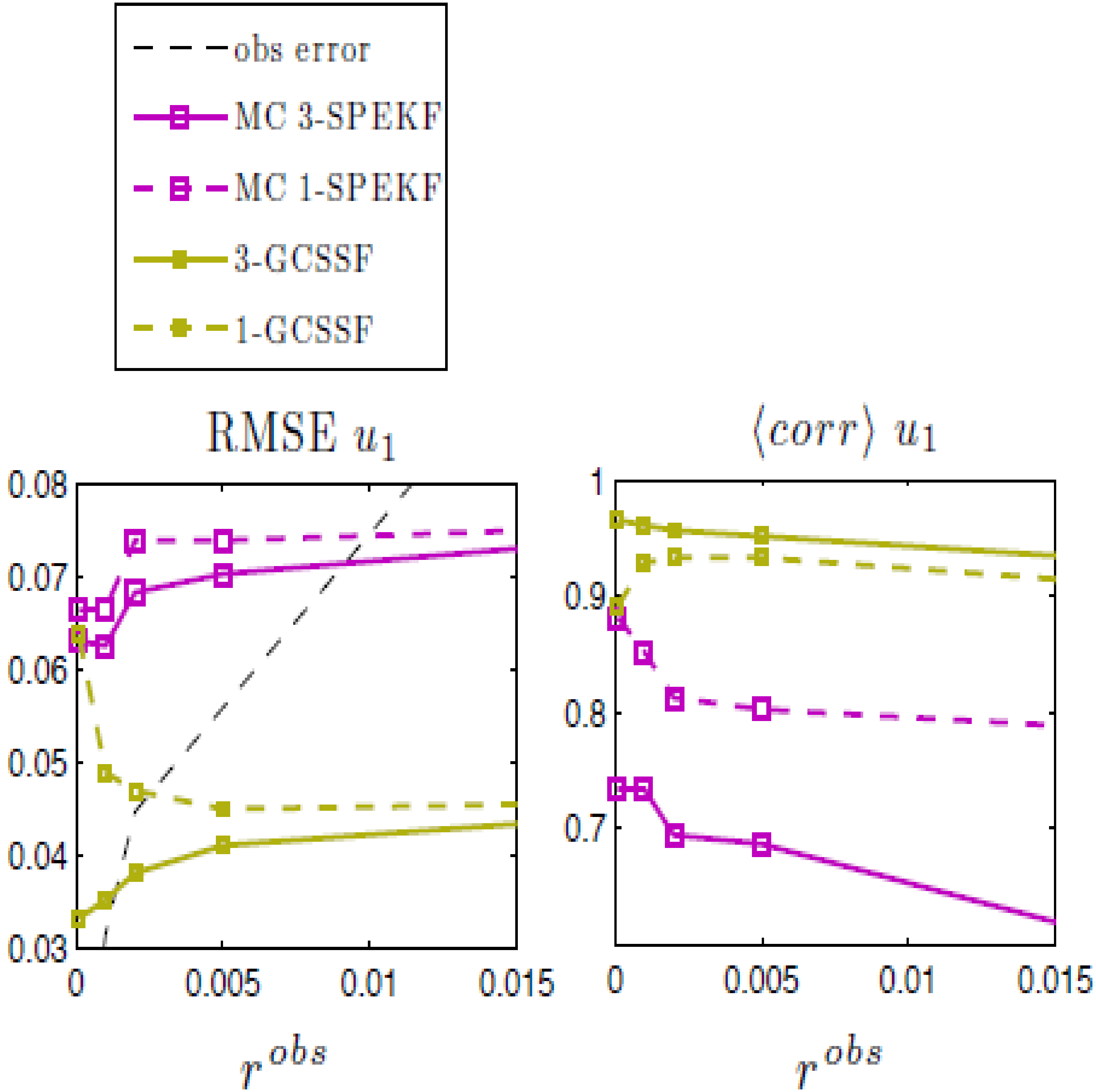
MC mSPEKF

(ens. size $N \sim \mathcal{O}(100)$);

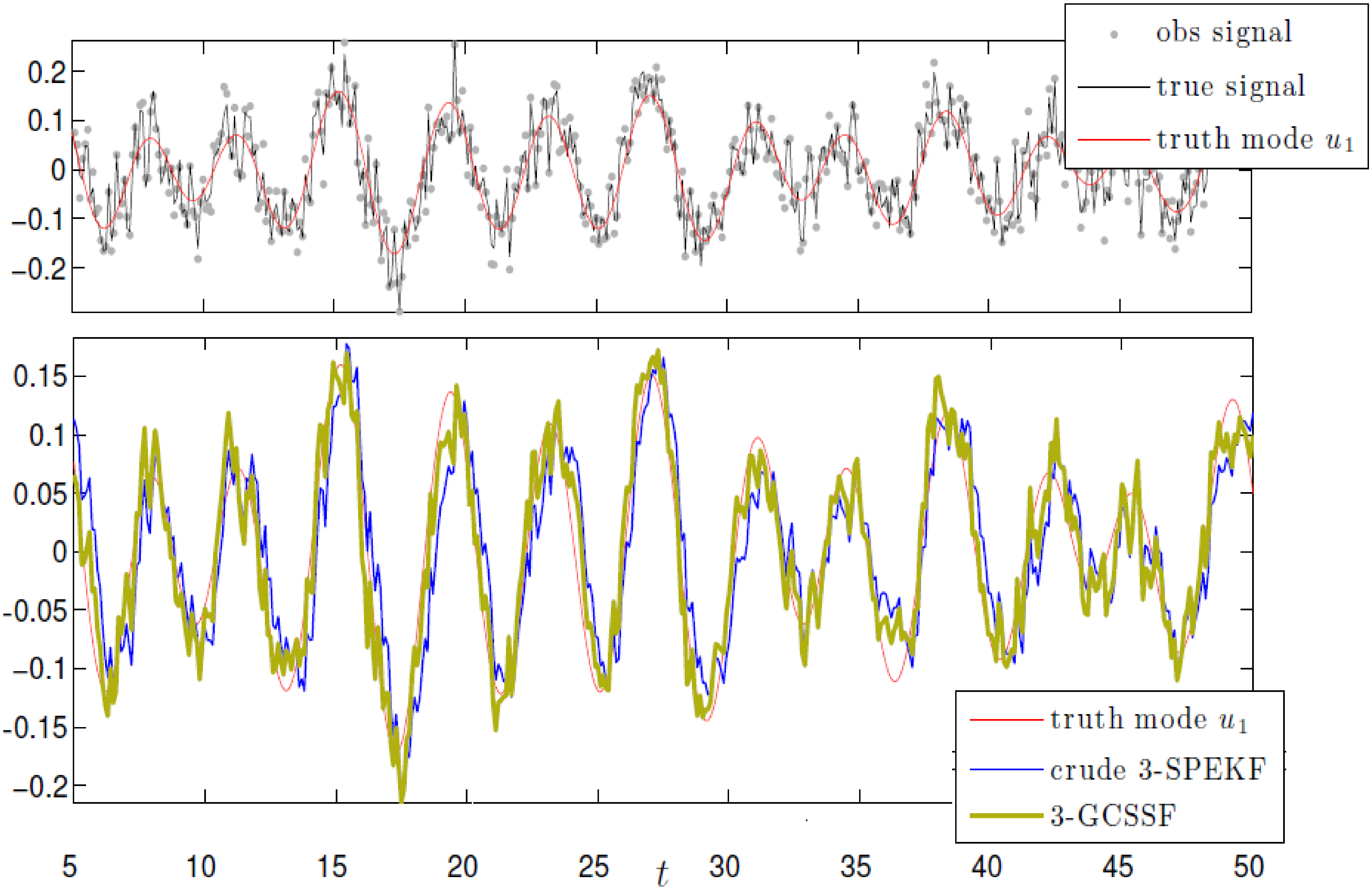
$\tau_{\mathcal{A}}^{\min} \ll \Delta t^{obs} \lesssim \tau_{\mathcal{A}}^{\max}$:

all filters are comparable

Comparison of the DSS results and one-mode filtering for recovering primary mode u_1 of TBHi



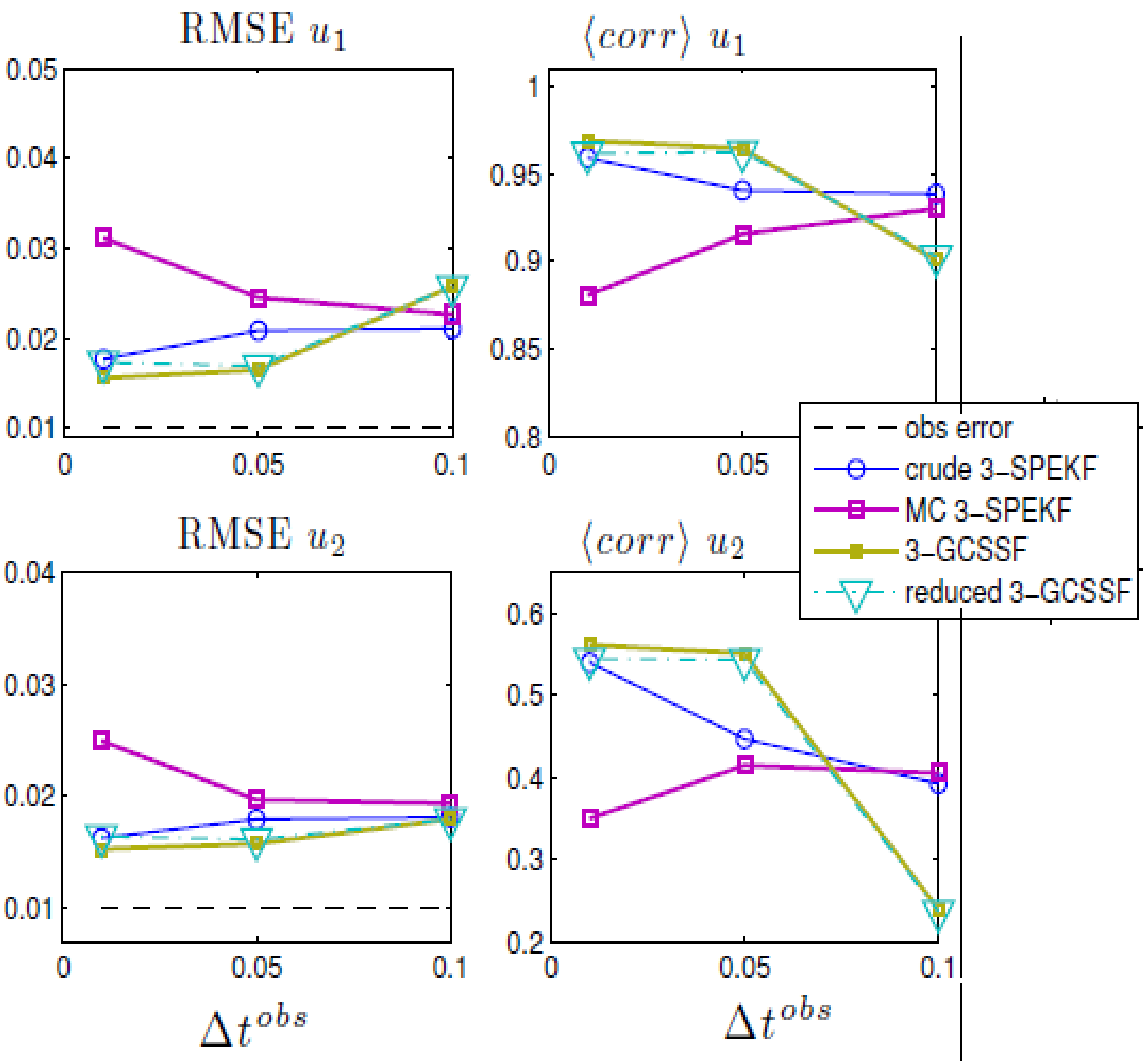
Path-wise example of DSS for the primary mode u_1 recovered from the aliased signal from **THBi** system involving u_1, u_{-34}, u_{36}



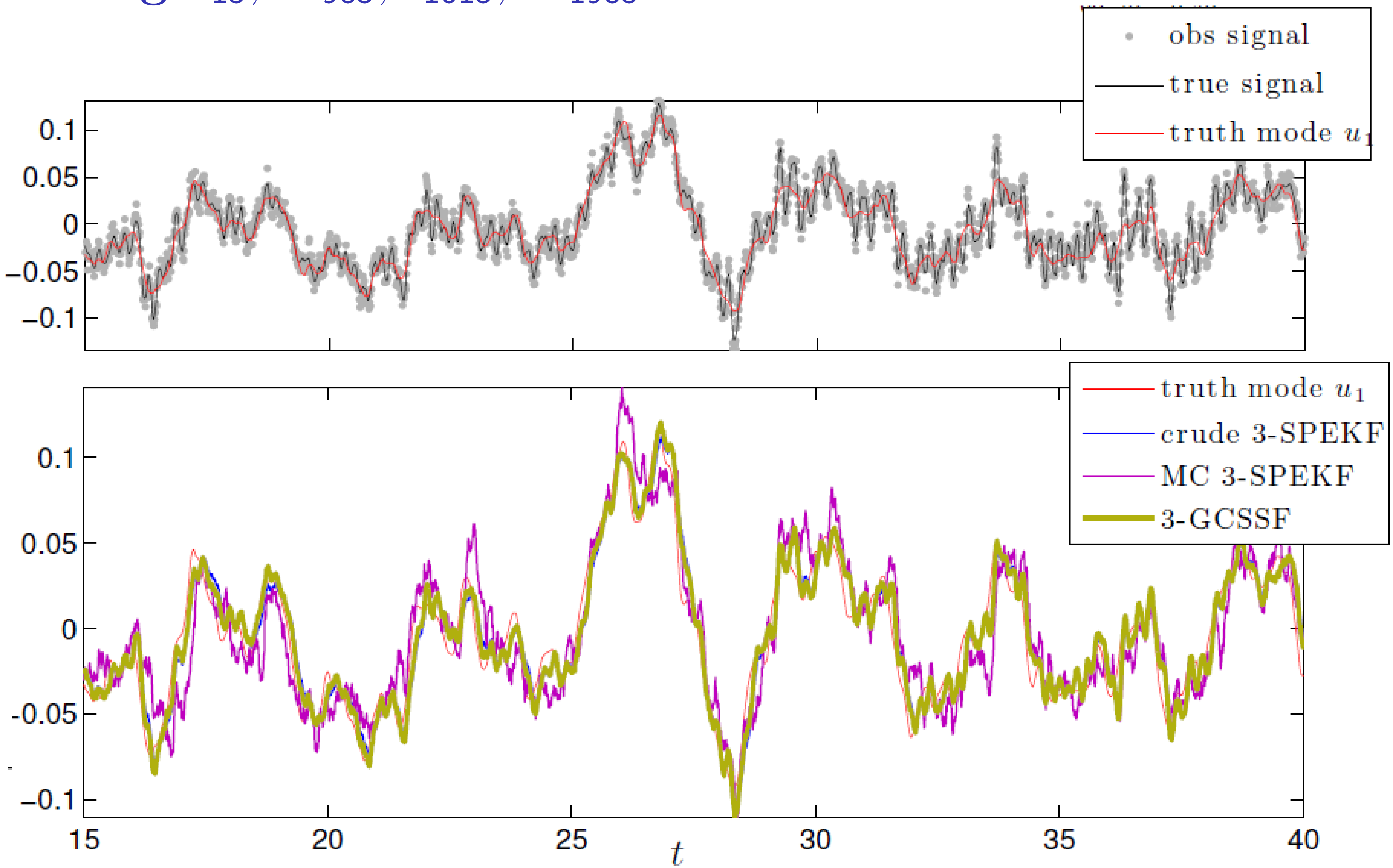
RMS_{3SPEKF}=0.053, $\langle corr \rangle_{3SPEKF}$=0.891 RMS_{3GCSSF}=0.038, $\langle corr \rangle_{3GCSSF}$=0.944

RMS error and pattern correlation for the MMT system as a function of observation time

$r^o = 0.1 \times 10^{-3}$ $E = (3.5, 0.3, 0.15) \times 10^{-3}$ $\tau_{corr} = (1, 0.015, 0.01)$

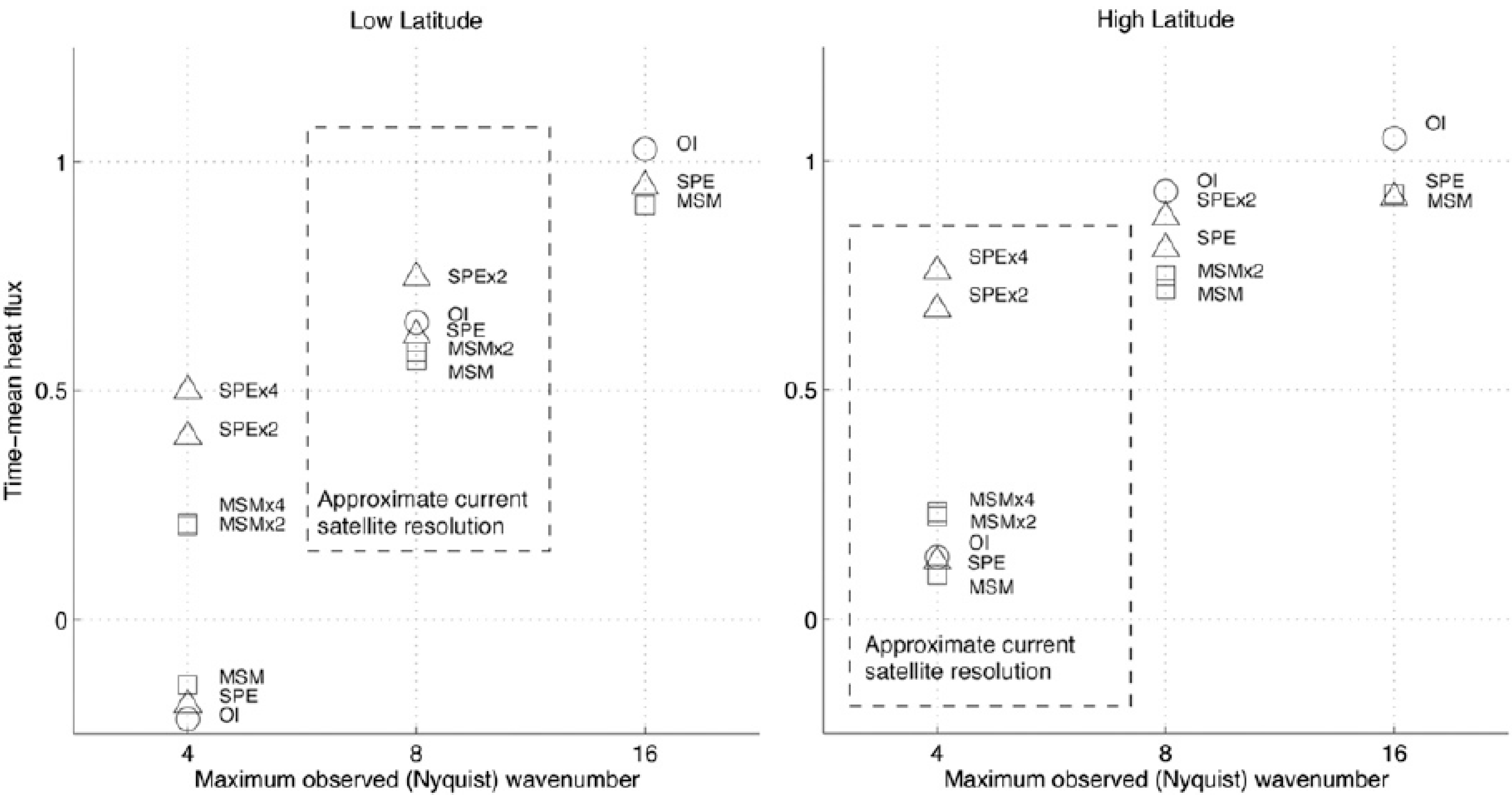


Path-wise example of DSS for the primary mode u_1 recovered from the aliased signal from MMT system involving $u_{15}, u_{-985}, u_{1015}, u_{-1985}$

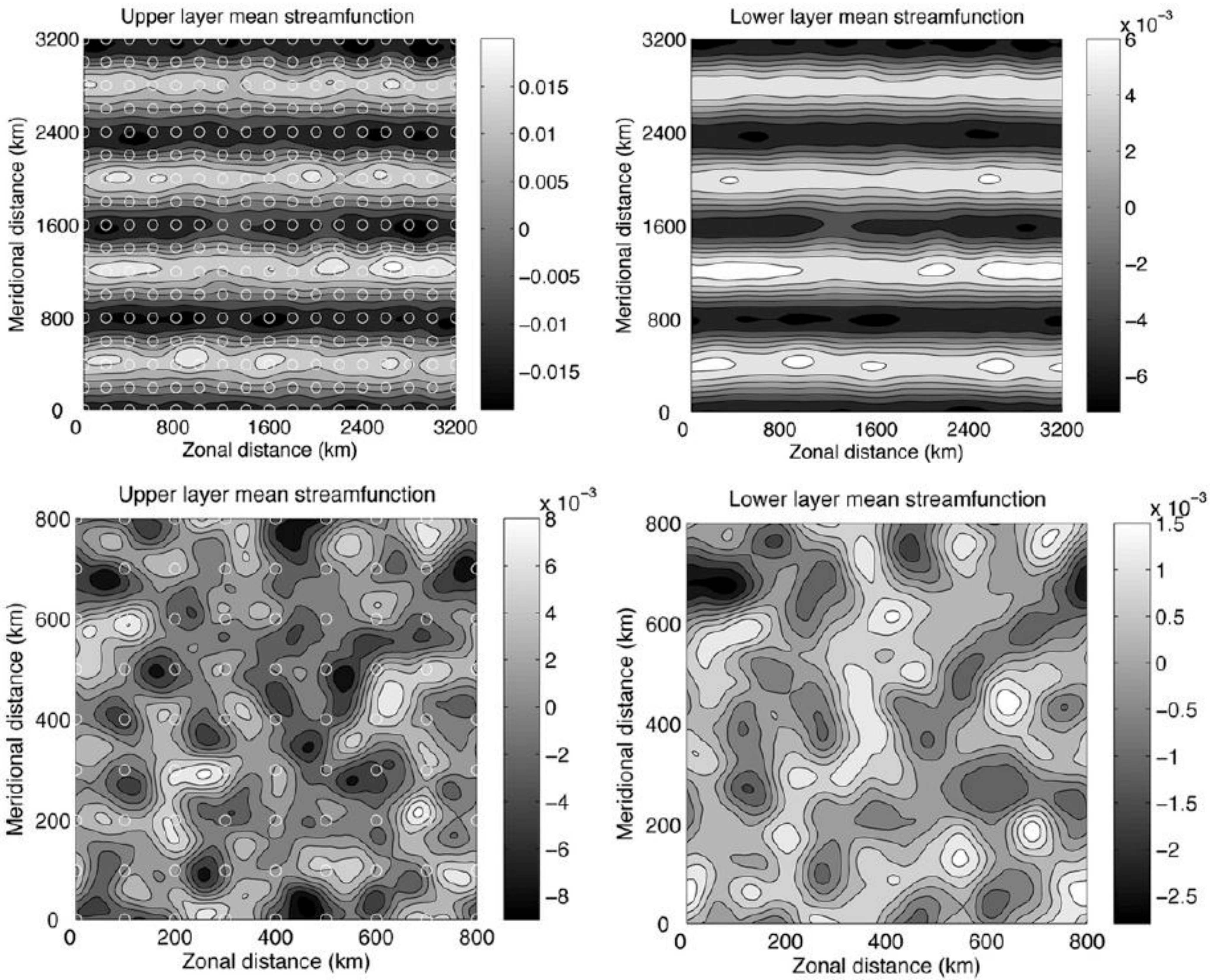


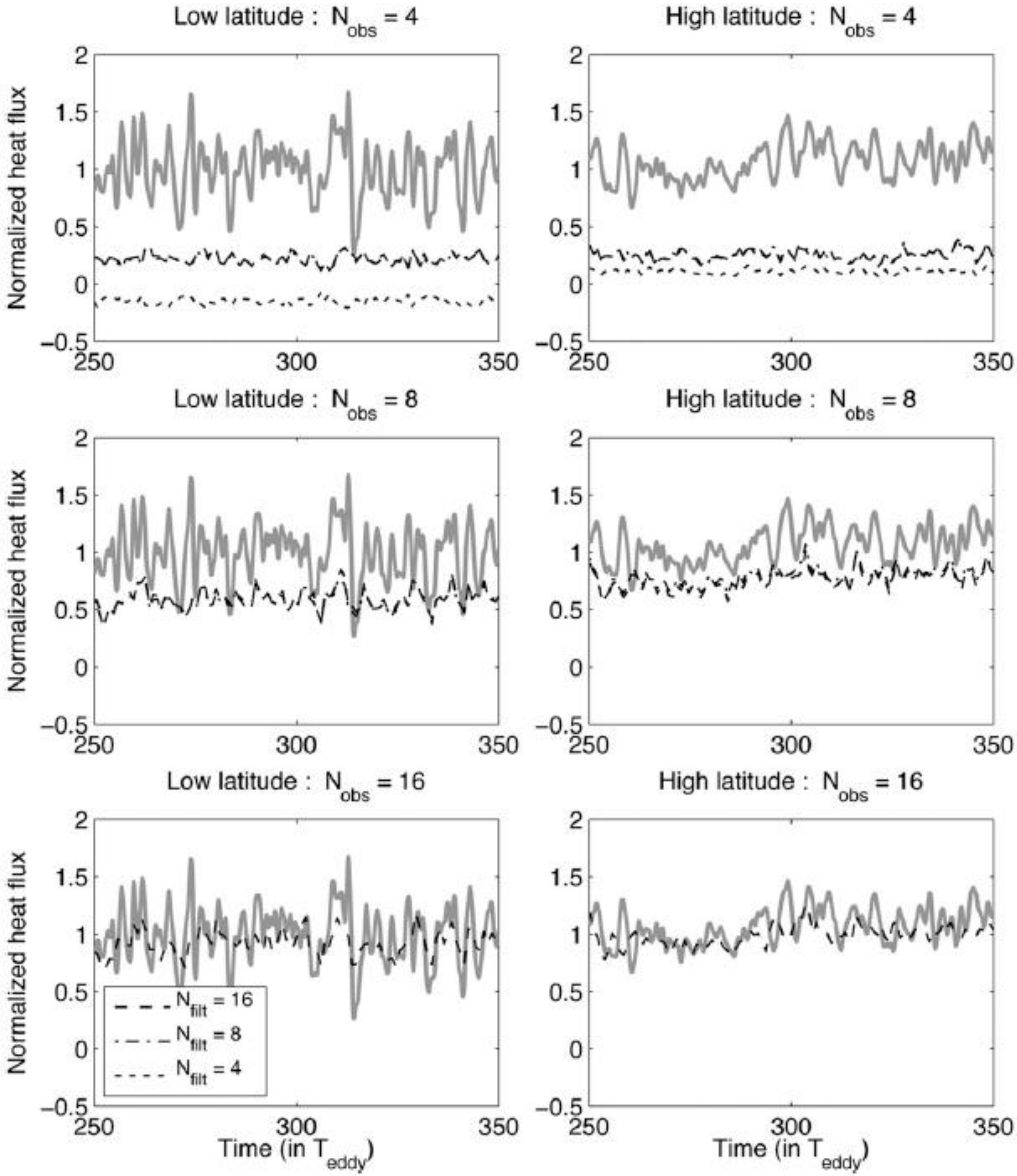
$RMS_{c3SPEKF}=0.018$, $\langle corr\rangle_{c3SPEKF}=0.959$ $RMS_{MC3SPEKF}=0.031$, $\langle corr\rangle_{MC3SPEKF}=0.881$ $RMS_{3GCSSF}=0.016$, $\langle corr\rangle_{3GCSSF}=0.968$

Performance of DSS

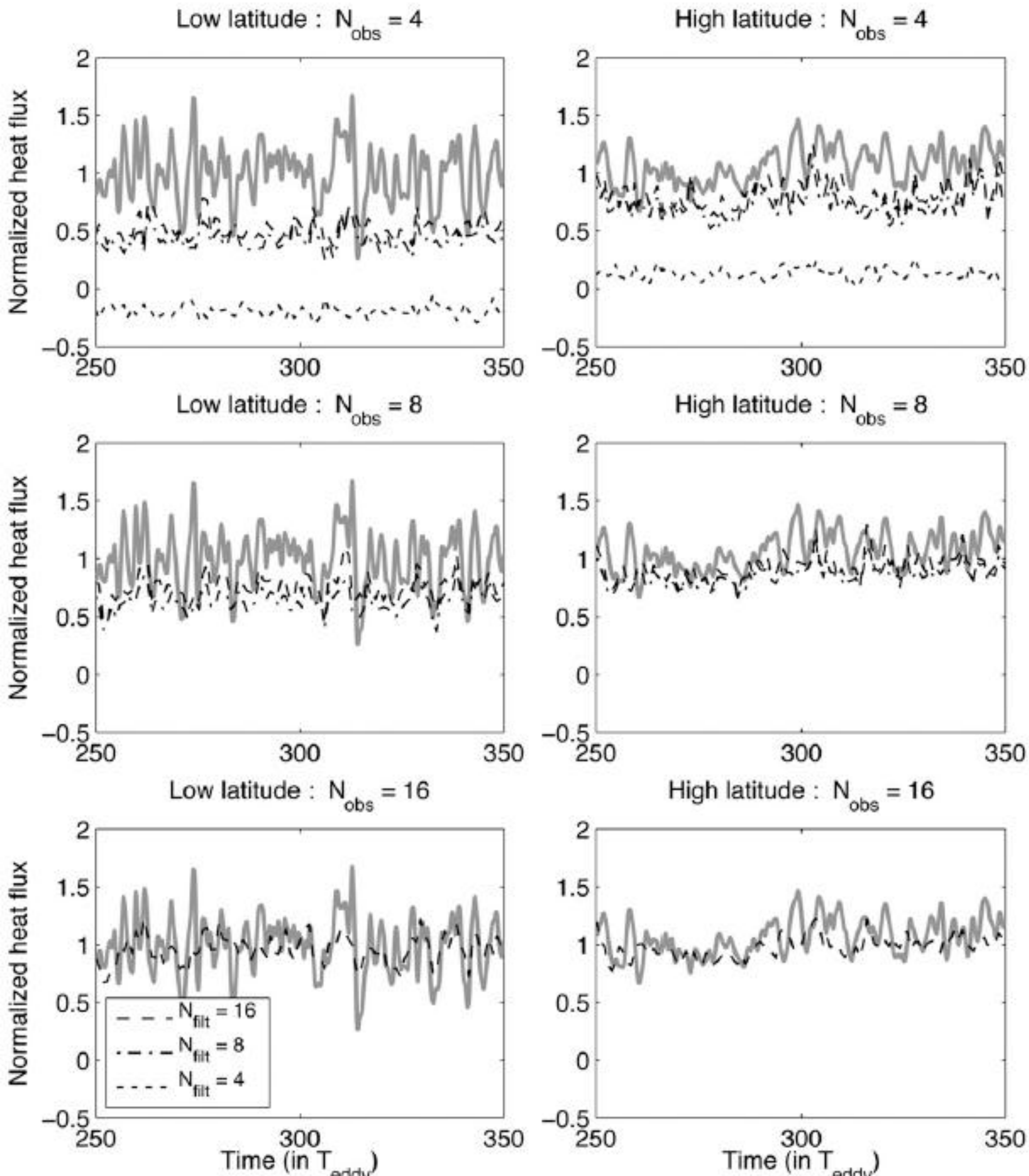


Upper- and lower-layer streamfunctions

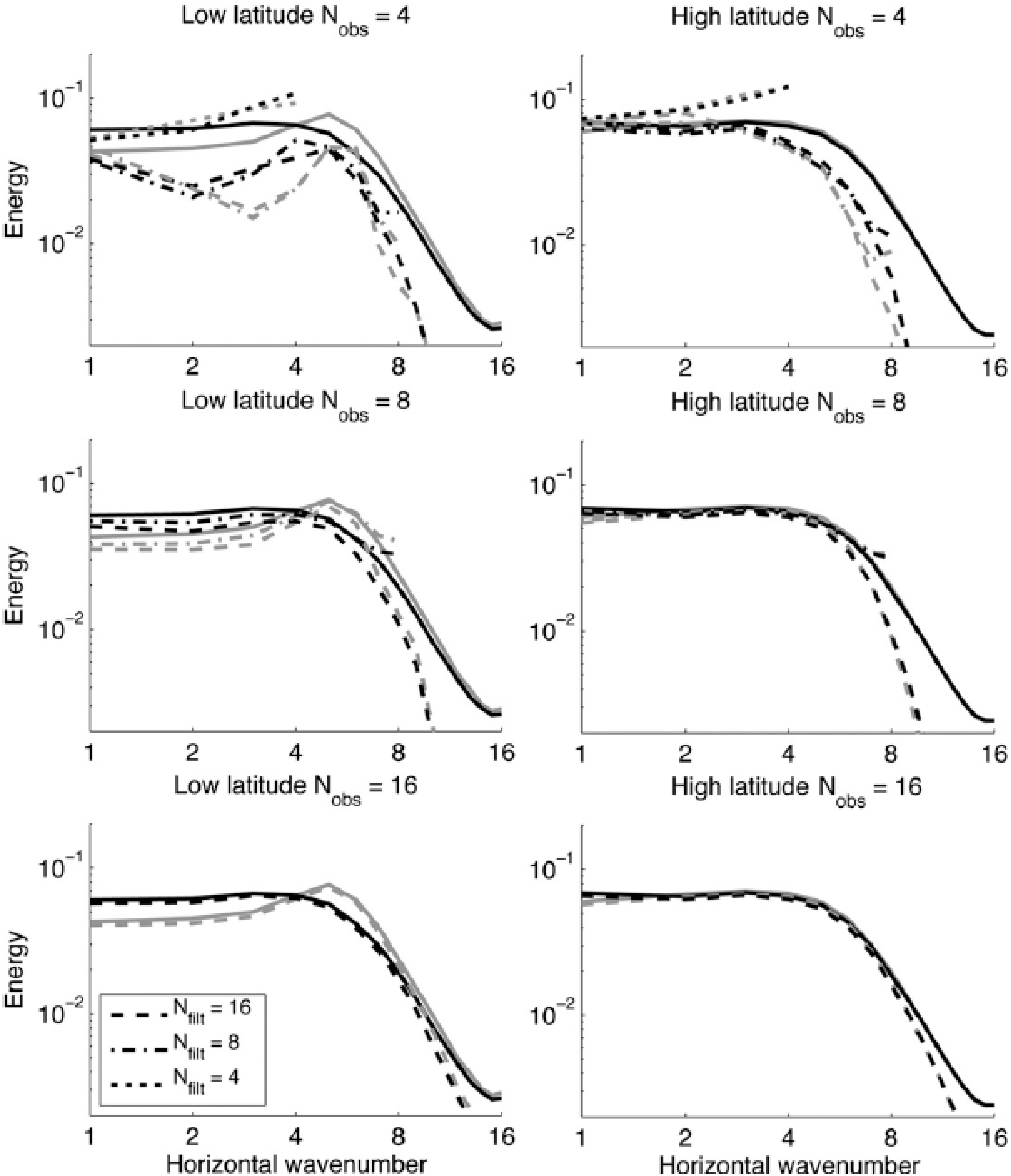




Time series of the poleward eddy heat transport estimated using MSM



Time series of the poleward eddy heat transport estimated using SPEKF



Normalized energy spectra in the zonal (black) and meridional (gray) directions, estimated using SPEKF