1. Examples of approximate Gaussian filters.

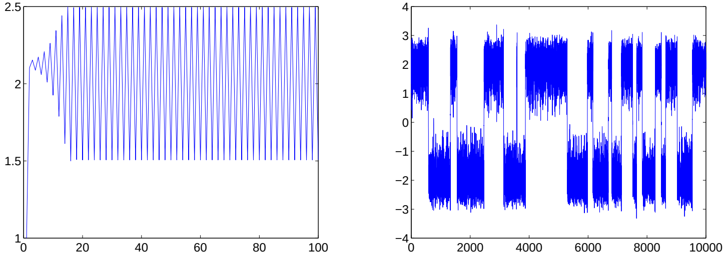

Figure: $v_{j+1} = \alpha \sin(v_j)[+\text{noise}]$

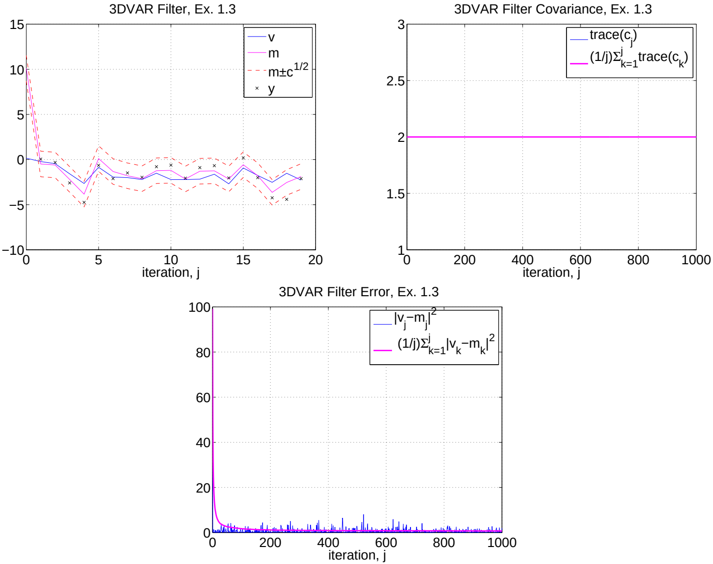

Figure: 3DVAR for Ex. 1.3.

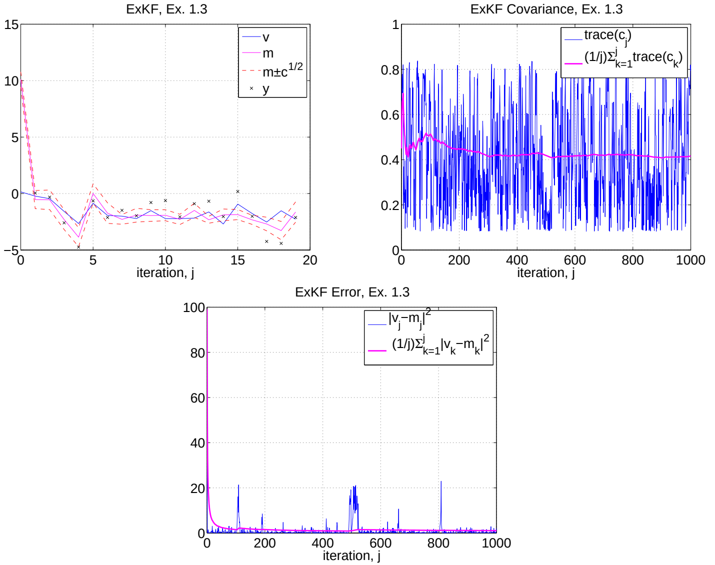

Figure: ExKF for Ex. 1.3.

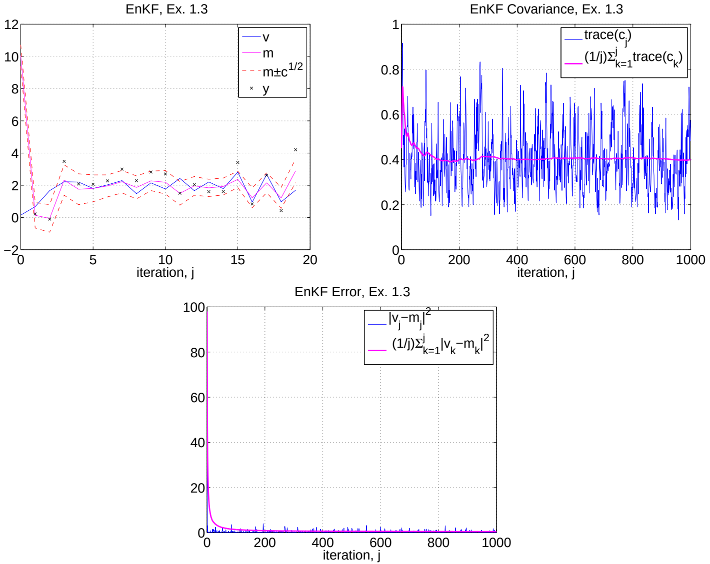

Figure: EnKF for Ex. 1.3.

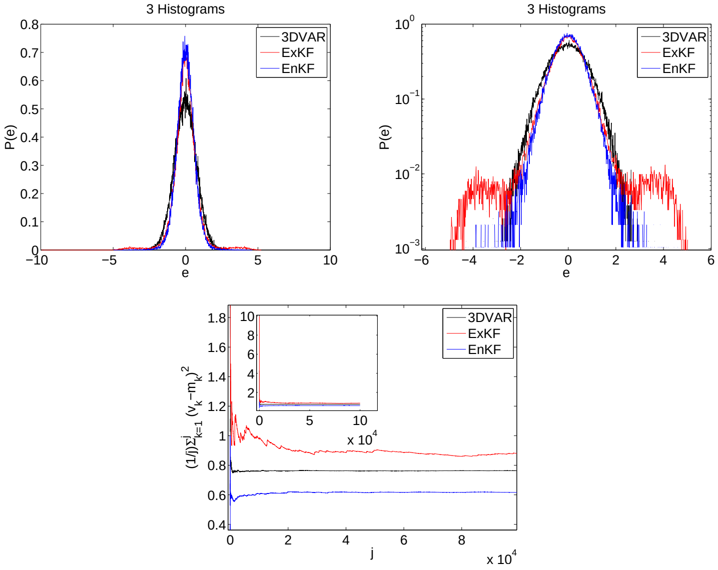

Figure: Convergence of e for each filter

Next Example to show recent research which combines the ideas on sampling and filtering which have been outlined pedagogically in the lecture notes. Based on:

LS11 K.J.H. Law and A.M. Stuart, Evaluating data assimilation algorithms. Monthly Weather Review 140(2012), 3757-3782. `http://arxiv.org/abs/1107.4118`

Betal13 C.E.A. Brett, K.F. Lam, K.J.H. Law, D.S. McCormick, M.R. Scott, A.M. Stuart, Accuracy and stability of filters for dissipative PDEs. Physica D 245(2013) 34-45. `http://arxiv.org/abs/1203.5845`

BLSZ13 D. Bloemker, K.J.H. Law, A.M. Stuart and K. Zygalalkis, Accuracy and stability of the continuous-time 3DVAR filter for the Navier-Stokes equation (Nonlinearity, To Appear). `http://arxiv.org/abs/1210.1594`

Forward Problem: Navier Stokes Equations

Let $f \in H$. Navier-Stokes as ODE on H :

$$\frac{dv}{dt} + \nu A v + B(v,v) = f, \quad v(0) = u$$

Here

$$H = \left\{ u \in L^2(\mathbb{T}^2) \middle| \nabla \cdot u = 0, \int_{\mathbb{T}^2} u\, dx = 0 \right\}, \text{ norm } |\cdot|$$

$$V = \left\{ u \in H^1(\mathbb{T}^2) \middle| \nabla \cdot u = 0, \int_{\mathbb{T}^2} u\, dx = 0 \right\}, \text{ norm } \|\cdot\|$$

Introduce semigroup notation in H (weak) or V (strong):

$$v(t) = \Psi(u;t), \quad \Psi(u) = \Psi(u;h), \quad \Psi^{(j)}(u) := \Psi(u;jh)$$

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Inverse Problem: Navier Stokes Equations

Define orthogonal projections onto low and high Fourier modes:

$$P_\lambda : H \mapsto \{\varphi_k(x), |k| \leq \lambda\},$$
$$Q_\lambda : H \mapsto \{\varphi_k(x), |k| > \lambda.\}$$

Observations:

$$y_j = P_\lambda \Psi^{(j)}(u) + \zeta_j, \quad \zeta_j \sim N(0, \Gamma)$$
$$Y_j = \{y_i\}_{i=1}^{j}.$$

Goal: find initial condition u given partial noisy observations Y_j.

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Relative Error in Mean c/w Posterior

Mean [LS11]

method	e_{mean}
4DVAR($t = 0$)	0.000731491
4DVAR($t = T$)	0.00130112
3DVAR	0.0634553
FDF	0.165732
LRExKF	0.00614573
EnKF	0.0596825

Relative Error in Variance c/w Posterior

Variance [LS11]

method	$e_{variance}$
4DVAR($t = 0$)	0.0932748
4DVAR($t = T$)	0.220154
3DVAR	6.34057
FDF	28.9155
LRExKF	0.195101
EnKF	0.516939

Relative Error c/w Posterior: Variance Inflation

From [LS11]:

method	e_{mean}	$e_{variance}$
3DVAR	0.458527	1.8214
[3DVAR]	0.27185	6.62328
LRExKF	0.632448	0.4042
[LRExKF]	0.201327	11.2449
EnKF	0.450555	0.583623
[EnKF]	0.279007	6.67466
FDF	0.189832	11.4573

- Numerics show comparison of various filters (*ad hoc*) against a full gold standard MCMC sampler (rigorously justified.). See [LS11].
- They show that, in typical data rich scenarios, the mean is well approximated by the filters, but that variance information is not.
- This effect is further complicated by *variance inflation* which is often used to stabilize filters.
- For theoretical explanation of role of variance inflation see [Betal13] and [BLSZ13].