Inverse hyperbolic problems with the boundary data on the part of the boundary.

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Second order hyperbolic equation.

Consider a hyperbolic equation of the form:

$$Lu \stackrel{def}{=} \frac{\partial^2 u(x,t)}{\partial t^2} + \sum_{j,k=1}^{n} \frac{1}{\sqrt{g(x)}} \left(-i\frac{\partial}{\partial x_j} + A_j(x)\right) \cdot \sqrt{g(x)} g^{jk}(x) \left(-i\frac{\partial}{\partial x_k} + A_k(x)\right) u(x,t) - V(x,t)u(x,t) = 0 \tag{1}$$

in $\Omega \times (0, T_0)$, where Ω is a bounded domain in $\mathbf{R}^n$, $n \geq 2$, with smooth boundary $\partial\Omega$, all coefficients in (1) are $C^\infty(\overline{\Omega})$ real-valued functions, $\|g^{jk}(x)\|^{-1}$ is the metric tensor in $\overline{\Omega}$, $g(x) = \det \|g^{jk}\|^{-1}$.

The Dirichlet-to-Neumann operator.

We consider the initial-boundary value problem:

$$(2) \qquad u(x,0) = u_t(x,0) = 0 \quad \text{in} \quad \Omega,$$

$$(3) \qquad u(x,t)\Big|_{\partial\Omega\times(0,T_0)} = f(x,t).$$

Let Γ_0 be an open subset of $\partial\Omega$. We shall consider $f(x,t)$ such that $\mathrm{supp}\, f \subset \Gamma_0 \times (0,T_0)$.

Then Dirichlet-to-Neumann (D-to-N) operator:

$$(4) \quad \Lambda f \stackrel{def}{=} \sum_{j,k=1}^{n} g^{jk}(x)\left(\frac{\partial u}{\partial x_j} + iA_j(x,t)u\right) \cdot \nu_k \left(\sum_{p,r=1}^{n} g^{pr}(x)\nu_p\nu_r\right)^{-\frac{1}{2}} \Big|_{\Gamma_0\times(0,T_0)},$$

where $u(x,t)$ is the solution of the initial-boundary value problem (1), (2), (3), $\nu = (\nu_1, ..., \nu_n)$ is the unit exterior normal vector at $x \in \partial\Omega$ with respect to the Euclidian metric. If $F(x) = 0$ is the equation on $\partial\Omega$ on some neighborhood then

$$\Lambda f \overset{def}{=} \sum_{j,k=1}^{n} g^{jk}(x)\left(\frac{\partial u}{\partial x_j} + iA_j(x,t)u\right)$$

$$\cdot F_{x_k}(x)\left(\sum_{p,r=1}^{n} g^{pr}(x)\nu_p\nu_r\right)^{-\frac{1}{2}}\Big|_{\Gamma_0\times(0,T_0)}$$

Note that domain Ω can be multi-connected and $\Gamma_0 \subset \partial\Omega$ can be not connected. An important example of domain Ω is the domain with obstacles when $\Omega = \Omega_0 \setminus \cup_{j=1}^{r}\overline{\Omega_j}$ where $\Omega_1, ..., \Omega_r$ are non-intersecting domains inside Ω_0 that are called obstacles, $\Gamma_0 = \partial\Omega_0$ and the zero Dirichlet boundary conditions are given on $\partial\Omega_j,\ 1 \le j \le r$.

Gauge Equivalence.

Let $G_0(\overline{\Omega})$ be a group of $C^\infty(\overline{\Omega})$ complex-valued functions $c(x)$ such that $|c(x)| \neq 0,\ c(x) = 1$ on $\overline{\Gamma_0}$. We say that potentials

$A(x) = (A_1(x), ..., A_n(x))$ and $A'(x) = (A'_1(x), ..., A'_n(x))$ are gauge equivalent if there exists $c(x) \in G_0(\overline{\Omega})$ such that

$$A'_j(x) = A_j(x) - ic^{-1}(x)\frac{\partial c}{\partial x_j},$$

$$x \in \overline{\Omega}, \quad 1 \leq j \leq n.$$

Note that if $Lu = 0$ and $u' = c(x)u$ then $L'u' = 0$ where L' is an operator of the form (1) with $A_j(x),\ 1 \leq j \leq n$, replaced by $A'_j(x),\ 1 \leq j \leq n$. We shall write for brevity

$$L' = c \circ L.$$

Main Theorem.

Theorem 1.1. *Let $L^{(1)}u^{(1)} = 0$ and $L^{(2)}u^{(2)} = 0$ be hyperbolic equations of the form (1) in domains $\Omega^{(1)}$ and $\Omega^{(2)}$ respectively. Assume that $\Gamma_0 \subset \partial\Omega^{(1)} \cap \partial\Omega^{(2)}$ is nonempty and open. Assume that the initial-boundary condition (2), (3) are satisfied with Ω replaced by $\Omega^{(p)}$, $p = 1, 2$.*

Suppose $\Lambda^{(1)} f = \Lambda^{(2)} f$ on $\Gamma_0 \times (0, T_0)$ for all smooth $f(x,t)$ with supports in $\Gamma_0 \times (0, T_0]$.

Suppose $T_0 > 2 \sup_{x \in \Omega^{(p)}} d_p(x, \Gamma_0)$, $p = 1, 2$, where $d_p(x, \Gamma_0)$ is the distance in $\Omega^{(1)}$ with respect to the metric $\|g_p^{jk}(x)\|^{-1}$ in $\Omega^{(p)}$ from $x \in \overline{\Omega^{(p)}}$ to Γ_0, $p = 1, 2$.

Then there exists a diffeomorphism

φ of $\overline{\Omega^{(1)}}$ onto $\overline{\Omega^{(2)}}$, $\varphi = I$ on Γ_0, and

$$\|g_2^{jk}\| = \varphi \circ \|g_1^{jk}\|.$$

Moreover, there exists a gauge transformation $c(x) \in G_0(\overline{\Omega^{(1)}})$ such that

$$c \circ \varphi \circ L^{(2)} = L^{(1)} \quad in \ \ \overline{\Omega^{(1)}}.$$

This theorem was proven by the BC-method (see M.Belishev, Inverse Problems, 1997, and Kachalov-Kurylev-Lassas, 2001). We give a new simpler proof of this result with the emphasis on the case of multi-connected domains with obstacles. We also consider a generalization on a case of the systems with Yang-Mills potentials.

More General Equations.

Consider a system of the form

$$\frac{\partial^2 u(x,t)}{\partial t^2} + \sum_{j,k=1}^{n} \frac{1}{\sqrt{g(x)}} \left(-i\frac{\partial}{\partial x_j} I_m + A_j(x) \right)$$

$$\cdot \sqrt{g(x)} g^{jk}(x) \left(-i\frac{\partial}{\partial x_k} I_m + A_k(x) \right) u(x,t)$$

$$+ V(x) u(x,t) = 0, \tag{5}$$

where $u(x,t), \; A_j(x), \; 0 \le j \le n, \; V(x)$ are $m \times m$ matrices, $\Omega = \Omega_0 \setminus \cup_{j=1}^{m} \overline{\Omega_j}$ is a domain with obstacles and the initial-boundary conditions (2), (3) are satisfied. Let $G_0(\overline{\Omega})$ be the gauge group of nonsingular C^∞ $m \times m$ matrices $C(x)$ in $\overline{\Omega}$ such that $C(x)\big|_{\partial\Omega_0} = I_m$. Matrices $A(x) = (A_1(x), ..., A_n(x)), \; V(x)$ are called Yang-Mills potentials.

We say that $(A(x), V(x))$ and $(A'(x), V'(x))$

are gauge equivalent if there exists $C(x) \in G_0(\overline{\Omega})$ such that

$$A'(x) = C^{-1}(x)A(x)C(x) - iC^{-1}(x)\frac{\partial C(x)}{\partial x},$$
$$V'(x) = C^{-1}(x)V(x)C(x). \tag{6}$$

Theorem 1.2. *Theorem 1.1 holds for the equations of the form (5) with Yang-Mills potentials.*

A particular case.

Consider the case when $T_0 = +\infty$, $g^{jk}(x) = \delta_{jk}$. Making the Fourier transform in t we get the Schrödinger equation with Yang-Mills potentials

$$(7) \qquad \sum_{j=1}^{n} \left(-i \frac{\partial}{\partial x_j} I_m + A_j(x) \right)^2 w(x)$$
$$+V(x)w(x) - k^2 w(x) = 0$$

When $m = 1$ we get the Schrödinger equation with electromagnetic potentials. The D-to-N operator for (7) has the form

$$\Lambda(k)h = \frac{\partial w}{\partial \nu} + i(A \cdot \nu)w \Big|_{\partial\Omega_0},$$

where $w \Big|_{\partial\Omega_0} = h$. Knowing the hyperbolic D-to-N operator we can recover $\Lambda(k)$ for all k except a discrete set K

and vice versa. Applying Theorem 2 we get that one can recover $A(x) = (A_1(x), ..., A_n(x))$, $V(x)$ modulo gauge transformation knowing $\Lambda(k)$ for all $k \in \mathbf{C} \setminus K$.

Description of the gauge equivalence classes and the Aharonov-Bonm effect.

Let $\Omega = \Omega_0 \setminus (\cup_{j=1}^r \overline{\Omega_j})$ be a domain with obstacles. Fix $x_0 \in \partial\Omega_0$. Let $\mathcal{P}$ be the space of all piecewise smooth paths in $\overline{\Omega}$ starting and ending at x_0. Consider an arbitrary path $\gamma \in \mathcal{P}$. Let $x = \gamma(\tau)$ be a parametric equation of γ, $0 \leq \tau \leq \tau_0$, and let $C(\tau, \gamma)$ be the solution of the system

$$(8) \qquad i\frac{\partial C(\tau,\gamma)}{\partial \tau} = \frac{d\gamma(\tau)}{d\tau} \cdot A(\gamma(\tau))C(\tau,\gamma), \quad C(0,\gamma) = I_m.$$

Denote by $C(\gamma, A) \in GL(m, \mathbf{C})$ the value of $C(\tau, \gamma)$ at $\tau = \tau_0$. Here $GL(m, \mathbf{C})$ is the group of nonsingular $m \times m$ matrices.

Note that when $m = 1$ we have $C(\gamma, A) = \exp(-i \int_\gamma A \cdot dx)$.

Wu and Yang (1975) called $C(\gamma, A)$ the gauge phase factor.

The image of the map $\gamma \to C(\gamma, A)$ of the group of paths $\mathcal{P}$ to $GL(m, \mathbf{C})$ is called the holonomy group of the connection $\sum_{j=1}^n A_j(x) dx_j$.

It is easy to show that

$$C(\gamma, A^{(1)}) = C(\gamma, A^{(2)})$$

for all $\gamma \in \mathcal{P}$ if and only if $A^{(1)}$ and $A^{(2)}$ are gauge equivalent in Ω.

As it was shown by Aharonov and B0hm (1959) the presence of distinct gauge equivalent classes of potentials can be detected in an experiment and this phenomenon is called Aharonov-Bohm effect.

A geometric optic approach.

Consider the Schrödinger equation with electromagnetic potentials

$$(9) \qquad \sum_{j=1}^{n}\left(-i\frac{\partial}{\partial x_j}+A_j(x)\right)^2 u(x) \\ +V(x)u(x)-k^2u(x)=0$$

in the domain $\Omega=\Omega_0\setminus(\cup_{j=1}^{n}\overline{\Omega_j})$ with obstacles. Assume that the D-to-N operator $\Lambda(k)$ is given for all $k\in\mathbf{C}\setminus K$. My earlier approach to the inverse problem for (9) was based on geometric optics constructions and the reduction to the integral geometry (tomography) problem. Such approach yields weaker results.

We say that $\gamma = \gamma_1 \cup \gamma_2 \cup ... \cup \gamma_N$ is a broken ray with legs $\gamma_1 . \gamma_2, ..., \gamma_N$ if $\gamma_k,\ 1 \leq k \leq N$ are geodesics, γ starts at point $x_0 \in \partial\Omega_0,\ \gamma$ has $N-1$ not tangenial points of reflection at the obstacles and γ ends at a point $x_N \in \partial\Omega_0$.

Consider two Schrödinger equations with electro-magnetic potentials $A^{(p)}(x)$, $V^{(p)}(x)$, $p = 1, 2$, with the Euclidian metric $g^{jk} = \delta_{jk}$ in a plane domain with convex obstacles.

Using the geometric optics solutions one can prove that if the D-to-N operators are equal then

$$\exp(i\int_\gamma A^{(1)}(x)\cdot dx) = \exp(i\int_\gamma A^{(2)}(x)\cdot dx), \tag{10}$$

$$\int_\gamma V^{(1)}(x)ds = \int_\gamma V^{(2)}(x)ds \tag{11}$$

for any broken ray. Having (10), (11) we can reduce the inverse problem for the Schrödinger equation to the inverse problem of the integral geometry of broken rays.

Some results in this direction were obtained in [E] (2004) for $n = 2$ under the geometrical restriction that there is no trapped rays. This condition is not satisfied when one has more than one smooth obstacle. However, there are piecewise smooth obstacles that satisfy these conditions. Despite that this approach is much more restrictive than the hyperbolic equations approach it has an advantage that it allows to prove the stability results in some cases.

Consider the following example:

An Example.

Let Ω_1 be a convex obstacle in $\mathbf{R}^2$ and let $f(x)$ be a smooth function in $\mathbf{R}^2 \backslash \Omega$, $f(x) = 0$ for $|x| > R$. It is well-known (Helgason) that if $\int_\gamma f(x) ds = 0$ for all lines γ not intersecting $\overline{\Omega}$ then $f(x) = 0$. But this problem is severly ill-posed. If one uses the broken rays too, i.e. if one compute $\int_\gamma f ds = F(\gamma)$ for all broken rays γ then the inverse problem is well-posed and there is a stability estimate (c.f. Mukhometov (1977), in the case of no obstacles).