A Survey of Computational High Frequency Wave Propagation II

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High Frequency Wave Propagation
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Numerical methods

- Direct methods
  - Wave equation (time domain)
  - Integral equation methods (frequency domain)
- Asymptotic methods
  - Physical optics
  - Geometrical optics
  - (Gaussian beams)
- Hybrid methods

Direct numerical methods for the wave equation

Scalar wave equation

$$u_{tt} - c(\boldsymbol{x})^2 \Delta u = 0, \qquad (t, \boldsymbol{x}) \in \mathbb{R}^+ \times \Omega, \qquad \Omega \subset \mathbb{R}^d$$

+ boundary and initial data.

- Discretize Ω, time and u.
- Many methods: FD (explicit, uniform staggered grids), FV, FEM (implicit or DG).
- Complexity $O(\omega^{(1/p+1)(d+1)})$ (including time).
- Issues: First/second order system. Treatment of boundaries/interfaces. Phase errors.

Direct integral equation methods for Helmholtz

Scattering problem for Helmholtz equation: $u = u_{\rm s} + u_{\rm inc}$, $c \equiv 1$

$$\Delta u_{\rm s} + \omega^2 u_{\rm s} = 0, \qquad x \in \mathbb{R}^d \setminus \bar{\Omega},$$

$$u_{\rm s} = -u_{\rm inc}, \qquad x \in \partial\Omega \quad + \text{ radiation condition}$$

Rewrite as integral equation, e.g.

$$u_{\rm inc}(x) = -\oint_{\partial\Omega} G(\omega|x - x'|)\frac{\partial u(x')}{\partial n} dx', \qquad \forall x \in \partial\Omega.$$

Discretize $\partial\Omega$ and $\frac{\partial u}{\partial n}$ $\Rightarrow$ $O(\omega^{d-1})$ unknowns.

Finite element/collocation methods, "method of moments".

Full matrix equation, direct solution, complexity $O(\omega^{3(d-1)})$.

Fast multipole methods, iterative solver, complexity $\approx O(\omega^{(d-1)})$.

Issues: $1^{\rm st}/2^{\rm nd}$ kind Fredholm equations. Conditioning. Corners.

Physical optics

Integral formulation of scattering problem

$$u_{\mathrm{s}}(x) = \oint_{\partial\Omega} G(\omega|x - x'|) \frac{\partial u(x')}{\partial n} dx', \qquad x \in \mathbb{R}^d \setminus \Omega,$$

Approximate $\frac{\partial u}{\partial n}$ by geometrical optics solution.

E.g. if $u_{\mathrm{inc}} = \exp(i\omega \boldsymbol{\alpha} \cdot \boldsymbol{x})$ is a plane wave, Ω convex, then

$$\frac{\partial u}{\partial n} \approx \frac{\partial(u_{\mathrm{inc}} + u_{\mathrm{s}}^{\mathrm{GO}})}{\partial n} = \begin{cases} 2i\omega\boldsymbol{\alpha} \cdot \hat{\boldsymbol{n}}(\boldsymbol{x}) e^{i\omega\alpha\cdot\boldsymbol{x}}, & \boldsymbol{x} \text{ illuminated,} \\ 0, & \boldsymbol{x} \text{ in shadow.} \end{cases}$$

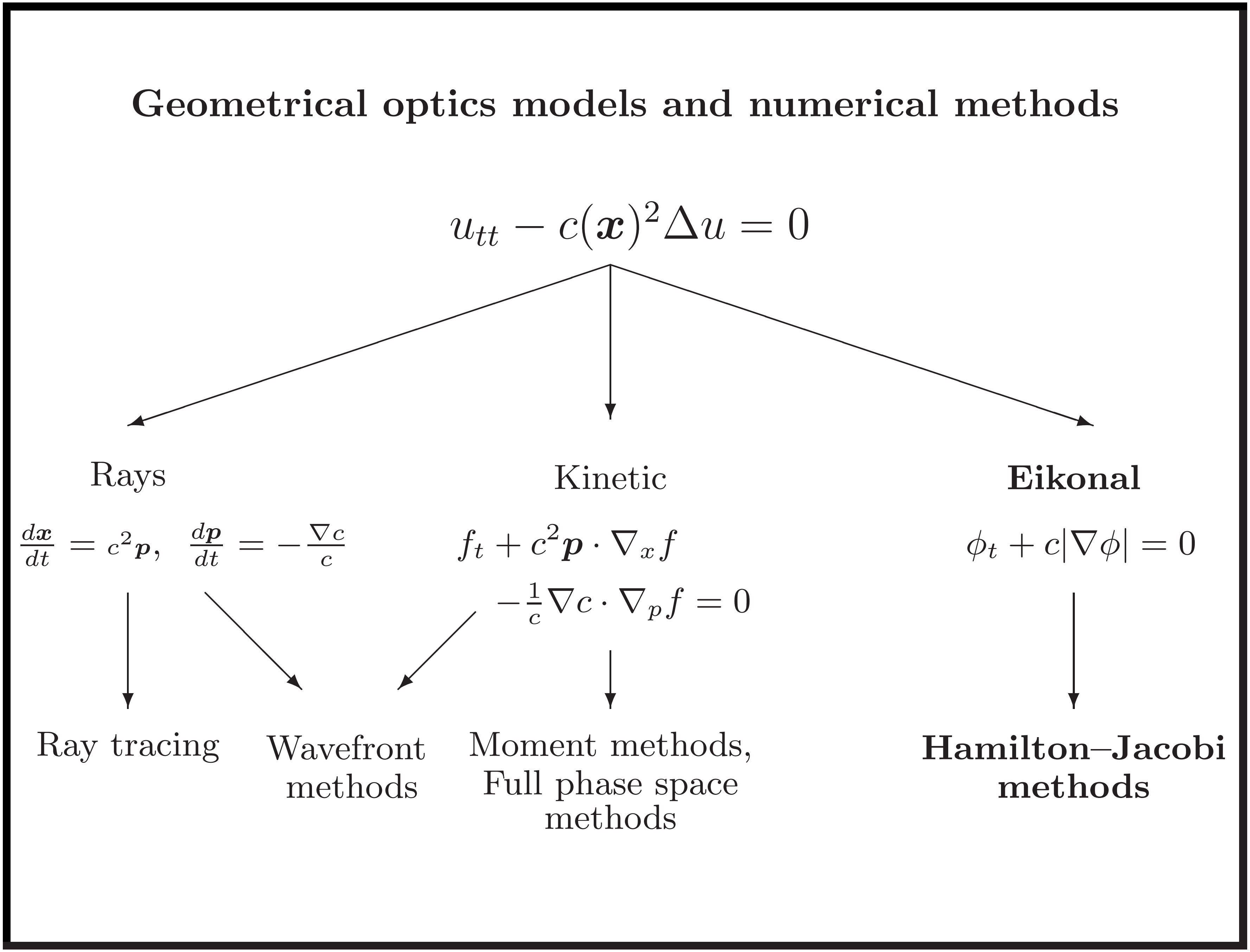
Cost of computing solution still depends on ω.

"Exact PO"

$$\frac{\partial u}{\partial n} = A(\boldsymbol{x}, \omega) e^{i\omega\alpha\cdot\boldsymbol{x}}/\omega$$

then $A(\boldsymbol{x}, \omega)$ smooth, uniformly in ω, except at shadow boundaries.

Discretize and solve $A(\boldsymbol{x}, \omega)$ at cost independent of ω. [Bruno]


Geometrical optics models and numerical methods
$u_{tt} - c(\boldsymbol{x})^2 \Delta u = 0$
Rays
$\frac{d\boldsymbol{x}}{dt} = c^2 \boldsymbol{p}, \quad \frac{d\boldsymbol{p}}{dt} = -\frac{\nabla c}{c}$
Kinetic
$f_t + c^2 \boldsymbol{p} \cdot \nabla_x f - \frac{1}{c} \nabla c \cdot \nabla_p f = 0$
Eikonal
$\phi_t + c|\nabla \phi| = 0$
Ray tracing
Wavefront methods
Moment methods, Full phase space methods
Hamilton–Jacobi methods

Eikonal equation

- Time-dependent version.

  Wave equation plus ansatz $u(t,x) \approx A(t,x)e^{i\omega\phi(t,x)}$ give

  $$\phi_t + c(x)|\nabla\phi| = 0.$$

  Upwind, high-resolution (ENO, WENO) finite difference methods [Osher, Shu, et al]

- Stationary version.

  Helmholtz equation plus ansatz $u(x) \approx A(x)e^{i\omega\varphi(x)}$ give

  $$|\nabla\varphi| = c(x)^{-1}.$$

  Fast marching [Sethian] or fast sweeping methods [Zhao, Tsai, et al].

  (Note, if IC and BC match, $\phi = \varphi - t$.)

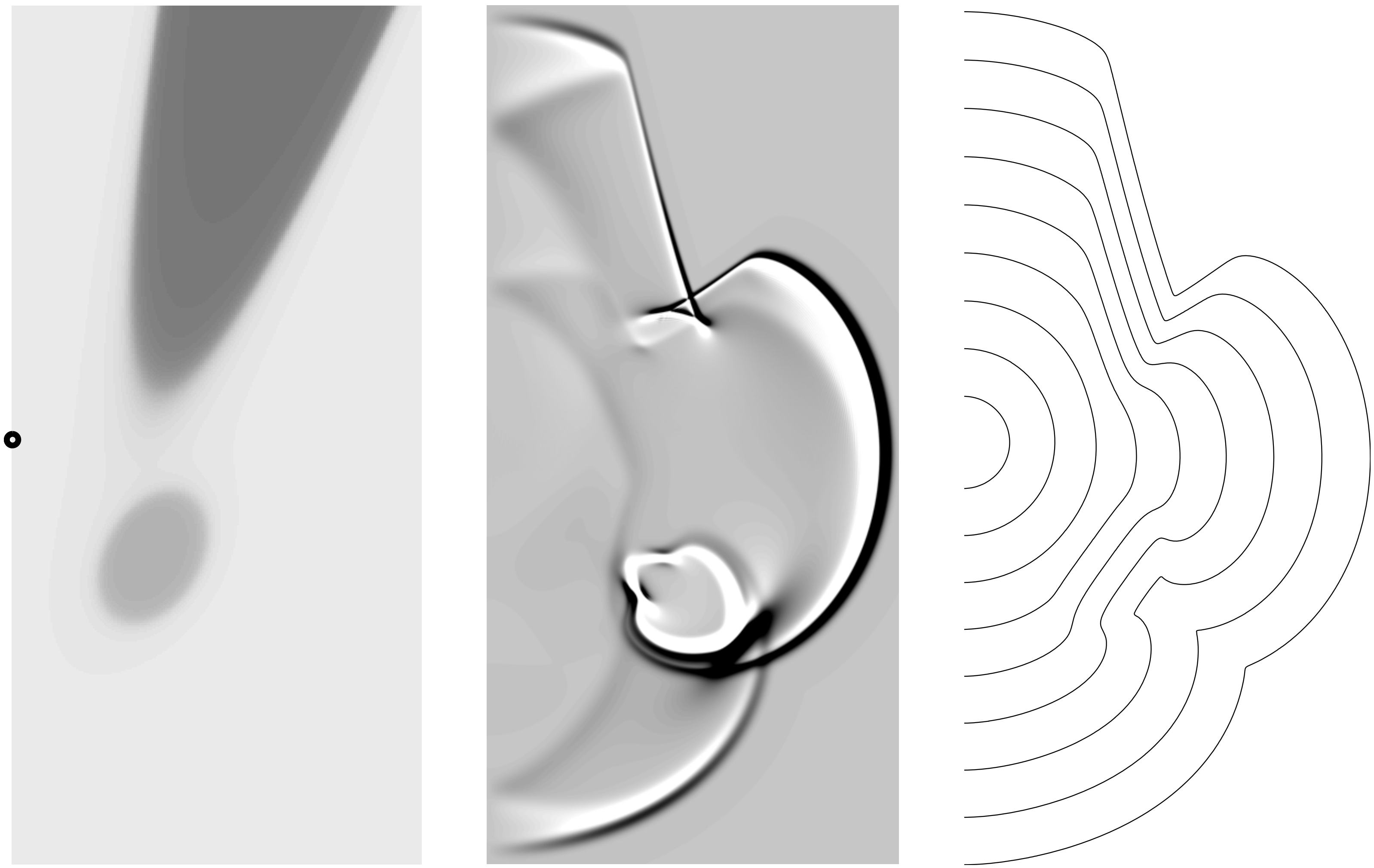
Eikonal equation

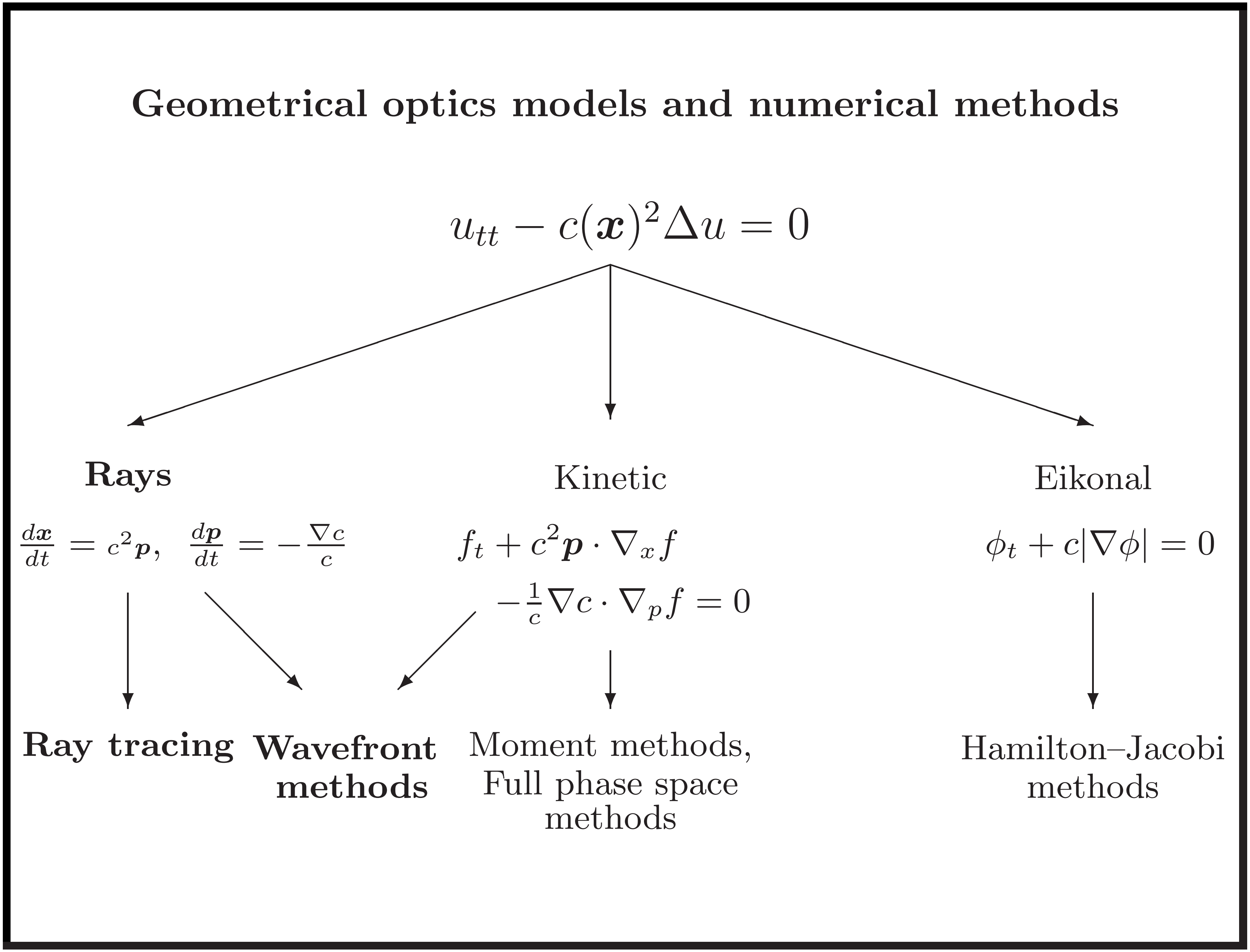
- Ansatz only treats *one* wave. In general crossing waves

$$u(x) \approx A_1(x)e^{i\omega\varphi_1(x)} + A_2(x)e^{i\omega\varphi_2(x)} + \cdots$$

- Nonlinear equation, no superposition principle
- Viscosity solution, kinks
- First arrival property: $\varphi_{\text{visc}}(x) = \min_n \varphi_n(x)$

Example: Eikonal solution

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Ray tracing

Rays are the (bi)characteristics $(\boldsymbol{x}(t), \boldsymbol{p}(t))$ of the eikonal equation, given by ODEs

$$\frac{d\boldsymbol{x}}{dt} = c(\boldsymbol{x})^2 \boldsymbol{p}, \qquad \frac{d\boldsymbol{p}}{dt} = -\frac{\nabla c(\boldsymbol{x})}{c(\boldsymbol{x})},$$

Hamiltonian system with $H = c(\boldsymbol{x})|\boldsymbol{p}|$ and $H \equiv 1$.

Solve with numerical ODE methods, e.g. Runge Kutta.

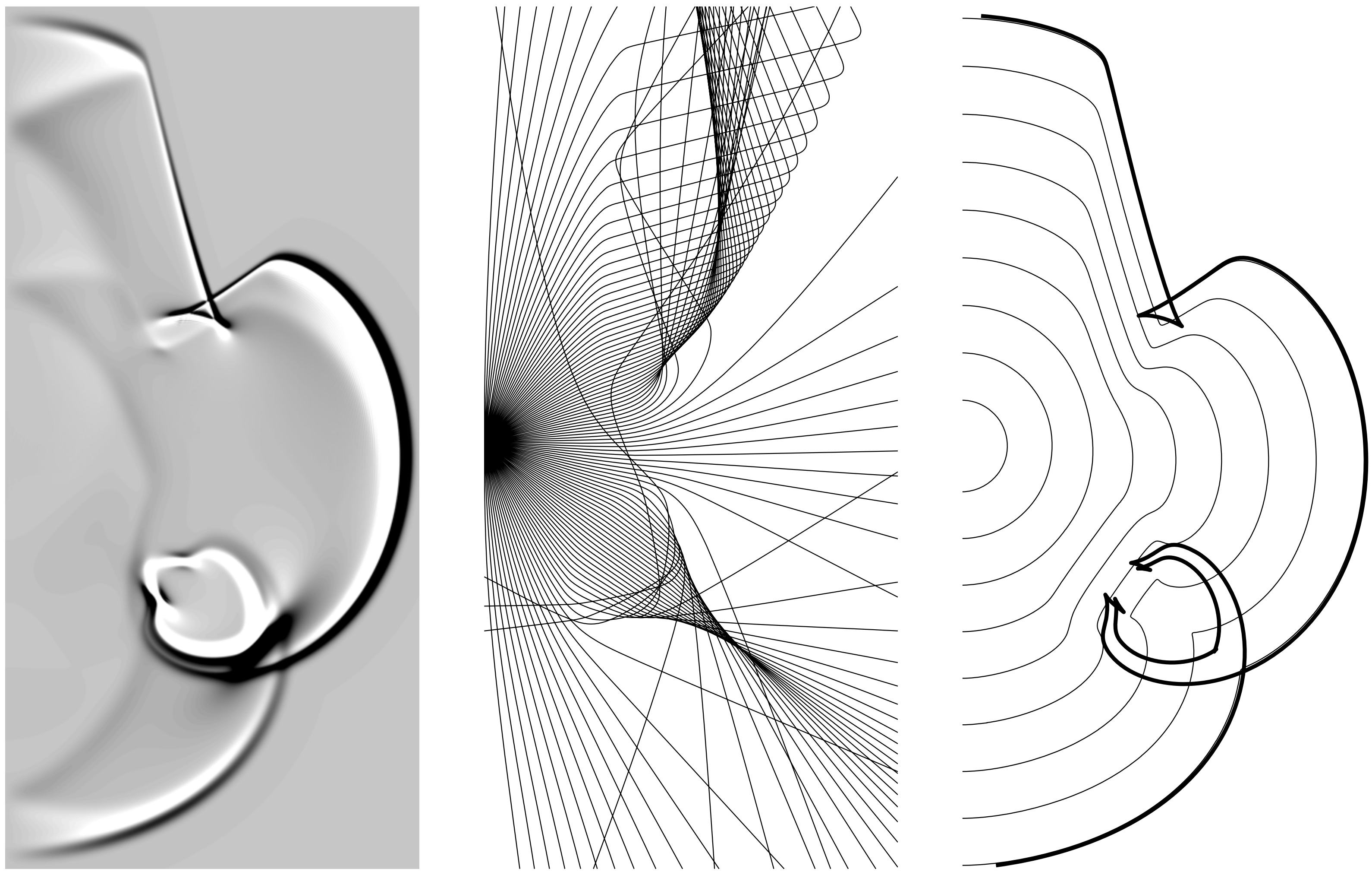
Note, if valid at $t = 0$, then for all $t > 0$:

- $\varphi(\boldsymbol{x}(t)) = t$, (phase $\sim$ traveltime)
- $\nabla\varphi(\boldsymbol{x}(t)) = \boldsymbol{p}(t)$, (local ray direction)
- $|\boldsymbol{p}(t)| = 1/c(\boldsymbol{x}(t))$, ($H = 1$ conserved, can reduce to $\boldsymbol{p} \in \mathbb{S}^{d-1}$)

There are also ODEs for the amplitude along rays.

Issues: Diverging rays. Interpolation onto regular grid.

Example: Ray/Wavefront solutions



Ray tracing boundary value problem

Start and endpoint of ray given.

- Piecewise constant $c(\boldsymbol{x})$

  Rays piecewise straight lines. Find refraction/reflection points at interfaces by Newton's method.

- Smoothly varying $c(\boldsymbol{x})$

  Ray tracing eq is a nonlinear elliptic boundary value problem

$$\frac{d}{dt}\left(c(\boldsymbol{x})^{-2}\frac{d\boldsymbol{x}}{dt}\right) = -\frac{\nabla c(\boldsymbol{x})}{c(\boldsymbol{x})},$$

$$\boldsymbol{x}(0) = \boldsymbol{x}_0,$$

$$\boldsymbol{x}(t^*) = \boldsymbol{x}_1.$$

  t^* additional unknown.

  Solve by shooting method or discretize PDE + Newton.

Multiple solutions difficult.

Wavefront tracking

Directly solve for wavefront given by $\varphi(x) = \text{const}$.

Suppose $\gamma(\alpha)$ is the initial wavefront, $\varphi(\gamma(\alpha)) = 0$.

Follow ensemble of rays

$$\frac{\partial \boldsymbol{x}(t,\alpha)}{\partial t} = c^2 \boldsymbol{p}, \qquad \boldsymbol{x}(0,\alpha) = \gamma(\alpha),$$

$$\frac{\partial \boldsymbol{p}(t,\alpha)}{\partial t} = -\frac{\nabla c}{c}, \qquad \boldsymbol{p}(0,\alpha) = \frac{\gamma'(\alpha)^{\perp}}{c|\gamma'(\alpha)|}.$$

Note: Moving front in normal direction a possibility

$$\boldsymbol{x}_t = c\frac{\boldsymbol{x}_\alpha^{\perp}}{|\boldsymbol{x}_\alpha|} \qquad (\text{since } 0 = \partial_\alpha \varphi(x(t,\alpha)) = x_\alpha \cdot \nabla\phi = x_\alpha \cdot p)$$

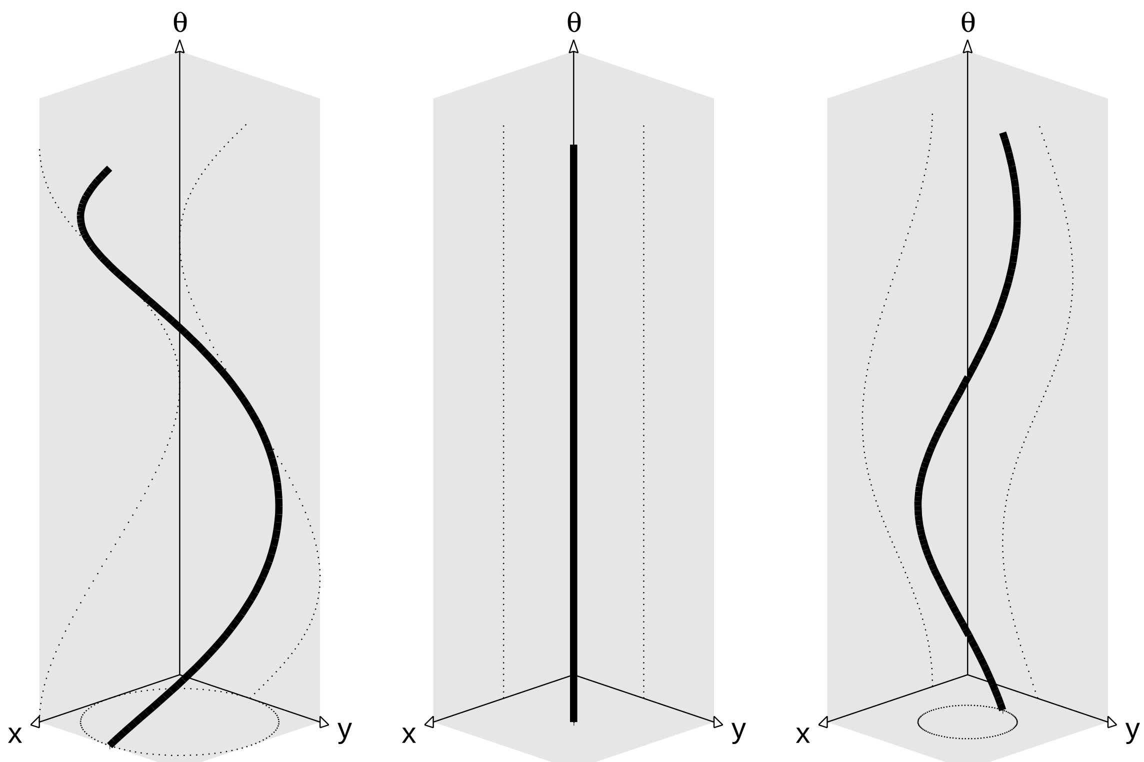
But not good since wavefront non-smooth!

Phase space

Phase space $(\boldsymbol{x}, \boldsymbol{p})$, where $\boldsymbol{p} \in \mathbb{S}^{d-1}$ is local ray direction

Observation: Wavefront is a smooth curve in phase space.

- 2D problems: 1D curve in 3D phase space (x, y, θ).
- 3D problems: 2D surface in 5D phase space $(x, y, z, \theta, \alpha)$.

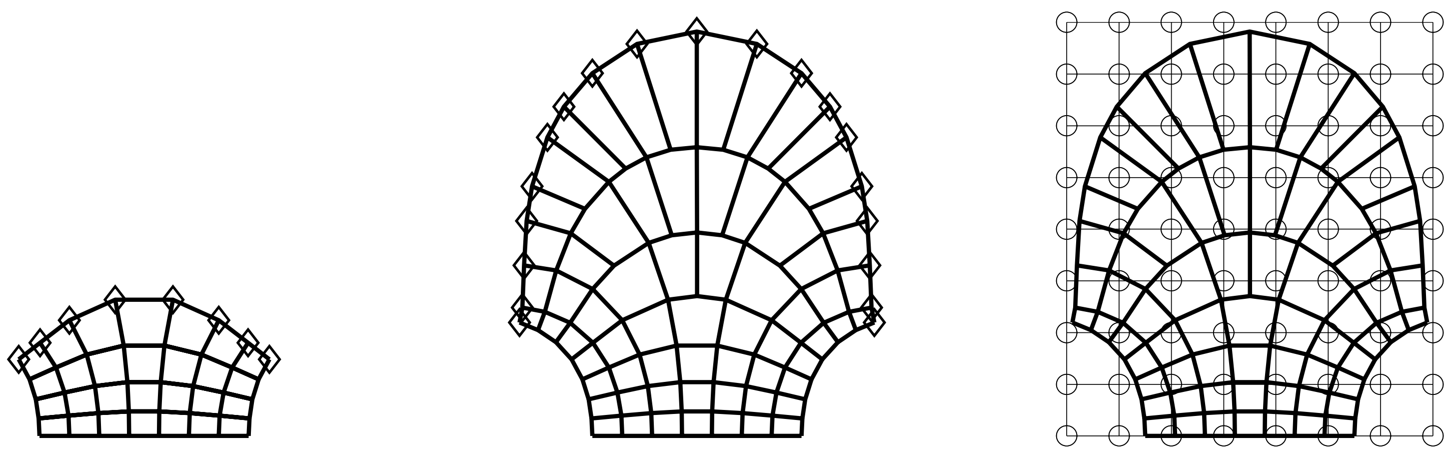


Wavefront construction

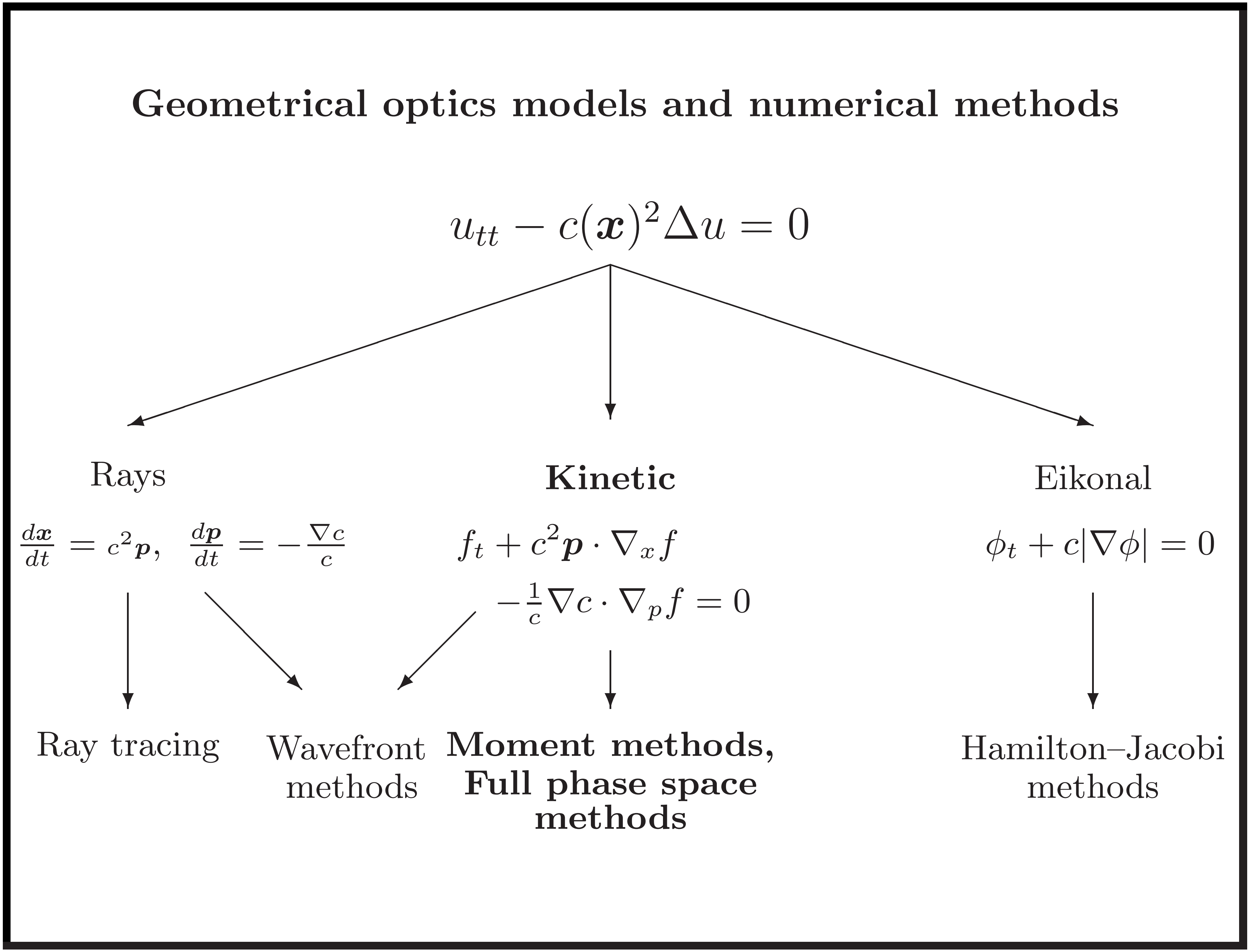
Propagate Lagrangian markers on the wavefront in phase space.

Insert new markers adaptively by interpolation when front resolution deteriorates.

Interpolate traveltime/phase/amplitude onto regular grid.



[Vinje, Iversen, Gjöystdal, Lambaré, . . .]



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Kinetic formulation

Let $f(t, \boldsymbol{x}, \boldsymbol{p})$ be the particle (photon) density in phase space.
Bicharacteristic equations $\Rightarrow$

$$f_t + c^2 \boldsymbol{p} \cdot \nabla_x f - \frac{\nabla c}{c} \cdot \nabla_p f = 0.$$

f supported on $|\boldsymbol{p}| = c(\boldsymbol{x})^{-1}$, $(H \equiv 1)$.

Can also be derived directly from wave eq. through e.g. Wigner measures [Tartar, Lions, Paul, Gerard, Mauser, Markowich, Poupaud, ...]

Relationship to wave equation solution:

$$u = Ae^{i\omega\phi} \quad \sim \quad f = A^2 \delta \left(\boldsymbol{p} - \nabla\phi \right).$$

Note: Loss of phase information.

Moment equations

- Derived from transport equation in phase space + closure assumption for a system of equations representing the moments. (C.f. hydrodynamic limit from Boltzmann eq.)
- PDE description in the "small" $(t, \boldsymbol{x})$-space.
- Arbitrary good superposition. N crossing waves allowed. (But larger N means a larger system of PDEs must be solved.)

[Brenier, Corrias, Engquist, OR] (wave equation),

[Gosse, Jin, Li, Markowich, Sparber] (Schrödinger)

Derivations, homogenous case ($c \equiv 1$)

Starting point is

$$f_t + \boldsymbol{p} \cdot \nabla_x f = 0.$$

Let $\boldsymbol{p} = (p_1, p_2)$. Define the moments,

$$m_{ij} = \int_{\mathbb{R}^2} p_1^i p_2^j f \, d\boldsymbol{p}.$$

From

$$\int_{\mathbb{R}^2} p_1^i p_2^j (f_t + \boldsymbol{p} \cdot \nabla_x f) d\boldsymbol{p} = 0,$$

we get the infinite (valid $\forall i, j \geq 0$) system of moment equations

$$(m_{ij})_t + (m_{i+1,j})_x + (m_{i,j+1})_y = 0.$$

Derivations, homogenous case, cont.

Make the closure assumption

$$f(\boldsymbol{x}, \boldsymbol{p}, t) = \sum_{k=1}^{N} A_k^2 \cdot \delta(|\boldsymbol{p}| - 1, \arg \boldsymbol{p} - \theta_k).$$

The moments take the form

$$m_{ij} = \sum_{k=1}^{N} A_k^2 \cos^i \theta_k \sin^j \theta_k.$$

Corresponds to a maximum of N waves at each point.

Choose equations for moments $m_{2k-1,0}$ and $m_{0,2k-1}$, $k = 1, \ldots, N$.

Gives closed system of $2N$ equations with $2N$ unknowns (the A_k's and θ_k's).

Moment equations, examples

Ex. $N = 1$

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}_t + \begin{pmatrix} \frac{u_1^2}{\sqrt{u_1^2+u_2^2}} \\ \frac{u_1 u_2}{\sqrt{u_1^2+u_2^2}} \end{pmatrix}_x + \begin{pmatrix} \frac{u_1 u_2}{\sqrt{u_1^2+u_2^2}} \\ \frac{u_2^2}{\sqrt{u_1^2+u_2^2}} \end{pmatrix}_y = 0.$$

where $u_1 = m_{10} = A^2 \cos\theta$ and $u_2 = m_{01} = A^2 \sin\theta$.

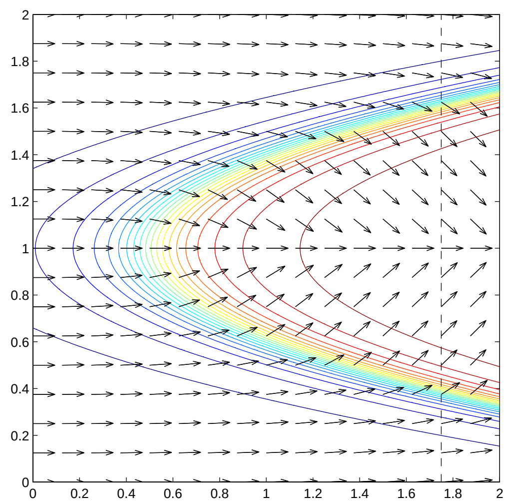
For $N \geq 2$,

$$\boldsymbol{F}_0(\boldsymbol{u})_t + \boldsymbol{F}_1(\boldsymbol{u})_x + \boldsymbol{F}_2(\boldsymbol{u})_y = 0.$$

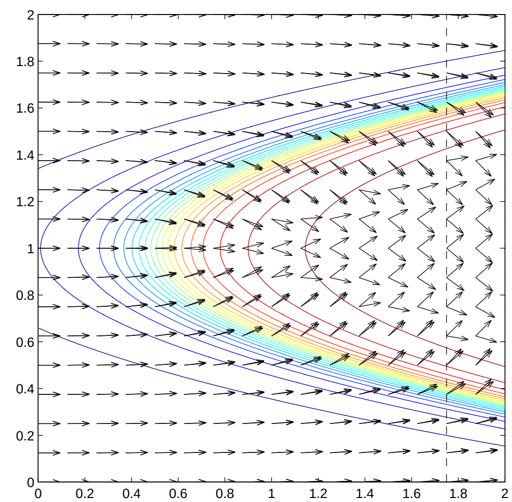
where $\boldsymbol{F}_0(\boldsymbol{u}), \boldsymbol{F}_1(\boldsymbol{u})$ and $\boldsymbol{F}_2(\boldsymbol{u})$ are complicated non-linear functions.

- PDE = weakly hyperbolic system of conservation laws, (with source terms when c varies)
- Flux functions in conservation law can be difficult to evaluate.

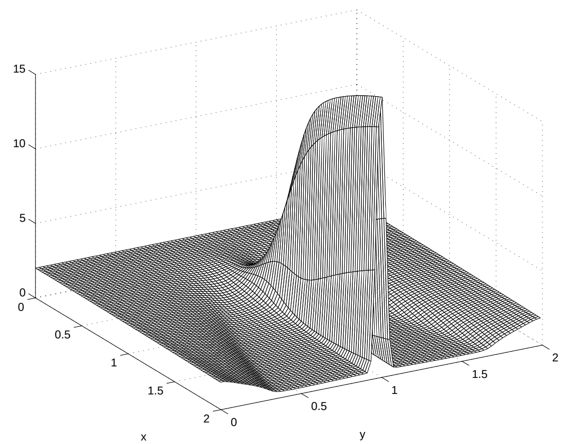
Wedge example



$N = 1$



$N = 2$



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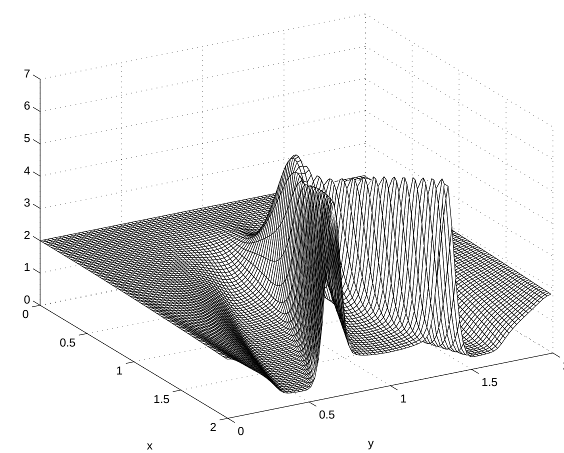


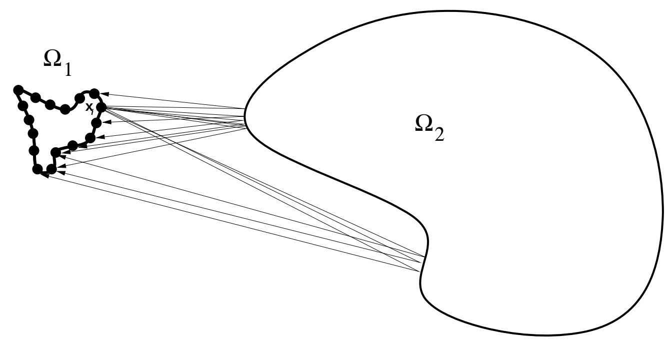
$N = 2$

Hybrid methods

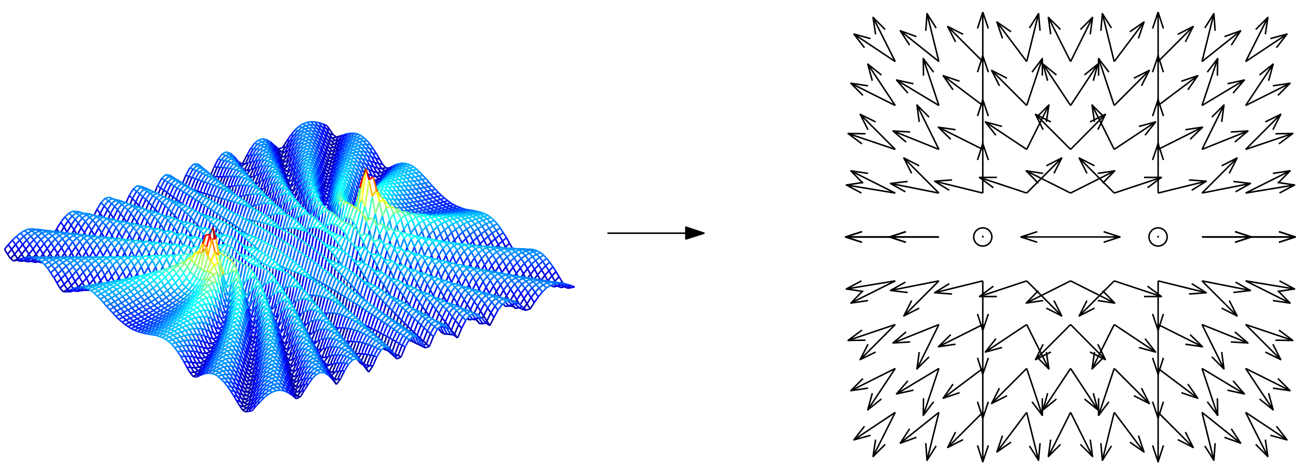
• Full Helmholtz or wave equation where variations in $c(x)$ and/or geometry on same scale as wavelength.

• GO elsewhere, typically for long range interactions.

Ex. antenna + aircraft.



Coupling of models.



Other methods

- Hamilton–Jacobi methods
  [Vidale, van Trier, Symes, Engquist, Fatemi, Osher, Benamou,. . .]
- Wavefront tracking using level sets in phase space
  [Osher, Tsai, Cheng, Liu, Jin, Qian, . . .]
- Wavefront tracking using segment projection
  [Engquist, OR, Tornberg]
- Full phase space methods
  [Sethian, Fomel, Symes, Qian]