Stable Semi-Discrete Schemes for the 2D Incompressible Euler Equations

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Maryland, May 2004

Outline

- Problem formulation
- Semi-discrete central schemes
- Other applications
- Discrete incompressibility and a maximum principle

The Incompressible 2D Euler and NS Equations

$$u_t + (u \cdot \nabla)u = -\nabla p + \nu \Delta u, \quad x \in \Omega \subset \mathbb{R}^2$$

$$\nabla \cdot u = 0$$

A vorticity formulation: $\omega = \nabla \times u$, $\vec{u} = (u, v)$.

1. A conservative form: $\omega_t + (u\omega)_x + (v\omega)_y = \nu \Delta \omega$.

2. A convective form: $\omega_t + u\omega_x + v\omega_y = \nu \Delta \omega$.

The divergence conditions $\Longrightarrow$ a streamfunction ψ: $u = \nabla^{\perp}\psi = (-\psi_y, \psi_x)$.

A Poisson equation: $\Delta \psi = \omega$.

Boundary conditions

1. No-slip boundary conditions for u: $u = 0$ on $\partial\Omega$.

2. No boundary conditions for ω (vorticity generation on the boundary, etc.)

3. Boundary conditions for ψ: $\psi(x,t) = \frac{\partial}{\partial n}\psi(x,t) = 0$, $x \in \partial\Omega$. (over-determined for the Poisson equation).

Pure Streamfunction Formulation

$$\frac{\partial \Delta\psi}{\partial t} + (\nabla^{\perp}\psi)\cdot\nabla(\Delta\psi) = \nu\Delta^2\psi, \qquad x \in \Omega$$

$$\psi(x,t) = \frac{\partial}{\partial n}\psi(x,t) = 0, \qquad x \in \partial\Omega.$$

THM (Uniqueness): Solutions are unique in $H_0^2(\Omega)$.

THM (Decay of solutions):

$$\||\nabla\psi(x,t)|\|_{L^2(\Omega)} \le e^{-\nu\lambda_\Omega t}\||\nabla\psi(x,0)|\|_{L^2(\Omega)}, \qquad \lambda_\Omega > 0.$$

Numerics for the Pure Streamfunction Formulation
(Ben-Artzi, Fishelov, Kupferman,...)

Time-discretization: Crank-Nicolson (2nd-order).

$$\begin{aligned}\left(\Delta - \frac{1}{4}\nu k \Delta^2\right)\psi^{n+\frac{1}{2}} &= \left(\Delta + \frac{1}{4}\nu k \Delta^2\right)\psi^{n} - \frac{1}{2}k[(u\cdot\nabla)\omega]^{n}, \\ \left(\Delta - \frac{1}{2}\nu k \Delta^2\right)\psi^{n+1} &= \left(\Delta + \frac{1}{2}\nu k \Delta^2\right)\psi^{n} - \frac{1}{2}k[(u\cdot\nabla)\omega]^{n+\frac{1}{2}}.\end{aligned}$$

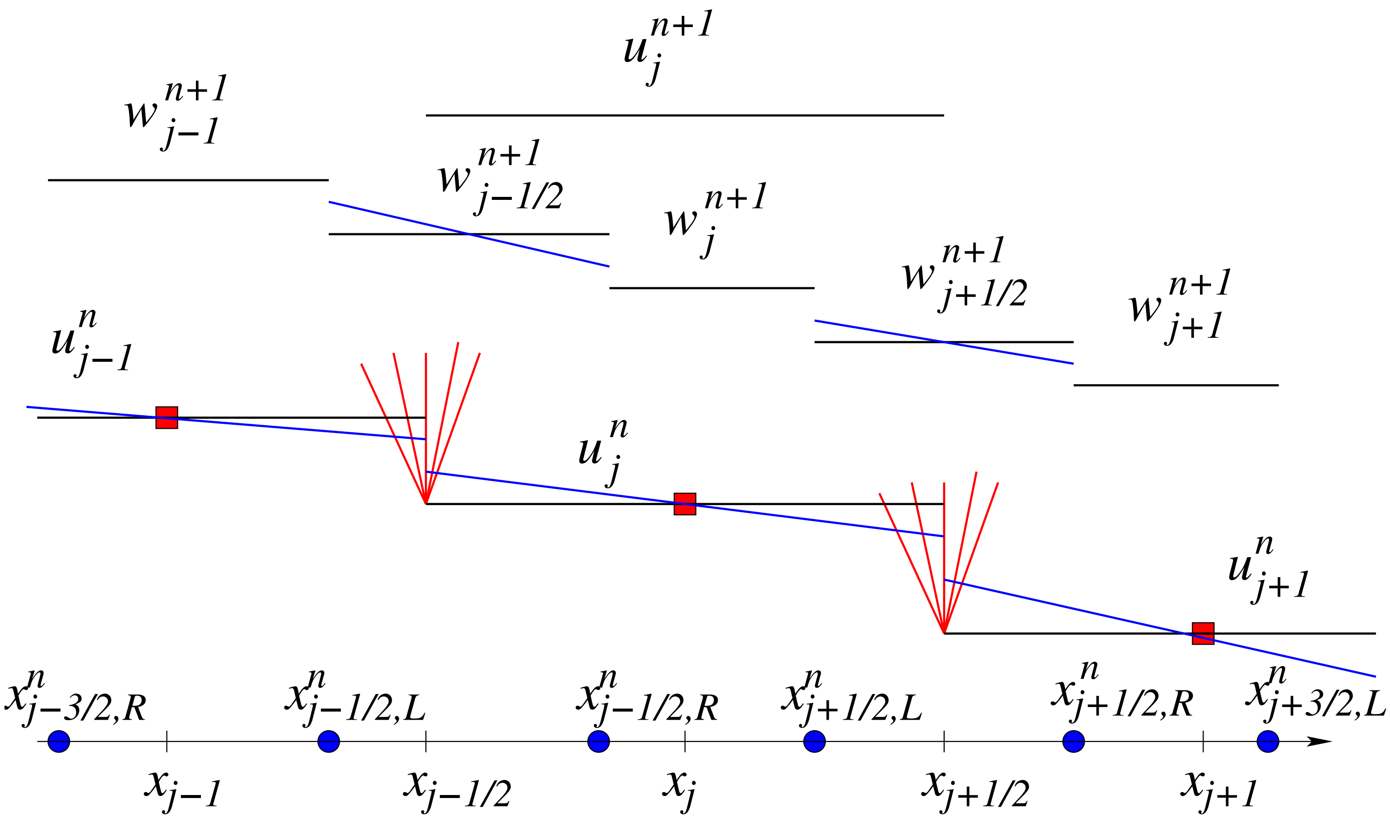
Spatial-discretization:

1. Laplacian (5-points), Bi-harmonic operator (a compact discretization due to Stephenson), The advection term, Linear solver $(\Delta - \alpha\Delta^2)\psi_{i,j} = RHS$.

2. Pure streamfunction vs. vorticity-streamfunction formulation.

Semi-Discrete Central-Schemes for Conservation Laws (Kurganov-Tadmor)

$$\boxed{u_t + f(u)_x = 0}$$



Reconstruction: In I_j, $\tilde{u}(x,t^n) = u_j^n + (u_x)_j^n(x - x_j)$.

Nonlinear limiters: MinMod with $1 \le \theta \le 2$

$$u_x = \mathcal{MM}\left(\theta\frac{u_j^n - u_{j-1}^n}{\Delta x}, \frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x}, \theta\frac{u_{j+1}^n - u_j^n}{\Delta x}\right).$$

Local speeds of propagation:

$$a_{j+\frac{1}{2}}(t) = \max_{u \in \mathcal{C}\left(u^-_{j+\frac{1}{2}}, u^+_{j+\frac{1}{2}}\right)} \rho\left(\frac{\partial f}{\partial u}(u)\right),$$

Evolution points: $x^n_{j+\frac{1}{2},R} = x_{j+\frac{1}{2}} + a^n_{j+\frac{1}{2}}\Delta t$.

An exact evolution: $w^{n+1}_{j+\frac{1}{2}} = \ldots$, $w^{n+1}_j = \ldots$.

Projection: $u_j = \frac{1}{\Delta x}\int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} \tilde{w}(\xi, t^{n+1})d\xi$.

In the limit

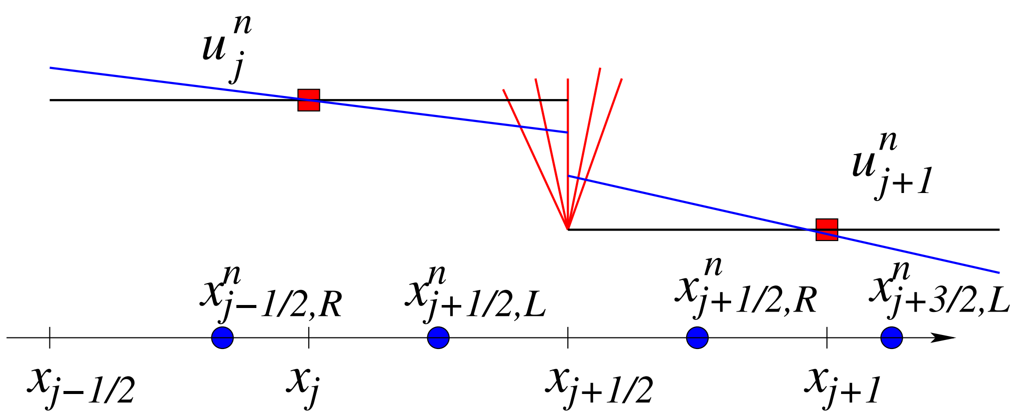
$$\lim_{\Delta t \to 0} \frac{u_j^{n+1} - u_j^n}{\Delta t}$$

The semi-discrete scheme:

$$\frac{d}{dt} u_j = -\frac{H_{j+\frac{1}{2}}(t) - H_{j-\frac{1}{2}}(t)}{\Delta x}.$$

The numerical flux:

$$H_{j+\frac{1}{2}} = \frac{f\left(u^+_{j+\frac{1}{2}}\right) + f\left(u^-_{j+\frac{1}{2}}\right)}{2} - \frac{a_{j+\frac{1}{2}}}{2}\left(u^+_{j+\frac{1}{2}} - u^-_{j+\frac{1}{2}}\right)$$



2D Semi-discrete Godunov-type schemes

$$\boxed{u_t + f(u)_x + g(u)_y = 0}$$

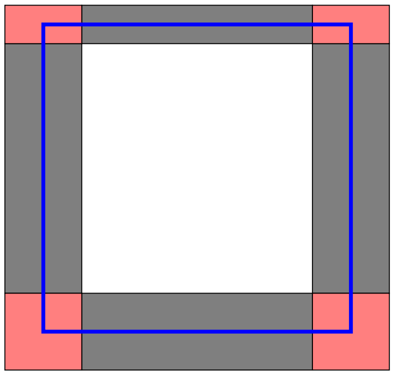
A Reconstruction:

$$\tilde{u}(x,y,t) = u_{j,k} + (u_x)_{j,k}(x - x_j) + (u_y)_{j,k}(y - y_k).$$

Local speeds of propagation:

$$a^x_{j+\frac{1}{2},k} = \max_{\pm} \rho\left(\frac{\partial f}{\partial u}\left(u^{\pm}_{j+\frac{1}{2},k}\right)\right), \qquad a^y_{j,k+\frac{1}{2}} = \max_{\pm} \rho\left(\frac{\partial g}{\partial u}\left(u^{\pm}_{j,k+\frac{1}{2}}\right)\right).$$

Evolution points $\Longrightarrow$ temporary averages



Projection to the original grid + limit as $\Delta t \to 0$

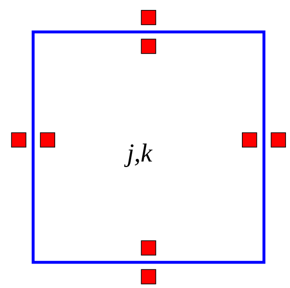
The semi-discrete scheme:

$$\frac{d}{dt}u_{jk}(t) = -\frac{H^x_{j+\frac{1}{2},k} - H^x_{j-\frac{1}{2},k}}{\Delta x} - \frac{H^y_{j,k+\frac{1}{2}} - H^y_{j,k-\frac{1}{2}}}{\Delta y}.$$

The numerical fluxes:

$$H^x_{j+\frac{1}{2},k}(t) = \frac{f\left(u^+_{j+\frac{1}{2},k}(t)\right) + f\left(u^-_{j+\frac{1}{2},k}(t)\right)}{2} - \frac{a^x_{j+\frac{1}{2},k}(t)}{2}\left[u^+_{j+\frac{1}{2},k}(t) - u^-_{j+\frac{1}{2},k}(t)\right]$$

$$H^y_{j,k+\frac{1}{2}}(t) = \frac{g\left(u^+_{j,k+\frac{1}{2}}(t)\right) + g\left(u^-_{j,k+\frac{1}{2}}(t)\right)}{2} - \frac{a^y_{j,k+\frac{1}{2}}(t)}{2}\left[u^+_{j,k+\frac{1}{2}}(t) - u^-_{j,k+\frac{1}{2}}(t)\right]$$



Comments

1. Third-order extensions (Kurganov, DL, Noelle, Petrova,...)
2. Different reconstructions
3. Numerical fluxes (local speeds of propagation, the projection step,...)

Additional terms: Parabolic, Source terms, ...

$$\boxed{u_t + f(u)_x = Q(u, u_x)_x}$$

$$\frac{d}{dt}u_j = -\frac{H_{j+\frac{1}{2}}(t) - H_{j-\frac{1}{2}}(t)}{\Delta x} + \frac{P_{j+\frac{1}{2}}(t) - P_{j-\frac{1}{2}}(t)}{\Delta x}.$$

The diffusion flux: $P_{j+\frac{1}{2}} = \frac{1}{2}\left[Q\left(u_j, \frac{u_{j+1}-u_j}{2}\right) + Q\left(u_{j+1}, \frac{u_{j+1}-u_j}{2}\right)\right].$

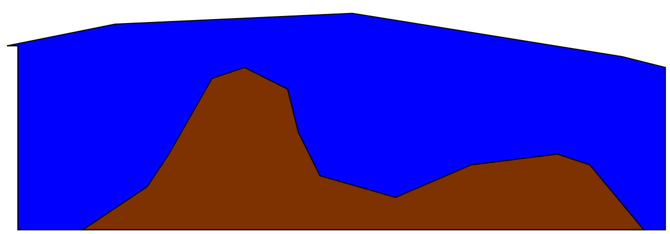
$$\boxed{u_t + f(u)_x = S(u)}$$

$$\frac{d}{dt}u_j = -\frac{H_{j+\frac{1}{2}}(t) - H_{j-\frac{1}{2}}(t)}{\Delta x} + \bar{S}_j.$$

The source term: $\bar{S}_j = ?$

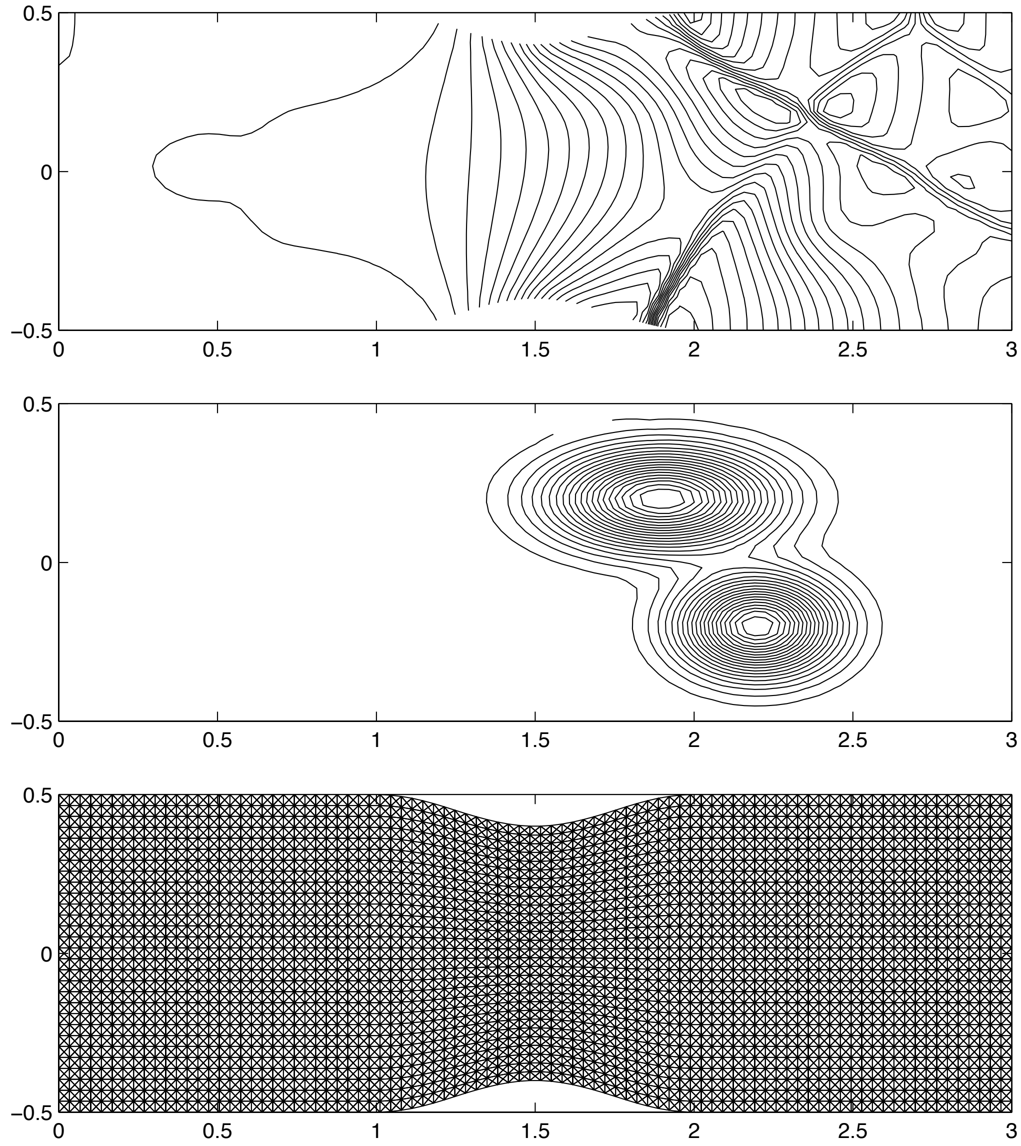
Example: Shallow Water Waves (Bryson + DL)

$$\begin{cases} h_t + (hu)_x + (hv)_y = 0, \\ (hu)_t + \left(hu^2 + \frac{1}{2}gh^2\right)_x + (huv)_y = -ghB_x, \\ (hv)_t + (huv)_x + \left(hv^2 + \frac{1}{2}gh^2\right)_x = -ghB_y. \end{cases}$$

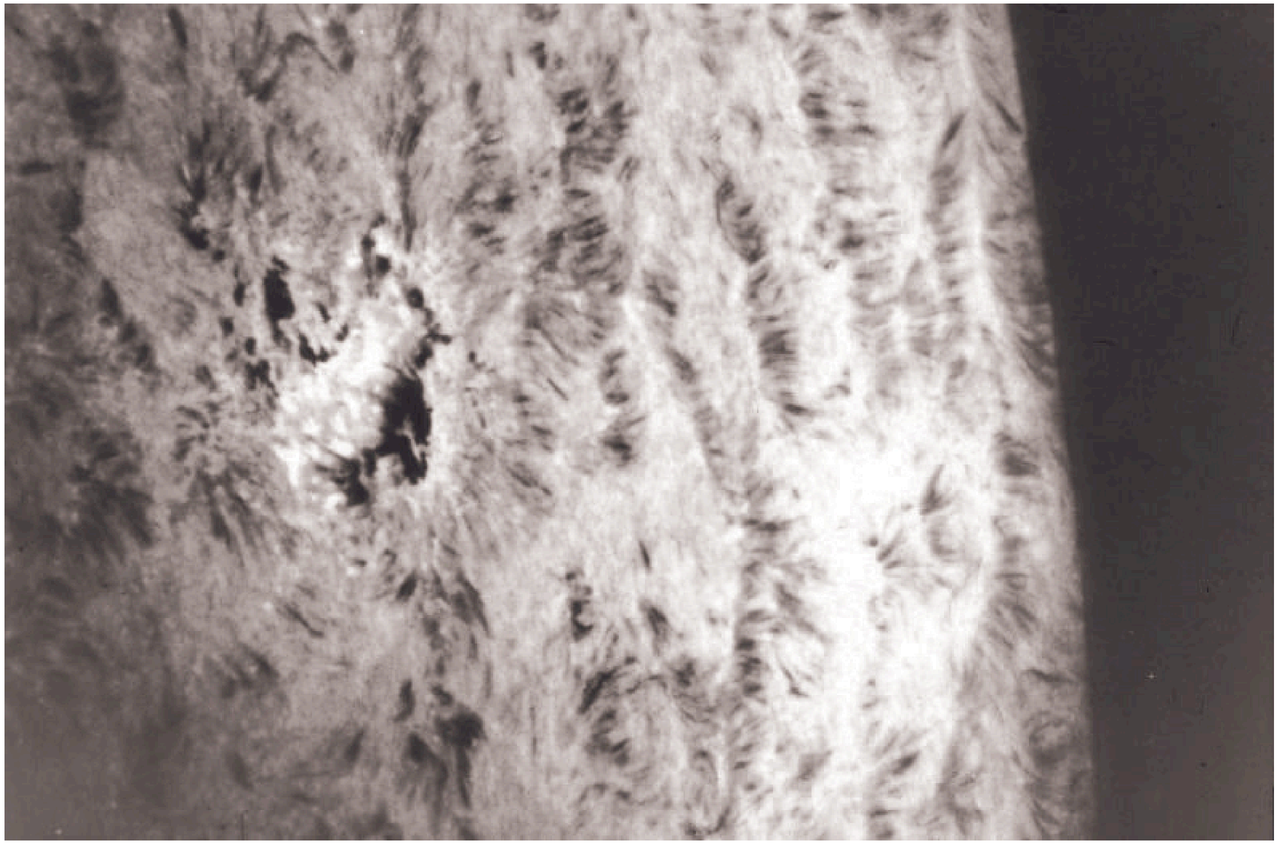


- Unstructured grids
- Source term discretization that preserves stationary steady-states

A Converging-Diverging Channel + Bottom Topography (Bryson-DL)



Example: Dynamics of the Solar Atmosphere (Bryson-Kosovichev-DL)



Spicules near the solar limb. They are blue-shifted (hence dark) in this narrow-band hydrogen-alpha image due to their motion towards the observer. (National Solar Observatory, Sacramento Peak)

Phenomena:

1. **Spicules.** Perhaps cooler photospheric material shooting up into the corona.

2. **Coronal Oscillations.** Particle oscillations in the corona have been observed to have periods of about 5 minutes above quiet regions and 3 minutes about sunspots (De Moortel et al., 2002).

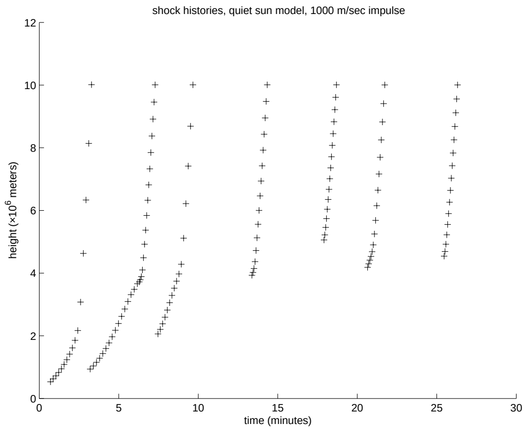
The model:

Euler equations for material in a flux tube of area $A(x,t)$ and gravity $g(x)$:

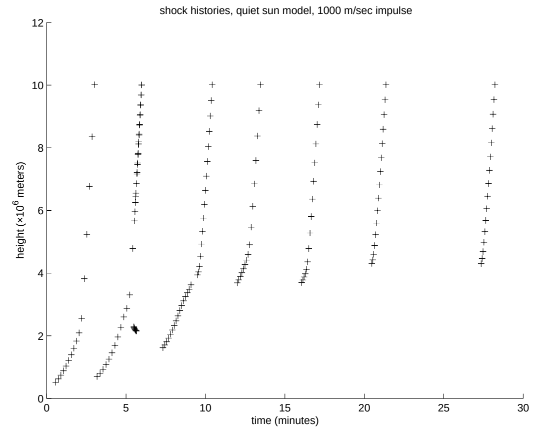
$$\begin{pmatrix} \rho \\ \rho u \\ E \end{pmatrix}_t + \begin{pmatrix} \rho u \\ \rho u^2 + p \\ (E+p)u \end{pmatrix}_x = \begin{pmatrix} -\rho u A^{-1} A_x - \rho A^{-1} A_t \\ -\rho u^2 A^{-1} A_x - \rho u A^{-1} A_t - g(x)\rho + F(x,t)\rho \\ -(E+p)uA^{-1}A_x - EA^{-1}A_t - g(x)\rho u \end{pmatrix}$$

$F(x,t)$: a velocity forcing term that perturb the base of the solar atmosphere.

Trajectories of Velocity Shocks, Quiet Sun Model (Bryson-Kosovichev-DL)



Constant flux tube



Expanding flux tube

An initial impulse amplitude of 1000 meters/second.

Numerical issues:

1. hydrostatic equilibrium at initialization and at the boundaries. Static $A = A(x)$.

2. 1D nonuniform grids.

Back to the Euler Equations (Vorticity)

The conservative formulation:

$$\boxed{\omega_t + (u\omega)_x + (v\omega)_y = 0}$$

Reconstruction (in $I_{j,k}$):

$$\tilde{\omega}(x,y) = \omega_{j,k} + (\omega_x)_{j,k}(x - x_j) + (\omega_y)_{j,k}(y - y_k).$$

The local speeds of propagation (as long as the velocities are in L^∞):

$$a^x_{j+\frac{1}{2},k} = |u_{j+\frac{1}{2},k}|, \qquad a^y_{j,k+\frac{1}{2}} = |v_{j,k+\frac{1}{2}}|.$$

(from the convective formulation ($\omega_t + u\omega_x + v\omega_y = 0$)).

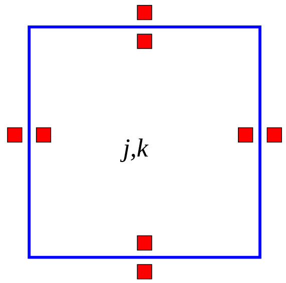
The scheme:

$$\frac{d}{dt}\omega_{jk}(t) = -\frac{H^x_{j+\frac{1}{2},k} - H^x_{j-\frac{1}{2},k}}{\Delta x} - \frac{H^y_{j,k+\frac{1}{2}} - H^y_{j,k-\frac{1}{2}}}{\Delta y}.$$

The numerical fluxes:

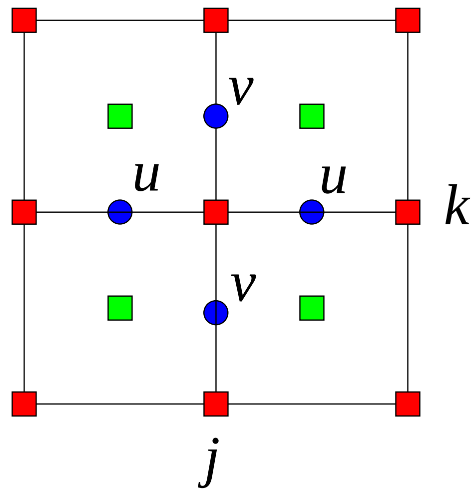
$$\begin{aligned} H^x_{j+\frac{1}{2},k} &= \frac{1}{2}\left(\omega^+_{j+\frac{1}{2},k} + \omega^-_{j+\frac{1}{2},k}\right) u_{j+\frac{1}{2},k} - \frac{a^x_{j+\frac{1}{2},k}}{2}\left(\omega^+_{j+\frac{1}{2},k} - \omega^-_{j+\frac{1}{2},k}\right), \\ H^y_{j,k+\frac{1}{2}} &= \frac{1}{2}\left(\omega^+_{j,k+\frac{1}{2}} + \omega^-_{j,k+\frac{1}{2}}\right) v_{j,k+\frac{1}{2}} - \frac{a^y_{j,k+\frac{1}{2}}}{2}\left(\omega^+_{j,k+\frac{1}{2}} - \omega^-_{j,k+\frac{1}{2}}\right). \end{aligned}$$

The remaining ingredient: the velocities $(u_{j\pm\frac{1}{2},k}, v_{j,k\pm\frac{1}{2}}) = ?$



Reconstructing the Velocities

The 5-point Laplacian: $\Delta\psi_{jk} = \omega_{jk}$.



Define the velocities:

$$u_{j+\frac{1}{2},k} = \frac{1}{2\Delta y}\left(\frac{\psi_{j,k+1}+\psi_{j+1,k+1}}{2} - \frac{\psi_{j,k-1}+\psi_{j+1,k-1}}{2}\right)$$

$$v_{j,k+\frac{1}{2}} = \frac{1}{2\Delta x}\left(-\frac{\psi_{j+1,k}+\psi_{j+1,k+1}}{2} + \frac{\psi_{j-1,k}+\psi_{j-1,k+1}}{2}\right).$$

A discrete incompressibility relation:

$$\frac{u_{j+\frac{1}{2},k} - u_{j-\frac{1}{2},k}}{\Delta x} + \frac{v_{j,k+\frac{1}{2}} - v_{j,k-\frac{1}{2}}}{\Delta y} = 0.$$

The Incompressible NS Equations

$$\boxed{\omega_t + (u\omega)_x + (v\omega)_y = \nu\Delta\omega}$$

$$\frac{d\omega_{j,k}}{dt} = -\frac{H^x_{j+1/2,k}(t) - H^x_{j-1/2,k}(t)}{\Delta x} - \frac{H^y_{j,k+1/2}(t) - H^y_{j,k-1/2}(t)}{\Delta y} + \nu Q_{j,k}(t)$$

$$\begin{aligned} Q_{j,k}(t) \quad &= \quad \frac{-\omega_{j+2,k}(t) + 16\omega_{j+1,k}(t) - 30\omega_{j,k}(t) + 16\omega_{j-1,k}(t) - \omega_{j-2,k}(t)}{12\Delta x^2} \\ &\quad + \frac{-\omega_{j,k+2}(t) + 16\omega_{j,k+1}(t) - 30\omega_{j,k}(t) + 16\omega_{j,k-1}(t) - \omega_{j,k-2}(t)}{12\Delta y^2}. \end{aligned}$$

- Third-order (KL, KP, KNP) - reconstructions (of ω, u, v), numerical flux, no theory.

A Maximum Principle

Theorem (DL): Consider the fully-discrete scheme:

$$\omega_{j,k}^{n+1} = \omega_{j,k}^{n} - \lambda \left(H^x_{j+\frac{1}{2},k} - H^x_{j-\frac{1}{2},k} \right) - \mu \left(H^y_{j,k+\frac{1}{2}} - H^y_{j,k-\frac{1}{2}} \right).$$

with the numerical fluxes

$$\begin{aligned} H^x_{j+\frac{1}{2},k} &= \frac{1}{2}\left(\omega^+_{j+\frac{1}{2},k} + \omega^-_{j+\frac{1}{2},k}\right) u_{j+\frac{1}{2},k} - \frac{a^x_{j+\frac{1}{2},k}}{2}\left(\omega^+_{j+\frac{1}{2},k} - \omega^-_{j+\frac{1}{2},k}\right), \\ H^y_{j,k+\frac{1}{2}} &= \frac{1}{2}\left(\omega^+_{j,k+\frac{1}{2}} + \omega^-_{j,k+\frac{1}{2}}\right) v_{j,k+\frac{1}{2}} - \frac{a^y_{j,k+\frac{1}{2}}}{2}\left(\omega^+_{j,k+\frac{1}{2}} - \omega^-_{j,k+\frac{1}{2}}\right). \end{aligned}$$

Assume that the velocities $u, v \in L^\infty$, that the derivatives are reconstructed with the MinMod limiter and that the following CFL condition holds

$$\max(\lambda \max_u |u|, \mu \max_v |v|) \le \frac{1}{8}.$$

Then

$$\max_{j,k}(\omega^{n+1}_{j,k}) \le \max_{j,k}(\omega^n_{j,k})$$

Proof

1. Writing ω^{n+1}_{jk} as a convex combination of $\omega^{\pm}_{j\pm\frac{1}{2},k}, \omega^{\pm}_{j,k\pm\frac{1}{2}}$:

$$\begin{aligned} \omega^{n+1}_{j,k} &= \omega^{+}_{j+\frac{1}{2},k}\left[\frac{\lambda}{2}\left(a^{x}_{j+\frac{1}{2},k} - u_{j+\frac{1}{2},k}\right)\right] \\ &\quad + \omega^{-}_{j+\frac{1}{2},k}\left[\frac{1}{4} + \frac{\lambda}{2}\left(u_{j+\frac{1}{2},k} - a^{x}_{j+\frac{1}{2},k} - 2\frac{\Delta^{x}_{j,k} u\omega}{\Delta^{x}_{j,k}\omega}\right)\right] + \ldots \end{aligned}$$

2. Use the properties of the $\mathcal{MM}$ limiter:

$$\max_{j,k}\left(\omega^{\pm}_{j\pm\frac{1}{2},k}, \omega^{\pm}_{j,k\pm\frac{1}{2}}\right) \leq \max_{j,k}(\omega^{n}_{j,k}).$$

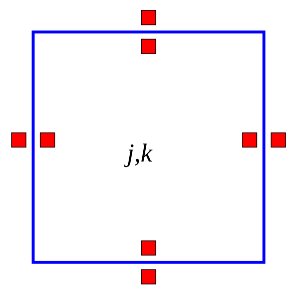
3. High-order in time: a convex combination of FE (TVD-RK).

Comments

1. Simpler than the fully-discrete case (DL-Tadmor). There: an exact discrete incompressibility was used to obtain the convex combination. Here - not needed...

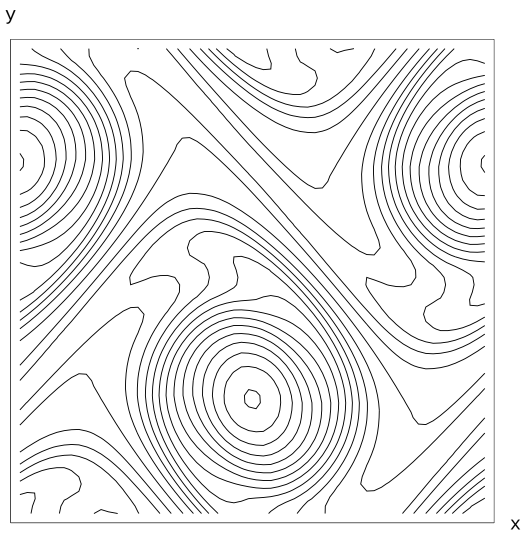
2. Clear upwinding structure:

$$\begin{aligned}\omega_{j,k}^{n+1} &= \omega_{j+\frac{1}{2},k}^{-}\left[\frac{1}{4}-\frac{\lambda}{2}\left(u_{j+\frac{1}{2},k}+a_{j+\frac{1}{2},k}^{x}\right)\right] \\ &\quad +\omega_{j+\frac{1}{2},k}^{+}\left[\frac{\lambda}{2}\left(a_{j+\frac{1}{2},k}^{x}-u_{j+\frac{1}{2},k}\right)\right]+\dots\end{aligned}$$

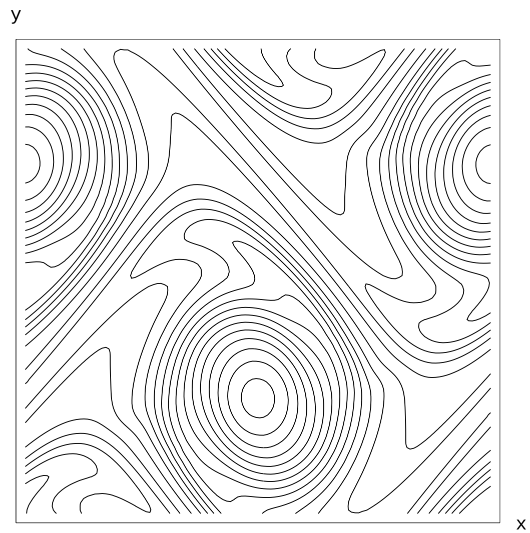


Incompressible NS, Double Shear-Layer Problem (Kurganov-DL)

3rd-order semi-discrete. $T = 10$. $\nu = 0.01$.



$N = 64 \times 64$



$N = 128 \times 128.$

Initial data:

$$u(x,y,0) = \begin{cases} \tanh\left(\frac{15}{\pi}\left(y - \frac{\pi}{2}\right)\right), & y \leq \pi, \\ \tanh\left(\frac{15}{\pi}\left(\frac{3\pi}{2} - y\right)\right), & y > \pi, \end{cases} \qquad v(x,y,0) = 0.05\sin(x).$$

Conclusion

- Simple
- Semi-discrete Godunov-type schemes
- Parabolic terms, source terms,...
- Applications
- A new maximum principle