Judging Model Reduction of Chaotic Systems via Shadowing Criteria

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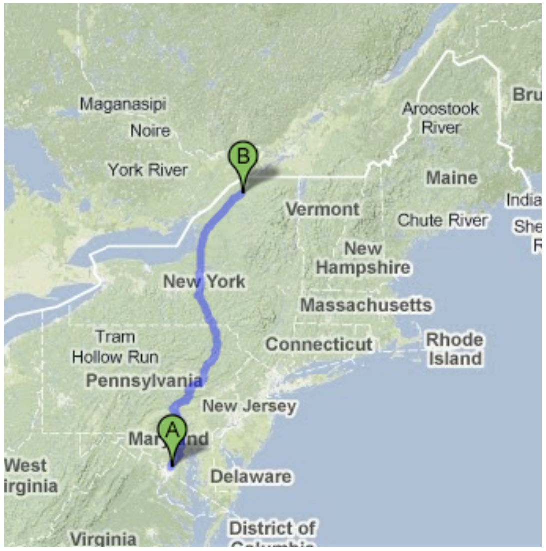
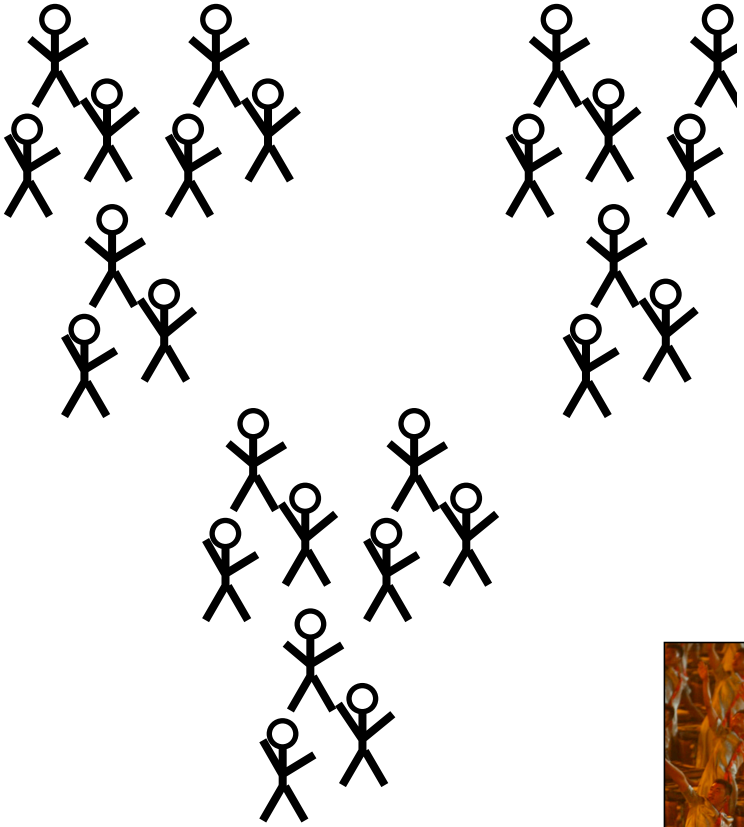
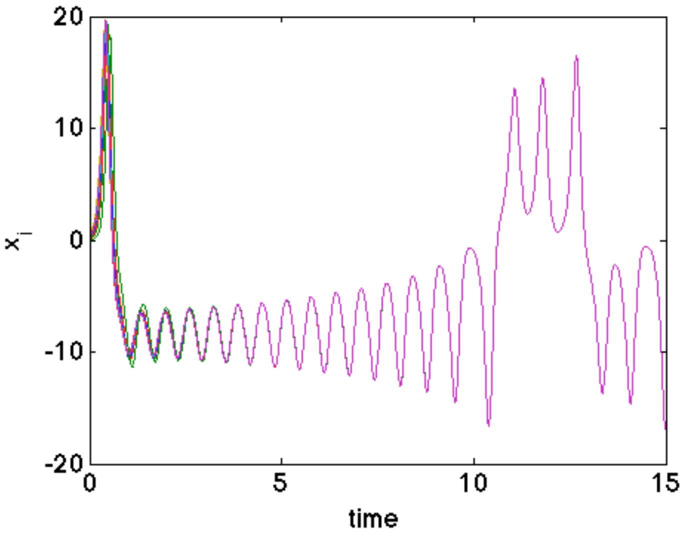


- **bolltem@clarkson.edu,** **http://www.clarkson.edu/~bolltem**

Example Dimension Reduction when many coupled oscillators

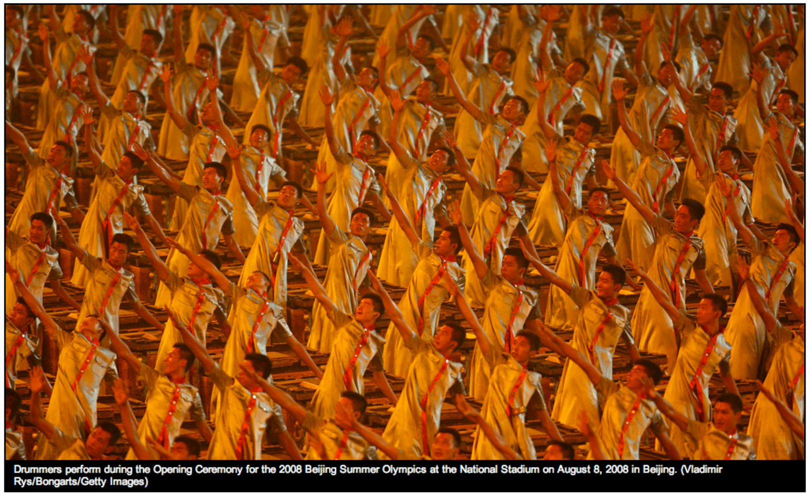


“Classical” Analysis

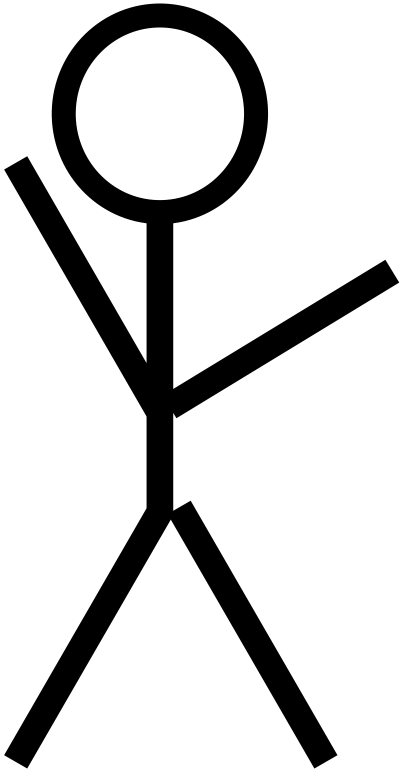


Master Stability Functions

Only for IDENTICAL oscillators.



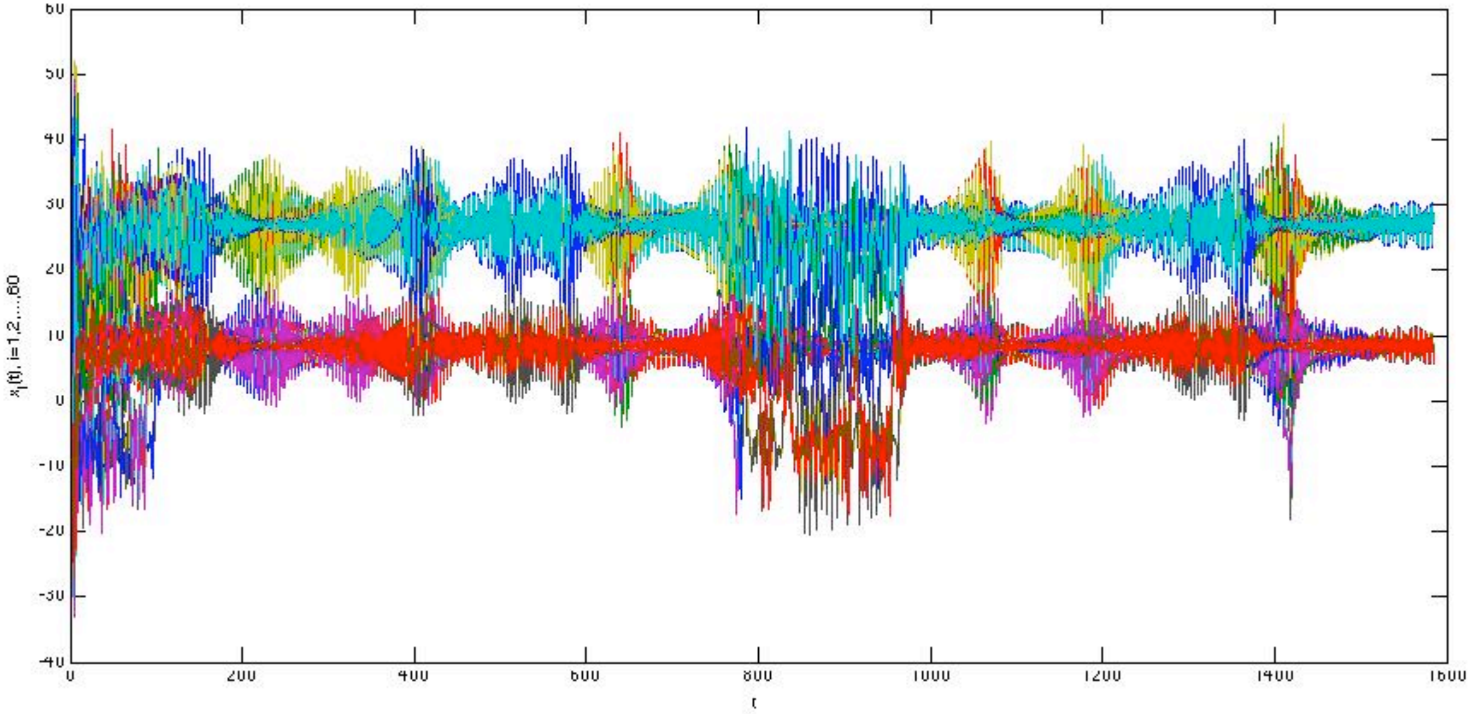
Drummers perform during the Opening Ceremony for the 2008 Beijing Summer Olympics at the National Stadium on August 8, 2008 in Beijing. (Vladimir Rys/Bongarts/Getty Images)



Example Dimension Reduction when many coupled oscillators

What model reduction/cooperation is here?!

In this signal?



-Cooperation and model reduction and many acting as one. Or as a few, In clusters.

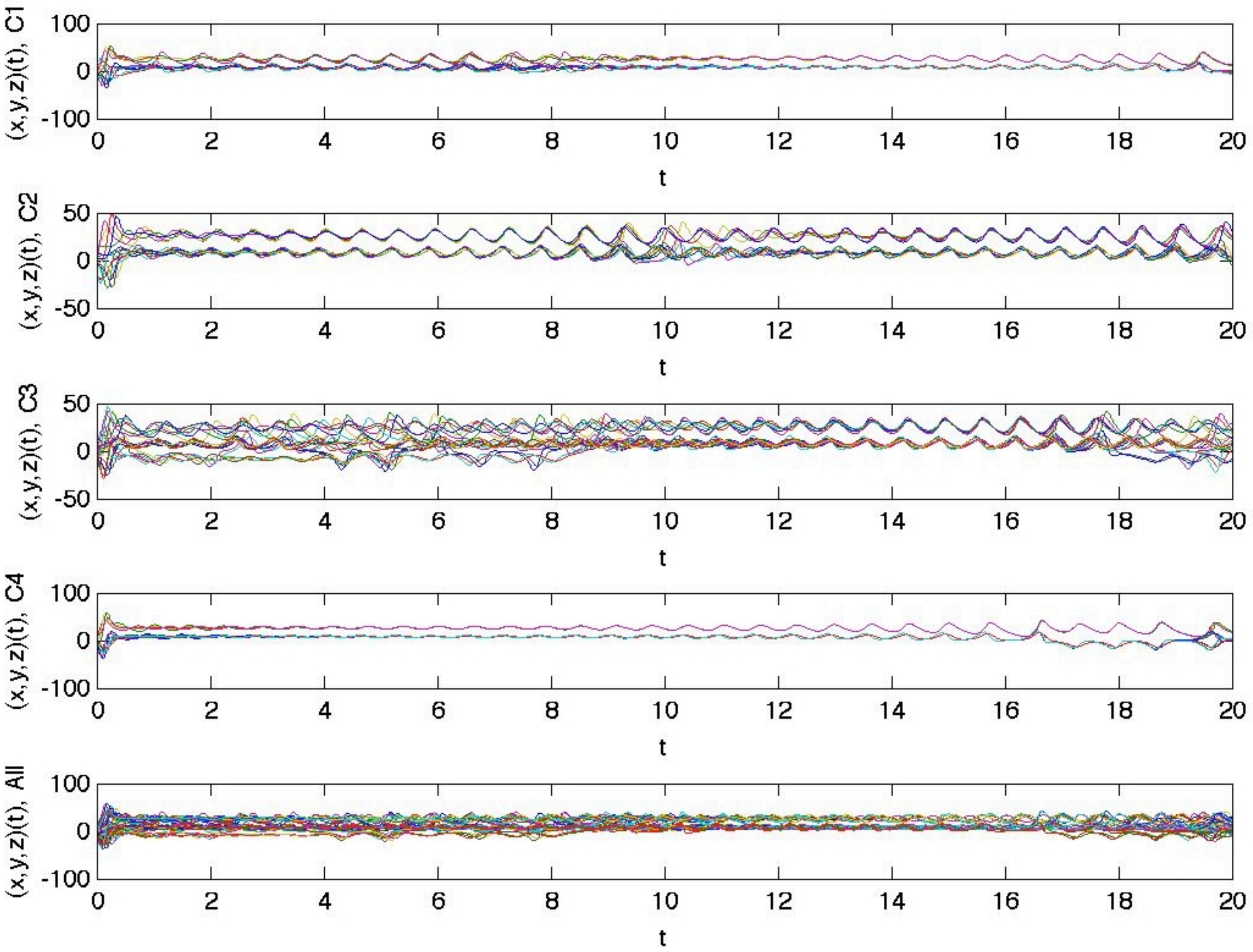
-And communities/partition/signals this tool is about the nonlinear averaging with respect to the appropriate partition and within, appropriate invariant manifold.

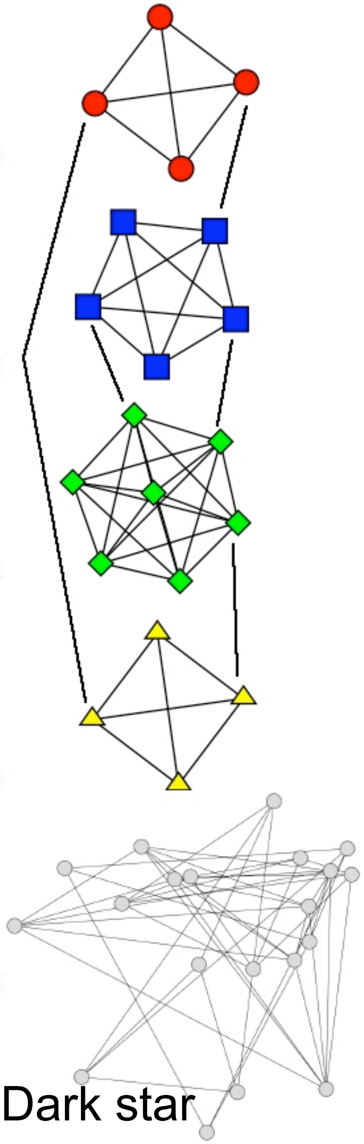
-agent model/swarm/Infectious Disease Dynamics.

-Hierarchical

We have networks, and then we have dynamics on Networks

Partition is key to begin the discussion of appropriate “averages”
– nonlinear “average” meaning error from an invariant manifold

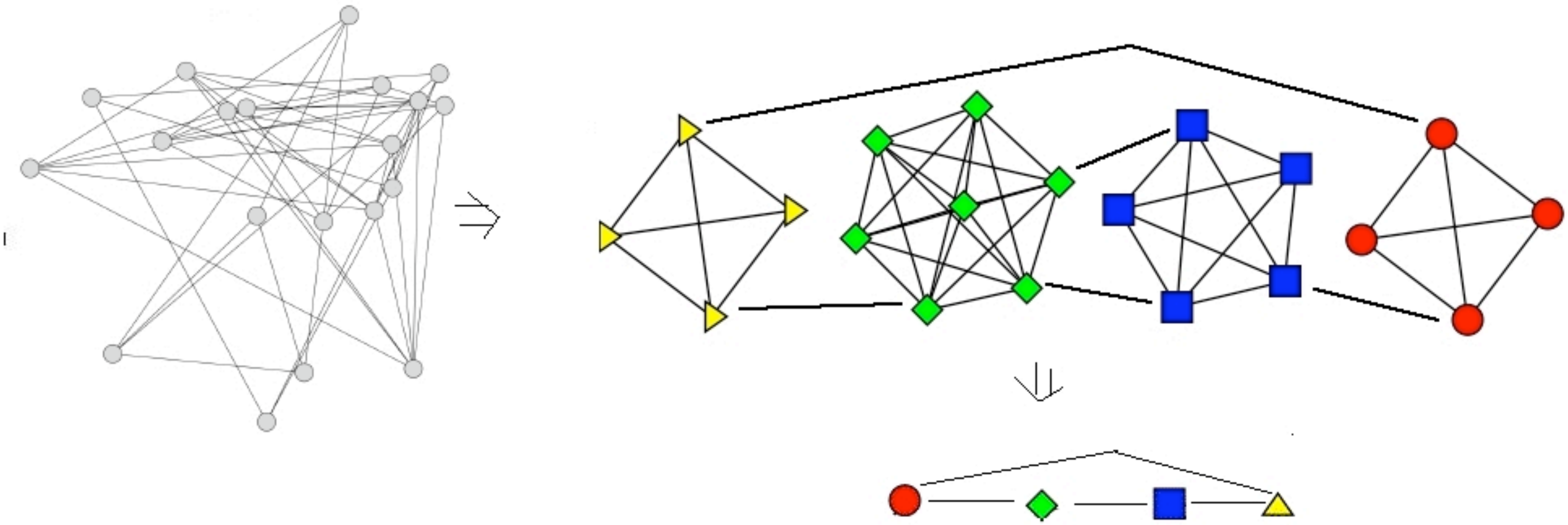




Dark star
Network

Appropriate partition partition, from which follows model reductions (sometimes dramatic simplification)

Which course grained scale is right?
Each: Simplify as appropriate.



-Coarse – Grained Models

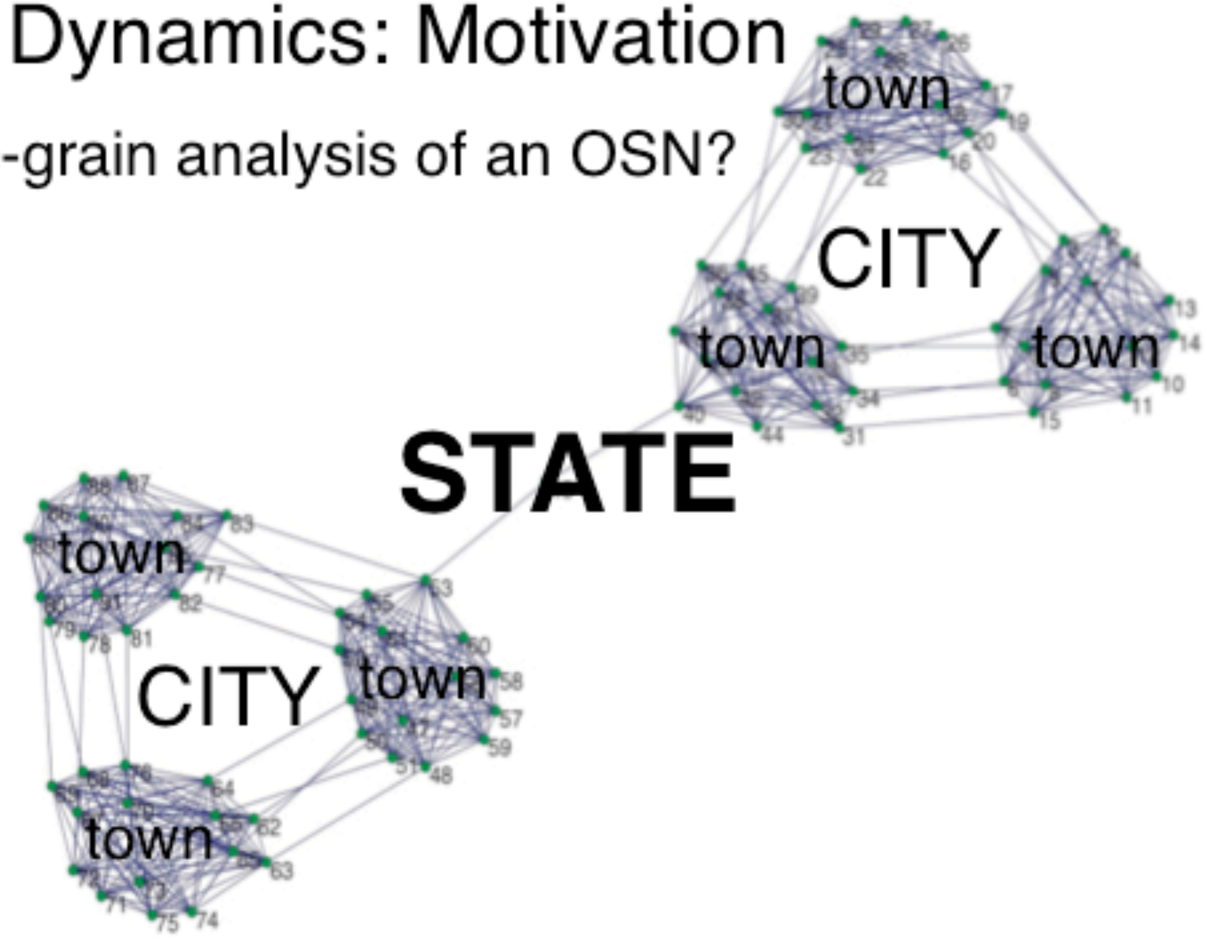
-Hierarchical

-Russian Doll of Hierarchical Models

-Jie Sun, Erik M. Bollt, and Takashi Nishikawa, "Master Stability Functions for Coupled Near-Identical Dynamical System," arXiv: 0811.0649, To appear Euro. Phys. Lett. (2009).

Multi-scale Dynamics: Motivation

Coarse-grain analysis of an OSN?



Perhaps a hierarchy of models/dynamical systems is appropriate, each available depending on the setting

J.P. Bagrow, E. M. Bollt, "A Local Method for Detecting Communities," cond-mat/0412482, Phys. Rev. E, 72 046108 (2005).

Two Themes here:

I. What is model reduction/dimension reduction?

-Series Truncation?

-Existence of a slow manifold?

-Inertial Manifold?

-Synchronization/cooperation?

II. How do I know if I did a good job?

- **-Error in a Banach space? –Residual.**
- **-Conjugacy/Diffeomorphism?**
- **-Shadowing time?**

The problem of model reduction requires comparison between the original model and the reduced order model in some appropriate ways.

For high dimensional chaotic system, direct comparison of two models is problematic – Even slight differences might cause considerable structural difference between orbits generated by the models respectively – not to mention Sens. Dep.

For a given high-dimensional system, there are often many different low-dimensional reduced models.

-For example, is it better to simply average the equations for individual units to obtain a reduced model for a coupled oscillator network,

-Or is it better to use a weighted average of the oscillator dynamics reflecting their various roles within the network?

-Would it be better to introduce an extra component into the reduced model to compensate for the loss of information due to dimensionality reduction?

To properly answer such questions, it is desirable and necessary to QUANTIFY the quality of a reduced model for a given system.

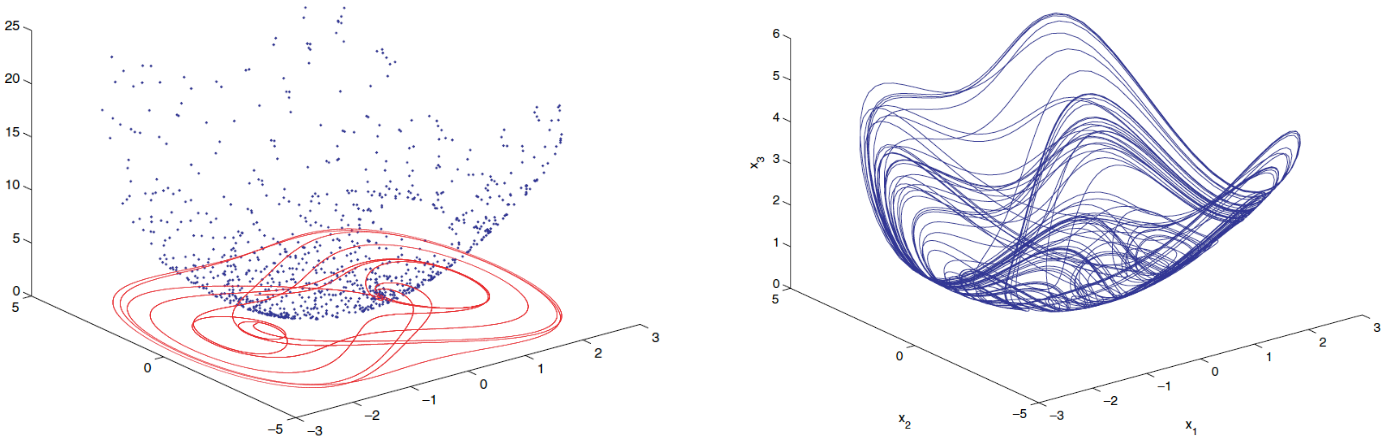
The difficulty comes partly from the fact of systems of different dimensions, making unnatural direct comparisons of either equations of motion or time series. Not to mention sensitive dependence to initial conditions.

A Dozen Slides or So
to tell you what I am not talking about…

I. What is model reduction/dimension reduction?

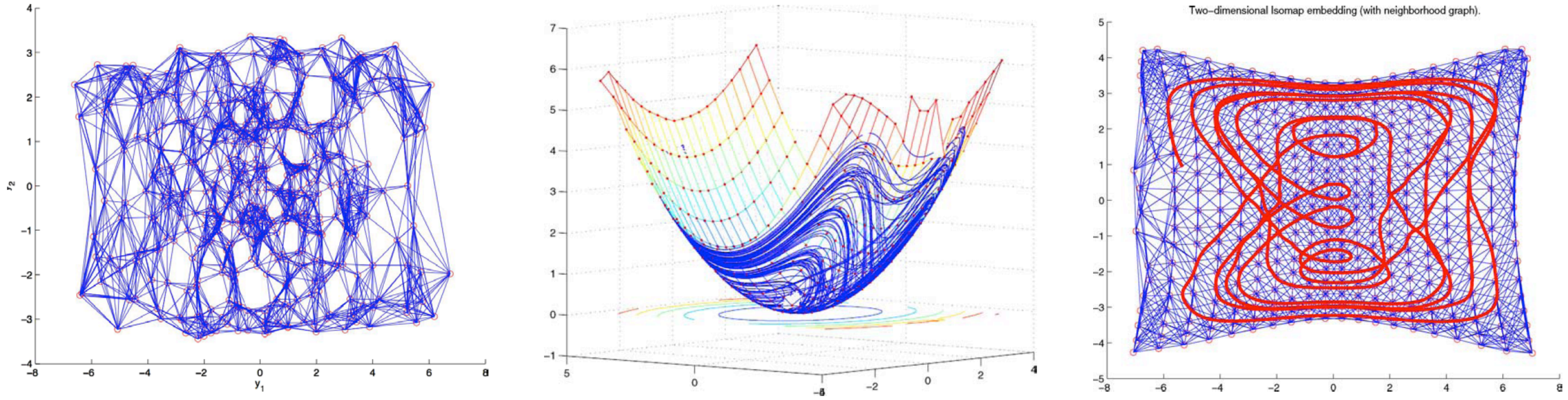
$$\begin{aligned}\dot{x}_1 &= x_2,\\ \dot{x}_2 &= \sin(x_3) - ax_2 - x_1^3 + x_1,\\ \dot{x}_3 &= 1,\\ \epsilon\dot{y} &= y - \alpha(x_1^2 + x_2^2).\end{aligned}$$

Example Dimension Reduction when slow manifold – Duffing on a paraboloid.



Looking for equations of motion in fewer variables in intrinsic coordinates

s’=f(s)=F(s,H(s))



Erik Bollt, "Attractor Modeling and Empirical Nonlinear Model Reduction of Dissipative Dynamical Systems," International Journal of Bifurcation and Chaos (IJBC) in Applied Sciences and Engineering, Vol. 17, No. 4 (2007) 1199-1219.

-Example Dimension Reduction when Series truncation.

Kuramoto–Shivasinky equations

$$u_t = (u^2)_x - u_{xx} - \nu u_{xxxx}, x \in [0, 2\pi],$$

periodically extended, $u(x,t) = u(x+2\pi, t)$

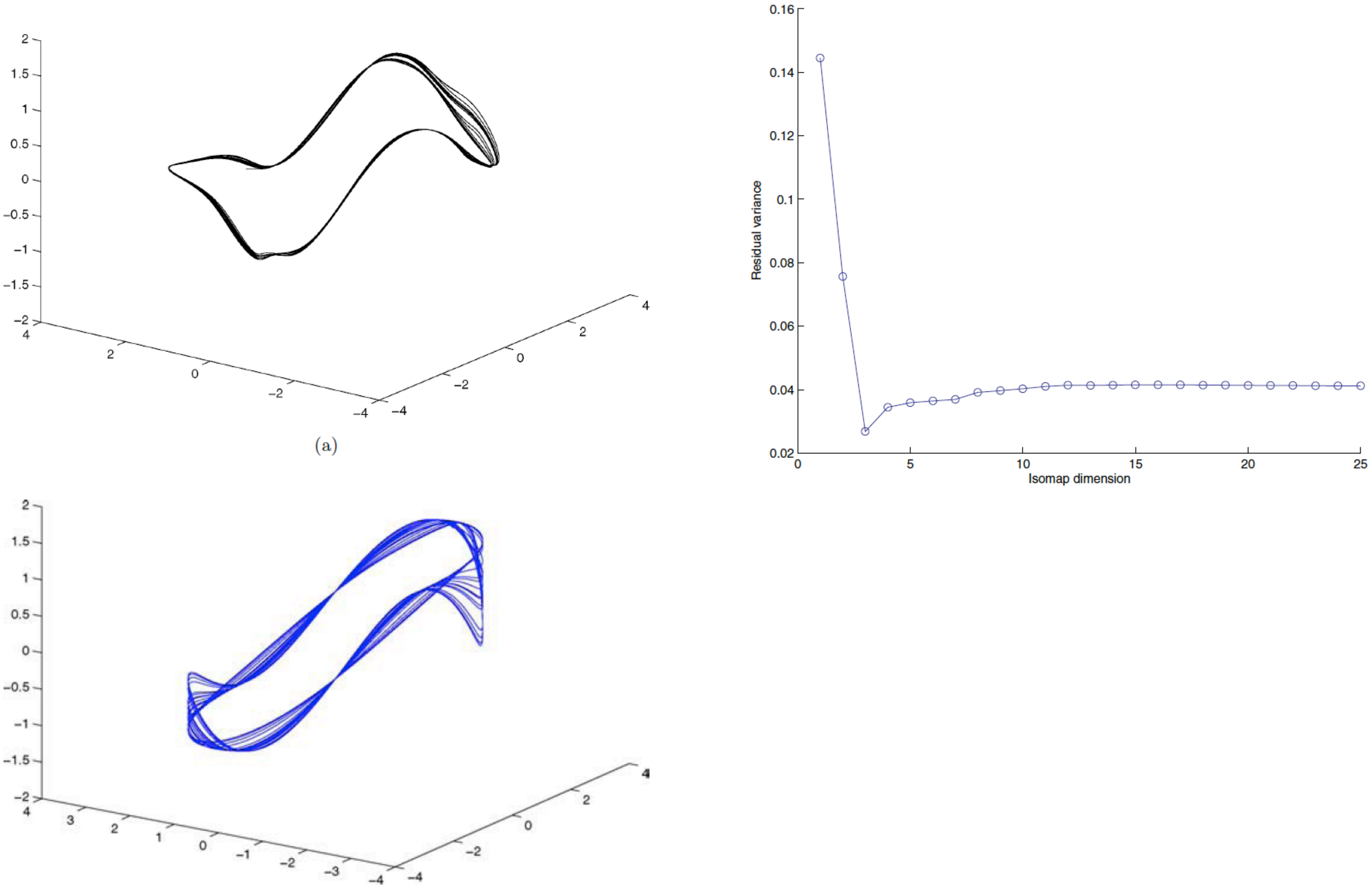
an ODE in a Banach space as follows

$$u(x,t) = \sum_{k=-\infty}^{\infty} b_k(t) e^{ikx}.$$ Assuming a real u forces $b_k = \overline{b}_k$.

Restricting to pure imaginary solutions yields, $b_k = ia_k$ for real a_k gives,

$$\dot{a}_k = (k^2 - \nu k^4) a_k + ik \sum_{k=-\infty}^{\infty} a_m a_{k-m},$$

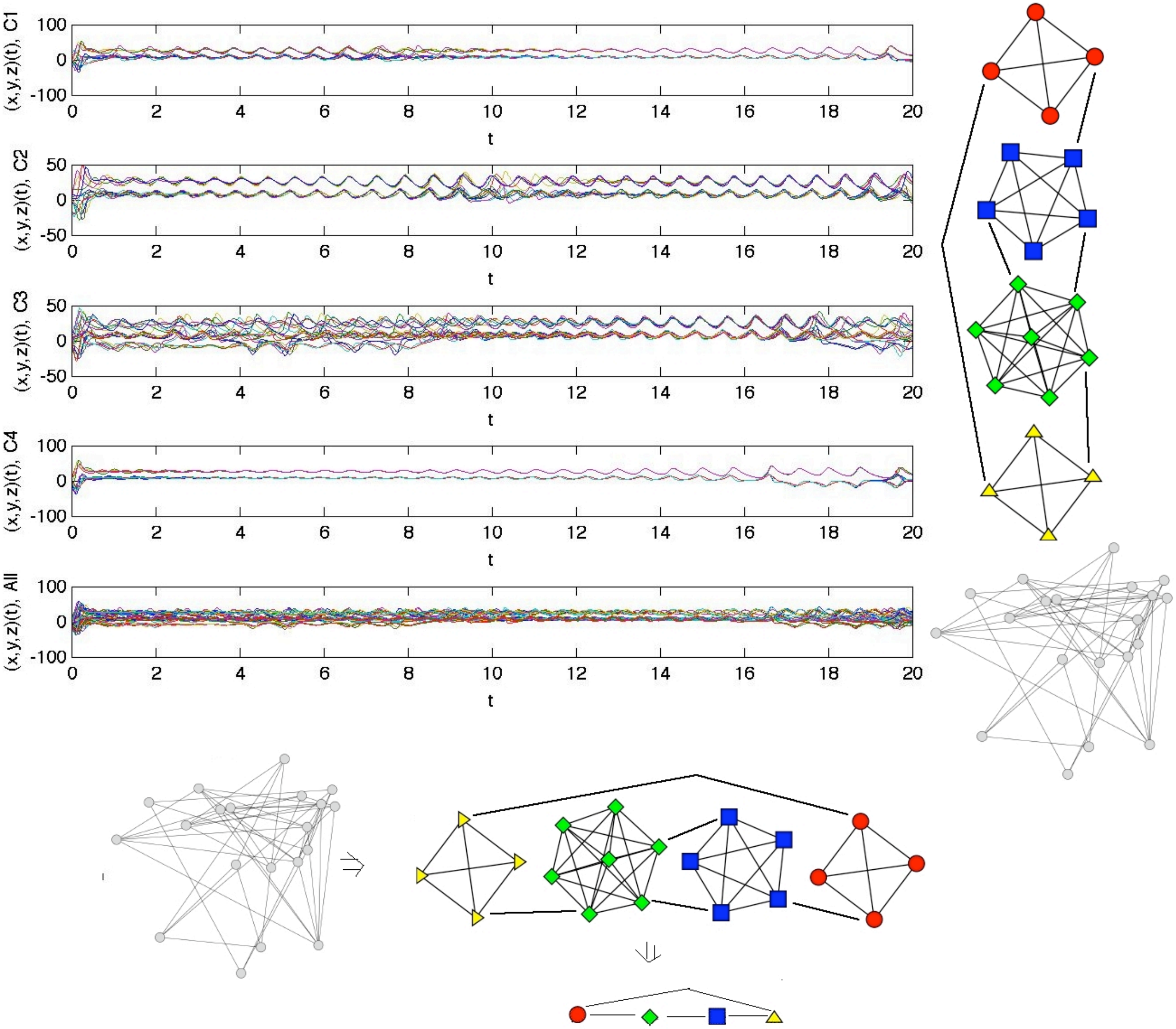
and restricting to odd solutions, $u(x,t) = -u(-x,t)$ gives $a_{-k} = a_k$. Finally, for computational reasons, it is always necessary to truncate at the Nth term,



(a)

Fig. 12. (a) Projection of the data of the KS ODE equations Eq. (51) onto three a_1, a_2, a_3. (b) Results of the ISOMAP algorithm embedding the data in three intrinsic variables.

Example Dimension Reduction when many coupled oscillators



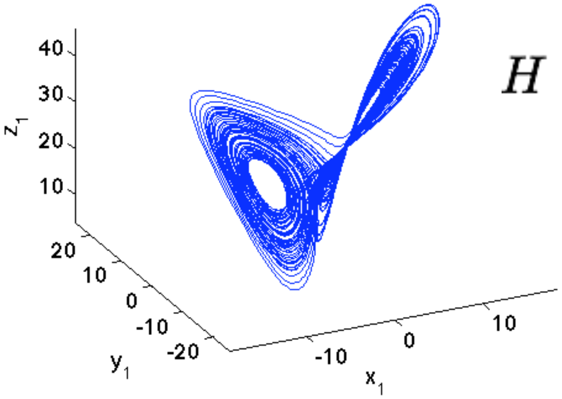
Jie Sun, Erik M. Bollt, Takashi Nishikawa, "Constructing Generalized Synchronization Manifolds by A Manifold Equation ," SIAM J. Appl. Dyn. Syst. Volume 8, Issue 1, pp. 202-221 (2009).

Complete and Nearly Sync.

Complete Sync. of Coupled Oscillator Network

oscillator network: coupled dynamical systems

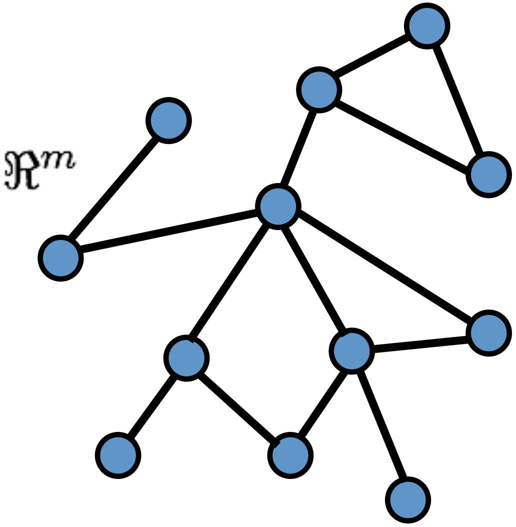
$f : \Re^m \rightarrow \Re^m$



$H : \Re^m \rightarrow \Re^m$

+

$A = [a_{ij}]^{n \times n}$



graph Laplacian

$L = [l_{ij}]^{n \times n}$

$l_{ii} = \sum_{j=1}^{n} a_{ij}$

$l_{ij} = -a_{ij} \ (i \neq j)$

$$\begin{aligned} \dot{w}_i &= f(w_i) + g \sum_{j=1}^{n} a_{ij}[H(w_j) - H(w_i)] \\ &= f(w_i) + g \sum_{j=1}^{n} l_{ij} H(w_j) \end{aligned}$$

Complete Synchronization: Master Stability Functions

Master Stability Functions

$$\dot{w}_i = f(w_i) - g\sum_{j}^{N} l_{ij} H(w_j) \qquad (i = 1, 2, ..., N.)$$

sync. dynamics: $\dot{s} = f(s)$

variational eqs:
For err from
Ident sync manif

$$\eta_i \equiv w_i - s$$

$$\dot{\eta}_i = Df(s)\eta_i - g\sum_{j=1}^{N} l_{ij} DH(s)\eta_j$$

Decouple the variational equations: $L = V\Lambda V^T$

$$\Lambda = diag[\lambda_1, ...\lambda_n] \quad V = [v_1, ..., v_n] \qquad 0 \equiv \lambda_1 < \lambda_2 \leq ... \leq \lambda_N$$

$$v_i = [v_{1i}, v_{2i}, ..., v_{ni}]^T$$

Change of variables: $\zeta_i \equiv v_{1i}\eta_1 + v_{2i}\eta_2 + ... + v_{ni}\eta_n$

$$\dot{\zeta}_i = \Big[Df(s) - g\lambda_i DH(s)\Big]\zeta_i$$

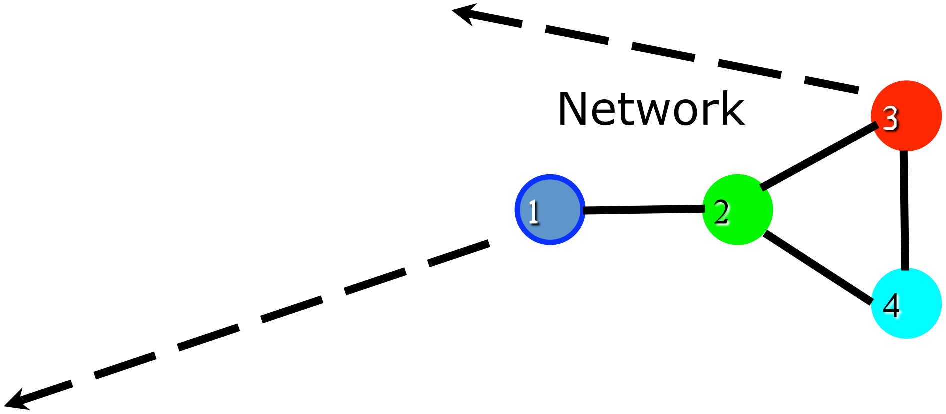
L. M. Pecora and T. L. Carroll,
"Master Stability Functions for Synchronized Coupled Systems" Phys. Rev. Lett. 80, 2109 (1998).,

Complete Synchronization: Master Stability Functions

Coupled Dynamical System

Coupled Network Dynamics

$$\begin{aligned}
\dot{x_3} &= -y_3 - z_3 && +g[(x_2 - x_3) + (x_4 - x_3)] \\
\dot{y_3} &= x_3 + 0.2y_3 && +g[(y_2 - y_3) + (y_4 - y_3)] \\
\dot{z_3} &= 0.2 + z(x_3 - c_3) && +g[(z_2 - z_3) + g(z_4 - z_3)]
\end{aligned}$$



$$\begin{aligned}
\dot{x_1} &= -y_1 - z_1 && +g(x_2 - x_1) \\
\dot{y_1} &= x_1 + ay_1 && +g(y_2 - y_1) \\
\dot{z_1} &= b + z(x_1 - c) && +g(z_2 - z_1)
\end{aligned}$$

A bunch of coupled Rosslers

Coupling Function

$$H([x, y, z]') = [x, y, z]'.$$

Graph Laplacian

$$L = \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 3 & -1 & -1 \\ 0 & -1 & 2 & -1 \\ 0 & -1 & -1 & 2 \end{bmatrix}$$

Nearly Sync.: Generalized MSFs

Generalized Master Stability Functions

Generalized master stability equations (GMSE):

$$\dot{\xi} = \Big[Df(s) - \alpha \cdot DH(s)\Big]\xi + \phi \qquad \phi \in \Re^{m}$$

$$\dot{\bar{w}} = \frac{1}{N}\sum_i f(w_i) + \bar{q} \quad \longleftrightarrow \quad \dot{s} = f(s) + \bar{q}$$

Generalized master stability functions (GMSF):

GMSF:
$$\Omega_2(\alpha, \phi) \equiv \lim_{T\to\infty} \sup_{t\geq T} \left(\frac{1}{t}\int_0^t ||\xi(\tau)||^2 d\tau\right)^{1/2}$$

$$\lim_{T\to\infty} \sup_{t\geq T} \left(\frac{1}{t}\int_0^t e^2(\tau) d\tau\right)^{1/2} \leq \frac{1}{\sqrt{N}}\Big[\sum_{i=2}^{N} \Omega_2^2(\alpha_i, \phi_i)\Big]^{1/2}$$

where:
$\alpha_i \equiv g\lambda_i$ and $\phi_i \equiv \Big[v_i^T \otimes I_m\Big]\boldsymbol{\delta q}$

-Jie Sun, Erik M. Bollt, Takashi Nishikawa, "Constructing Generalized Synchronization Manifolds by A Manifold Equation ," SIAM J. Appl. Dyn. Syst. Volume 8, Issue 1, pp. 202-221 (2009).

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Measuring the sync. error of the system:

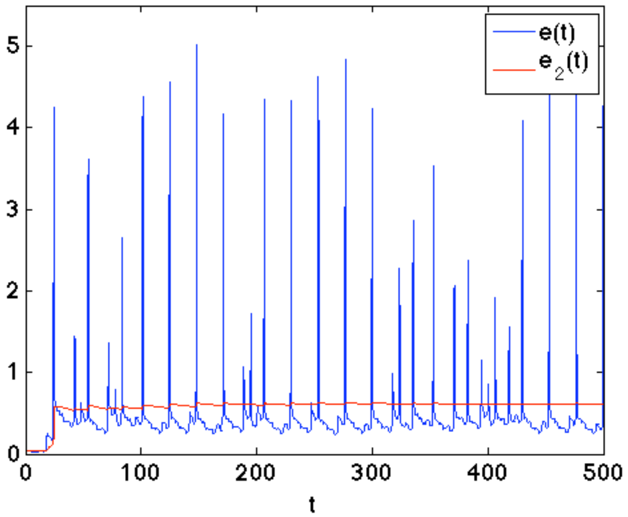
Def: the spatial-temporal average error of the system at time t, as:

$$e_2(t) \equiv \Big(\sum_i \frac{1}{t} \int_0^t ||w_i(\tau) - \bar{w}(\tau)||^2 d\tau \Big)^{1/2}$$

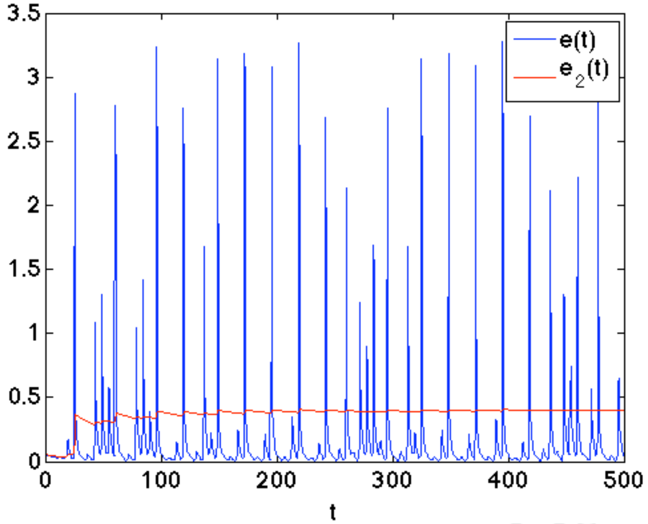
Def: A collection of oscillators $w_1, \ldots, w_N$ are ϵ-synchronized

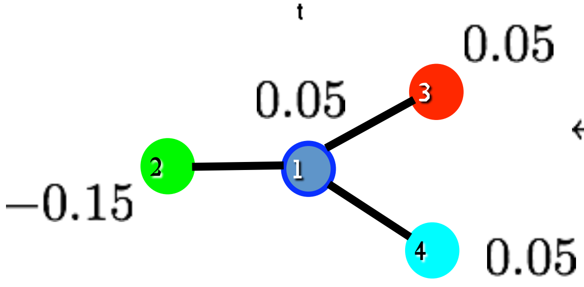
(w.r.t norm $||.||$) if: (usually choose $||.||$ as Euclidean norm.)

$$\lim_{T \to \infty} \sup_{t \geq T} e_2(t) \leq \epsilon$$

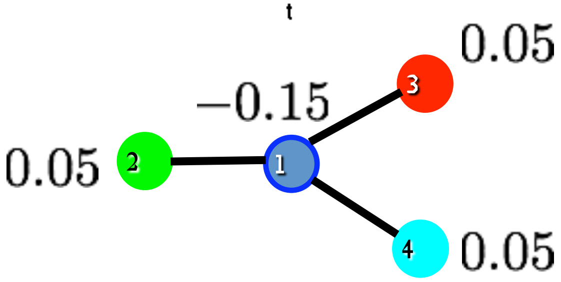


$g = 0.15$





$\longleftarrow q_i(3) \longrightarrow$



Judging DIMENSION REDUCTION based on errors?

Complete Synchronization $$\lim_{t\to\infty} ||w_t^{(i)} - w_t^{(j)}|| \to 0, \forall i, j.$$

Linear Methods

Nearly Synchronization $$\lim\sup_t ||w_t^{(i)} - \bar{w}_t|| \approx 0,$$

Nonlinear Methods

Generalized Synchronization $$\lim_{t\to\infty} ||h^{(i)}(w_t^{(i)}) - h^{(j)}(w_t^{(j)})|| \to 0, \forall i, j.$$

-Is it too much to ask that the error goes to zero in some measure?

-Maybe we should just ask that the model creates plausible data?

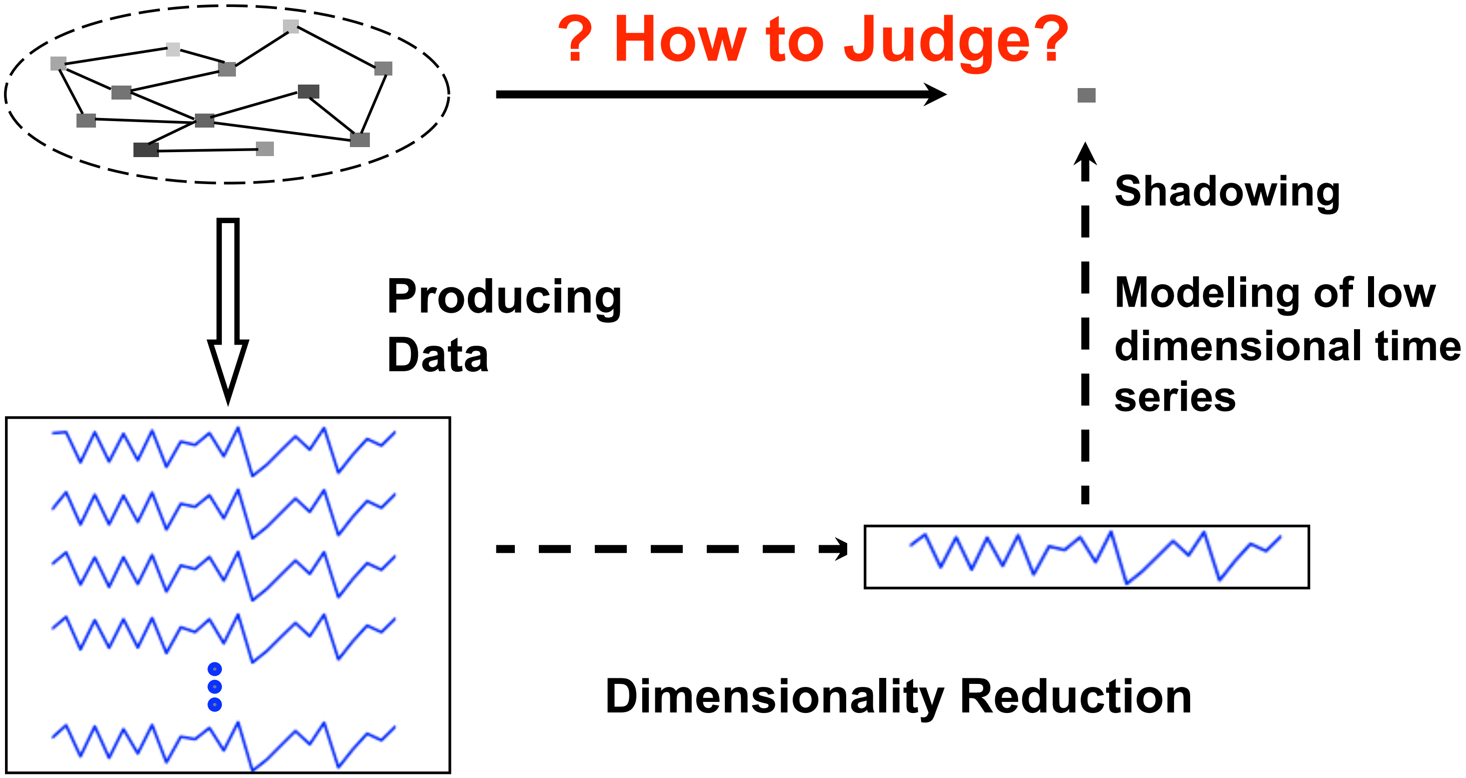
II. How do I know if I did a good job?

JUDGING MODEL REDUCTION

Two steps:

1) Measuring the loss of information due to dimensionality reduction of the time series,
   - -Residuals relative to some model reduction manifold – PCA/POD or ISOMAP

2) Measuring how good the reduced system is as a model for the reduced time series.



? How to Judge?

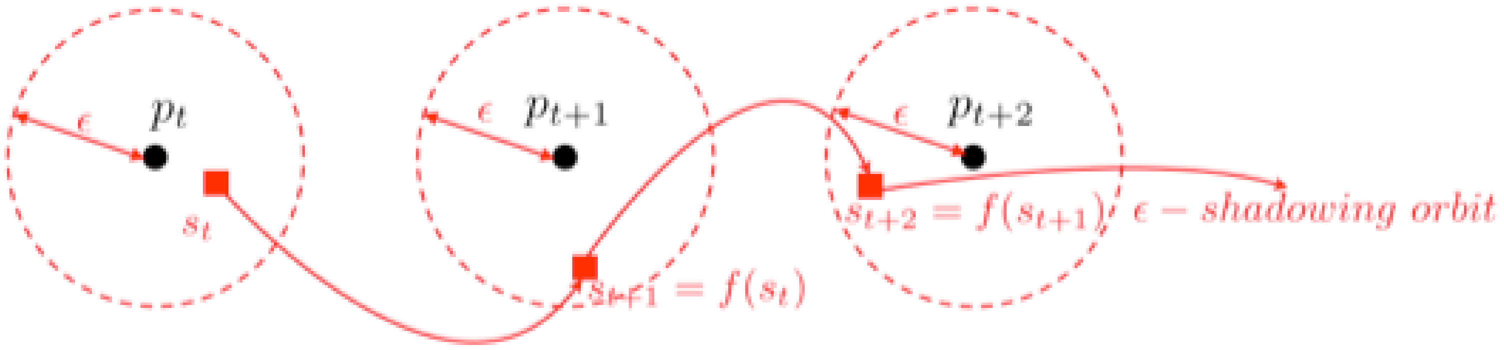
Shadowing

Modeling of low dimensional time series

Producing Data

Dimensionality Reduction

Shadowing Illustration



II. How do I know if I did a good job?
Two Themes here:

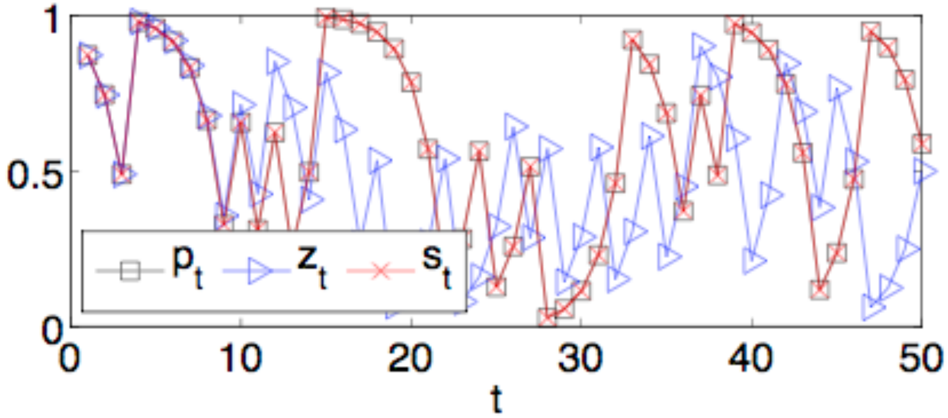
-Data is reproducible in the sense of “shadowable for a long time.”

-We judge model based on optimal shadowing distance.

-D. V. Anosov, Proc. Steklov Inst. Math 90 (1967).
-R. Bowen, J. Diff. Eqns. 18, 333 (1975).
-S. M. Hammel, J. A. Yorke, and C. Grebogi, Bull. Amer. Math. Soc. 19, 465 (1988).
-C. Grebogi, S. M. Hammel, J. A. Yorke, and T. Sauer, Phys. Rev. Lett. 65, 1527 (1990).
-K. Palmer, Shadowing in Dynamical Systems: Theory and Applications (Springer, 2000).

SHADOWING

Shadowing Example



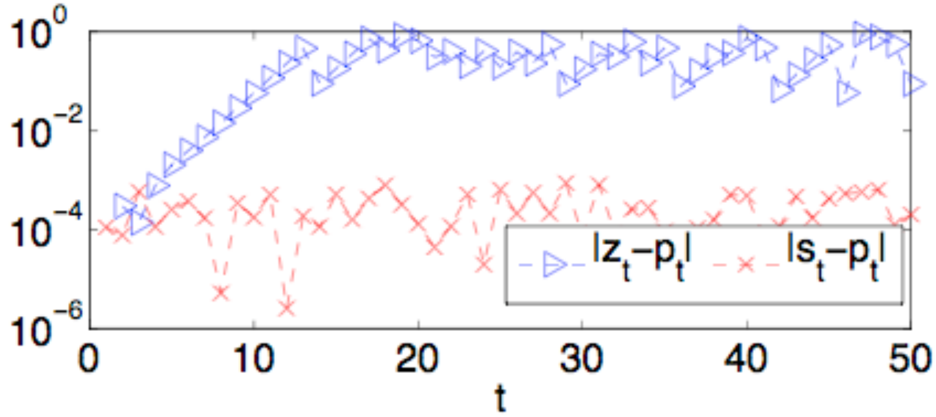
$p_{t+1} = 4p_t(1 - p_t) + \delta_t$ **noisy orbit** $\delta_t \sim 2^{-10}$

$z_{t+1} = 4z_t(1 - z_t)$ **true orbit with same initial condition**

$p_1 = z_1 = 0.872486372083970...$

$s_{t+1} = 4s_t(1 - s_t)$ **true orbit with a magic initial condition**

$s_1 = 0.872375078713858...$



define *optimal shadowing distance* ϵ_{opt}

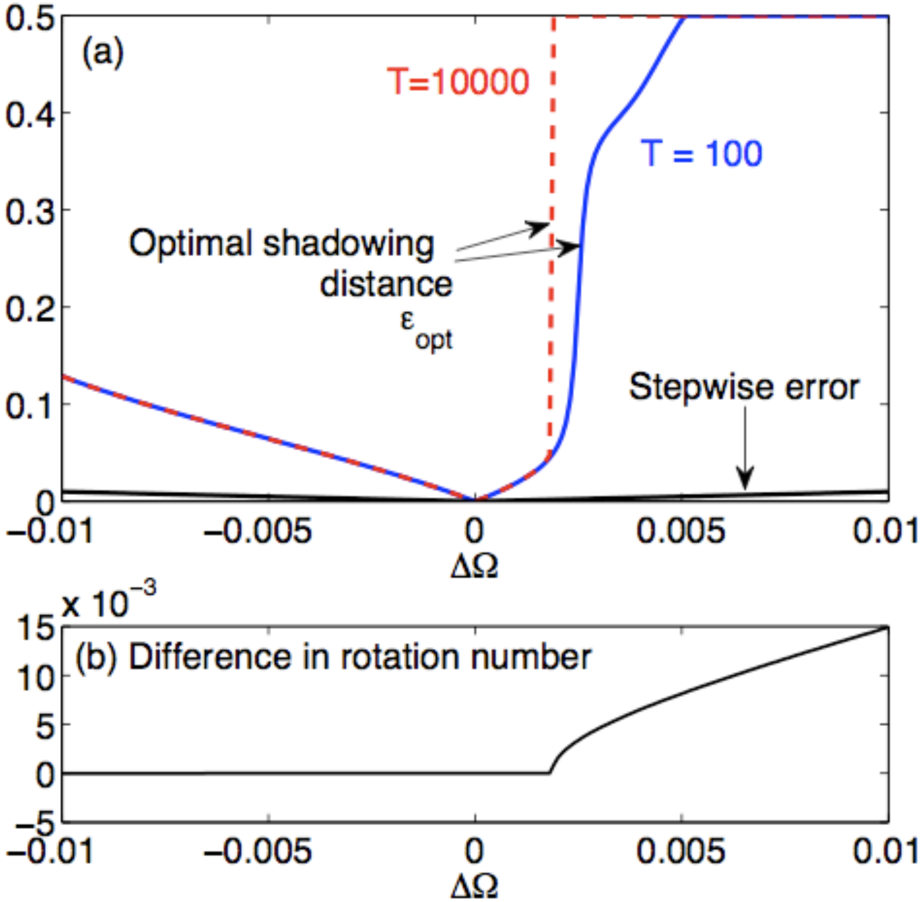
$$\epsilon_{opt} \equiv \inf_{x_1 \in D} \sup_t ||x_t - p_t||,$$

where $\{x_t\}_{t=1}^{T}$ is the trajectory of the reduced model

$$x_{t+1} = f(x_t),\ x_t \in D \subset \mathbb{R}^d$$

$\{p_t\}_{t=1}^{T}$ is the reduced time series

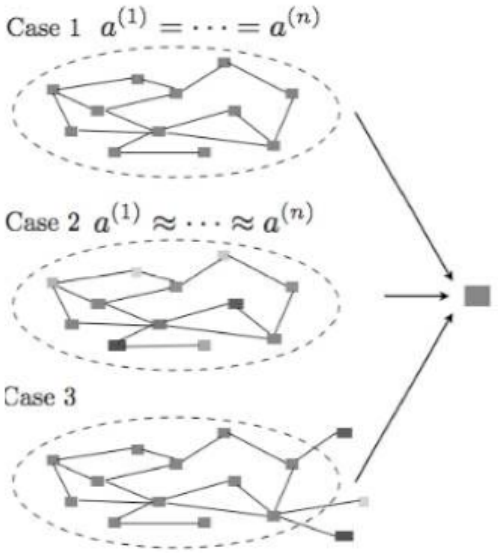
how long the model is valid.



$$p_{t+1} = p_t + \Omega - 0.12 \sin(2\pi p_t) \text{ with } \Omega = 0.35$$

$$x_{t+1}^{(i)} = f(x_t^{(i)}, a^{(i)}) - \sigma \sum_{j=1}^{n} l_{ij} f(x_t^{(j)}, a^{(j)}),$$

($n \times d$)-dimensional complex system,



1. If the oscillators are identical, in what sense can we model the network by a single oscillator?

$\lim_{t\to\infty} ||x_t^{(i)} - x_t^{(j)}|| \to 0$

2. If the oscillators are *non-identical*, in what sense can we model the network by a single oscillator?

3. In what sense can we model a *nearly synchronized cluster* in the network by a single oscillator?

a single oscillator model may not exactly represent the true collective behavior of the coupled system.

choose the average trajectory $\bar{x}_t \equiv \sum_i x_t^{(i)}/n$ as a low dimensional representation

$$x_{t+1}^{(i)} = f(x_t^{(i)}, a^{(i)}) - \sigma \sum_{j=1}^{n} l_{ij} f(x_t^{(j)}, a^{(j)}),$$

$(n \times d)$-dimensional complex system,

a single oscillator model may not exactly represent the true collective behavior of the coupled system.

choose the average trajectory $\bar{x}_t \equiv \sum_i x_t^{(i)}/n$ as a low dimensional representation

$$\bar{x}_{t+1} = \frac{1}{n} \sum_{i=1}^{n} f(x_t^{(i)}, a^{(i)}) - \frac{\sigma}{n} \sum_{i,j=1}^{n} l_{ij}$$

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with $\bar{a} \equiv \sum_i a^{(i)}/n$, one obtains $s_{t+1} = f(s_t, \bar{a})$

Even in a situation where the oscillators are nearly identical and nearly synchronized

$\limsup_t ||x_t^{(i)} - \bar{x}_t|| \approx 0$,

error can accumulate over time and depend critically on the distribution of heterogeneity

optimal shadowing distance ϵ_{opt} provides a quantitative measure

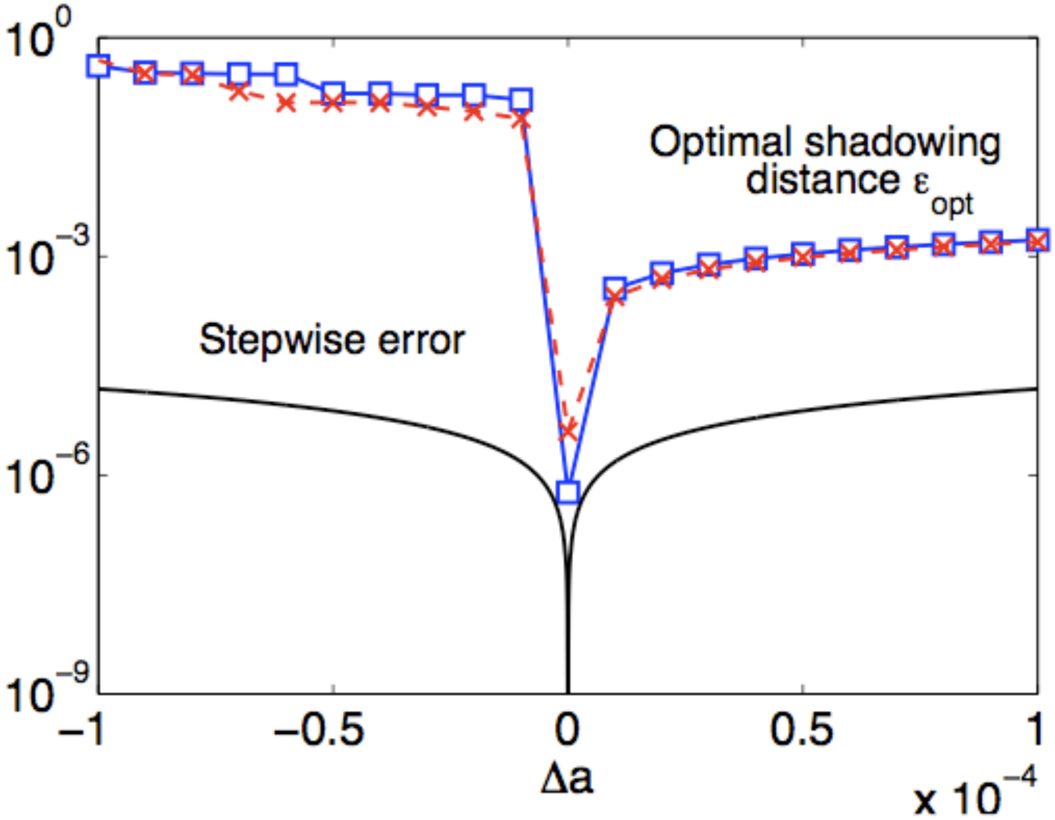
Erdős-Rényi 1000 nearly synchronized logistic maps

$f(x, a) = ax(1-x)$ $[3.9998, 4]$

$\sqrt{\sum_{t=1}^{T-1} |f(\bar{x}_t, a) - \bar{x}_{t+1}|^2/T}.$

vs

ϵ_{opt}



$\Delta a = a - 3.999$

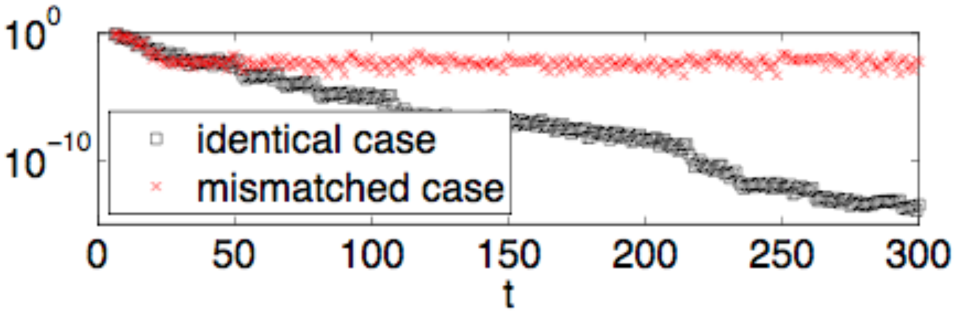
Coupled Henon oscillators through an Erdos-Renyi network (n=200, m=1993).

$$f[w_t^{(i)}, a^{(i)}] = [1 + y_t^{(i)} - a^{(i)}(x_t^{(i)})^2, bx_t^{(i)}]$$

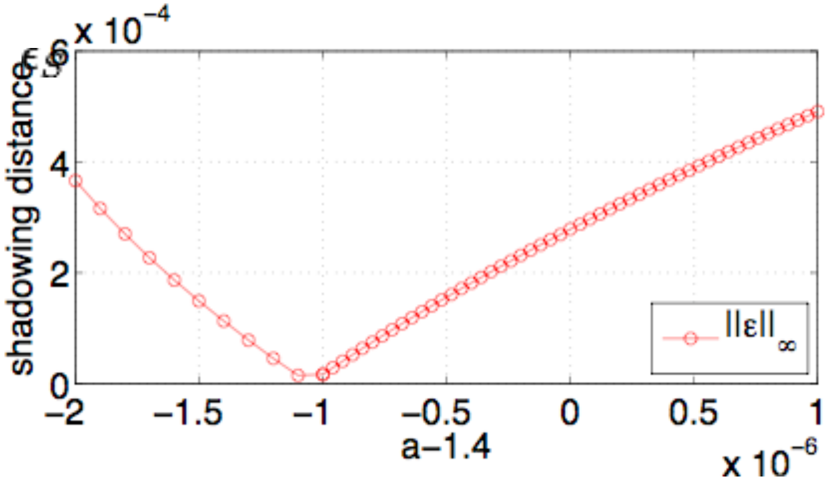
Identical: $a^{(1)} = ... = a^{(n)} = 1.4, b = 0.3$

Mismatched: $a^{(i)} \sim N(1.4, 0.0013^2)$

Synchronization Error $\langle w(i) - \bar{w} \rangle = \epsilon_{DR}$

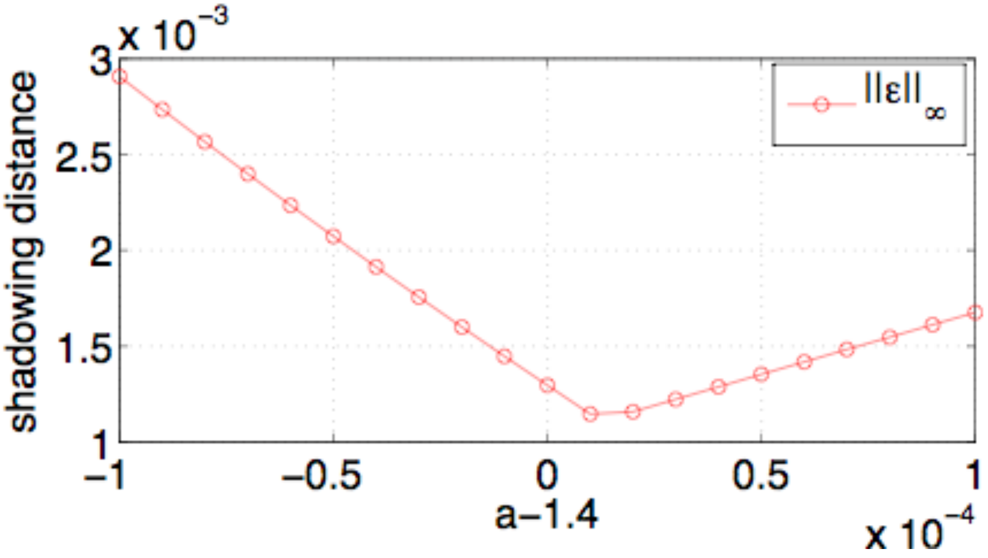


Shadowing Error



Coupled Henon oscillators through an Erdos-Renyi network (n=500, m=12348) with outlier. $a^{(1)} = ... = a^{(n)} = 1.4, b = 0.3$

Shadowing Error ϵ_S



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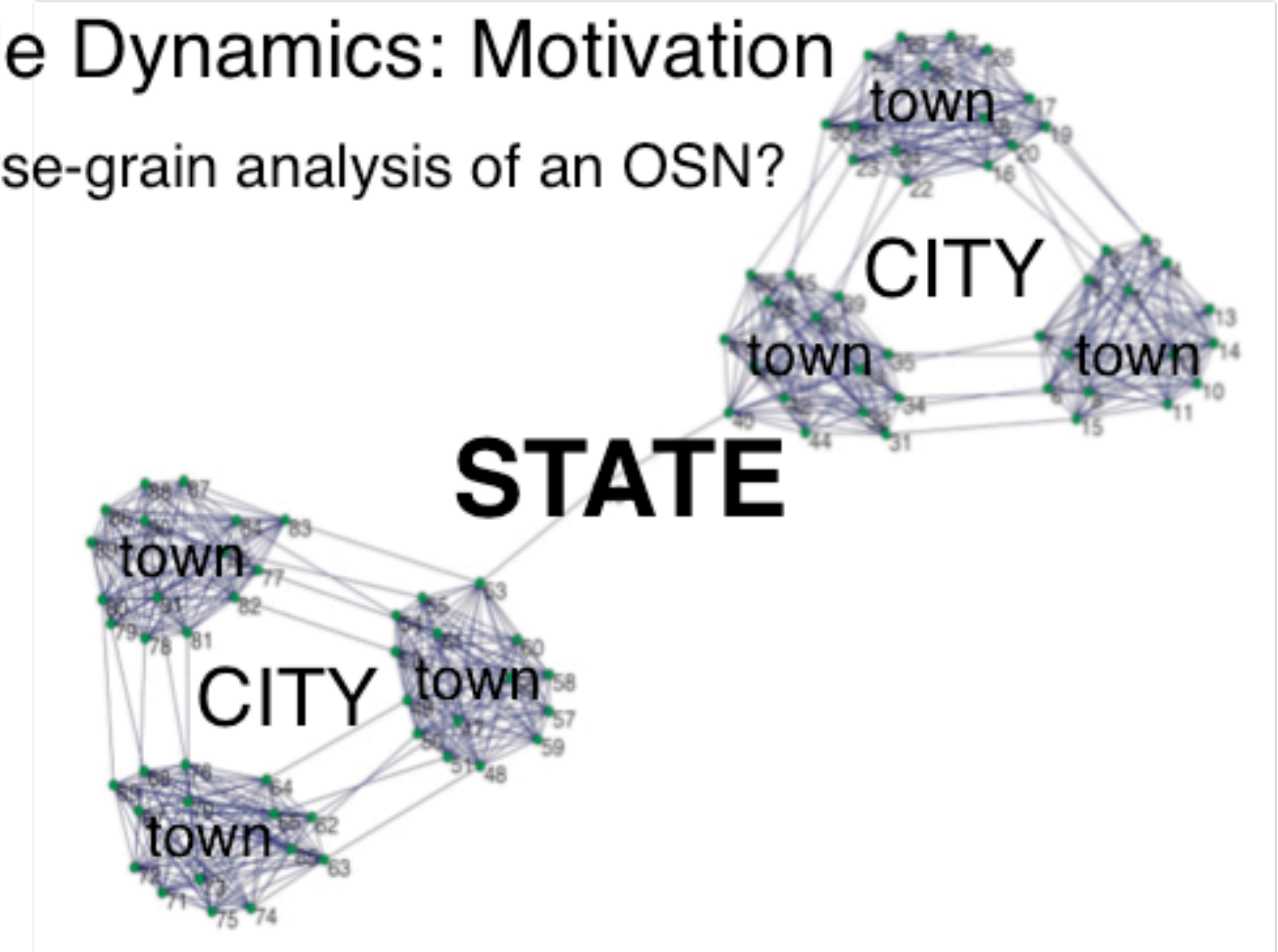
-K. Palmer *Shadowing in Dynamical Systems: Theory and Applications* (Springer, 2000.)

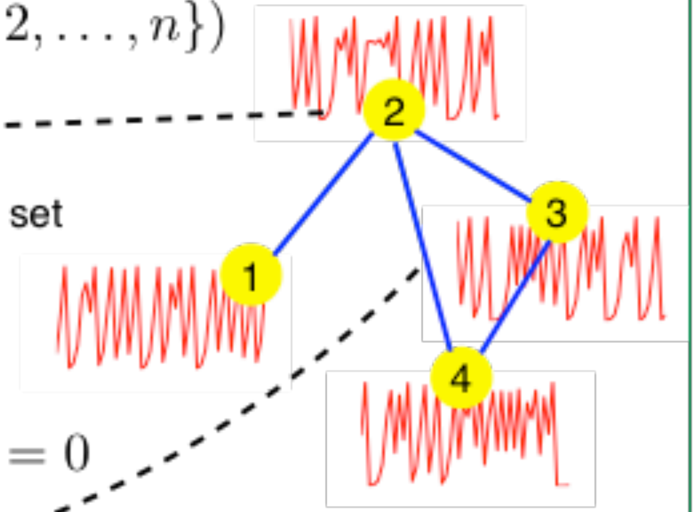
Multi-scale Dynamics: Motivation

Coarse-grain analysis of an OSN?



Coupled Oscillator Network (OSN)

Single oscillator dynamics: $\dot{\theta}_i = f_i(\theta_i) \qquad (i \in \{1, 2, \ldots, n\})$



$$\theta_i \in X \subset \mathbb{R}^m$$

$f_i : X \to X$ compact set

Coupling function: $h(\theta_j - \theta_i)$

$h : \mathbb{R}^m \to \mathbb{R}^m$ such that $h(0) = 0$

Coupled Oscillator Network (OSN)

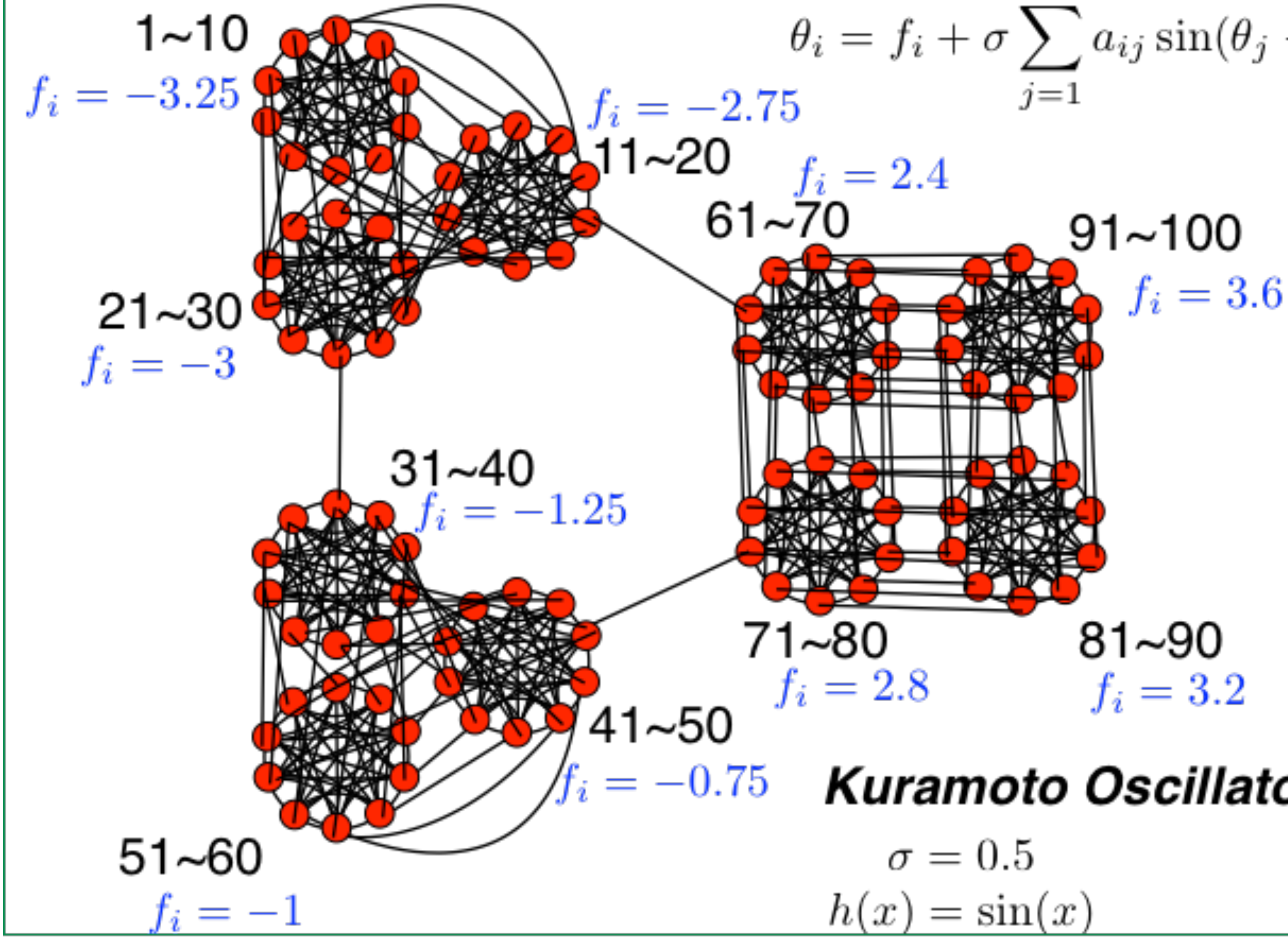
individual dynamics + coupling function + graph structure

$$\dot{\theta}_i = f_i(\theta_i) + \sigma \sum_{j=1}^{n} a_{ij} h(\theta_j - \theta_i)$$

coupling strength

Multi-scale Dynamics: an Example OSN

$$\theta_i = f_i + \sigma \sum_{j=1}^{n} a_{ij} \sin(\theta_j - \theta_i)$$

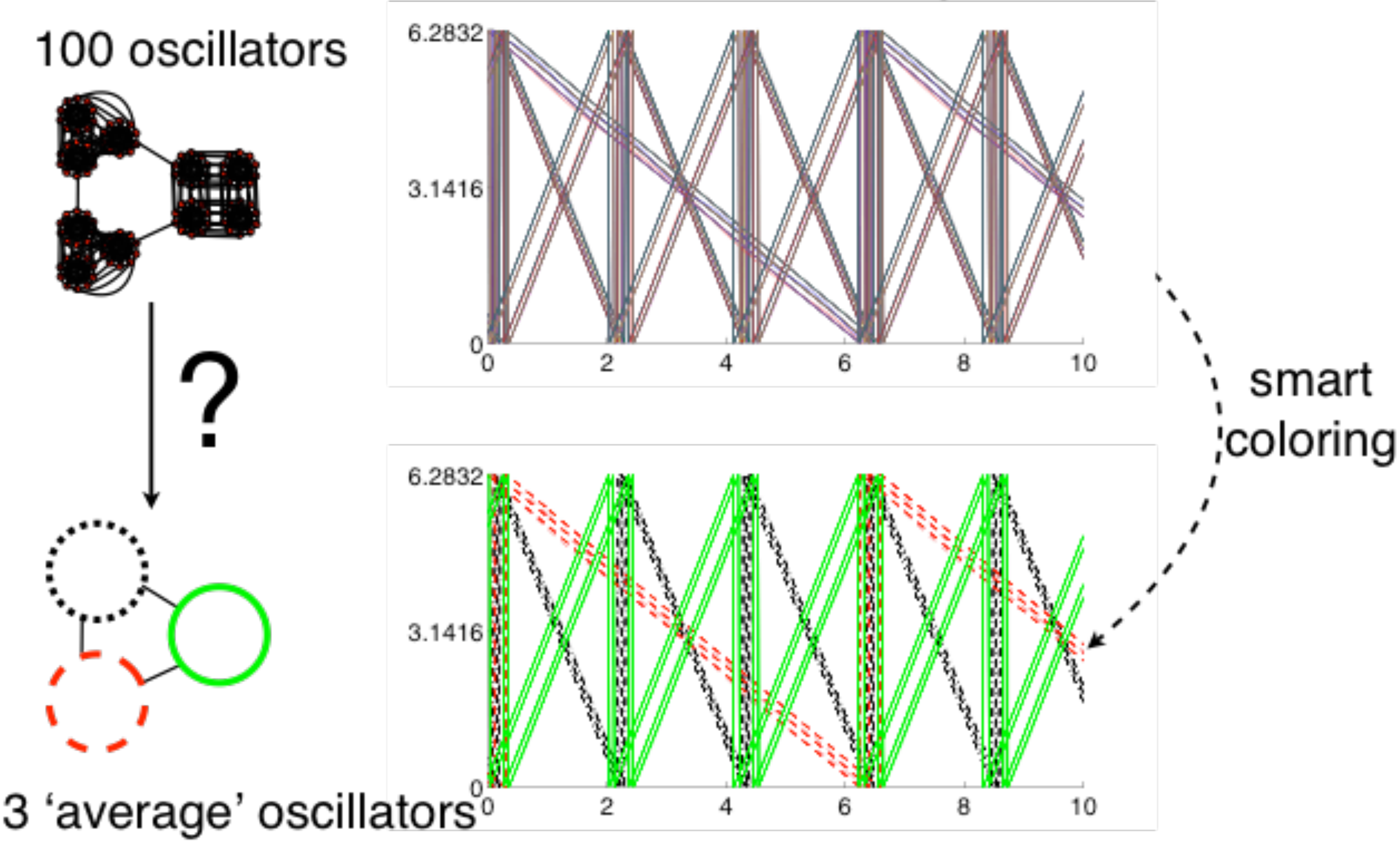


Kuramoto Oscillators

$\sigma = 0.5$

$h(x) = \sin(x)$

Time Series of the Example OSN



100 oscillators

?

3 ‘average’ oscillators

smart coloring

Model Reduction of an OSN

$$\dot{\theta}_i = f_i(\theta_i) + \sigma \sum_{j=1}^{n} a_{ij} h(\theta_j - \theta_i)$$

$\downarrow$ grouping/partition

average motion of group ℓ

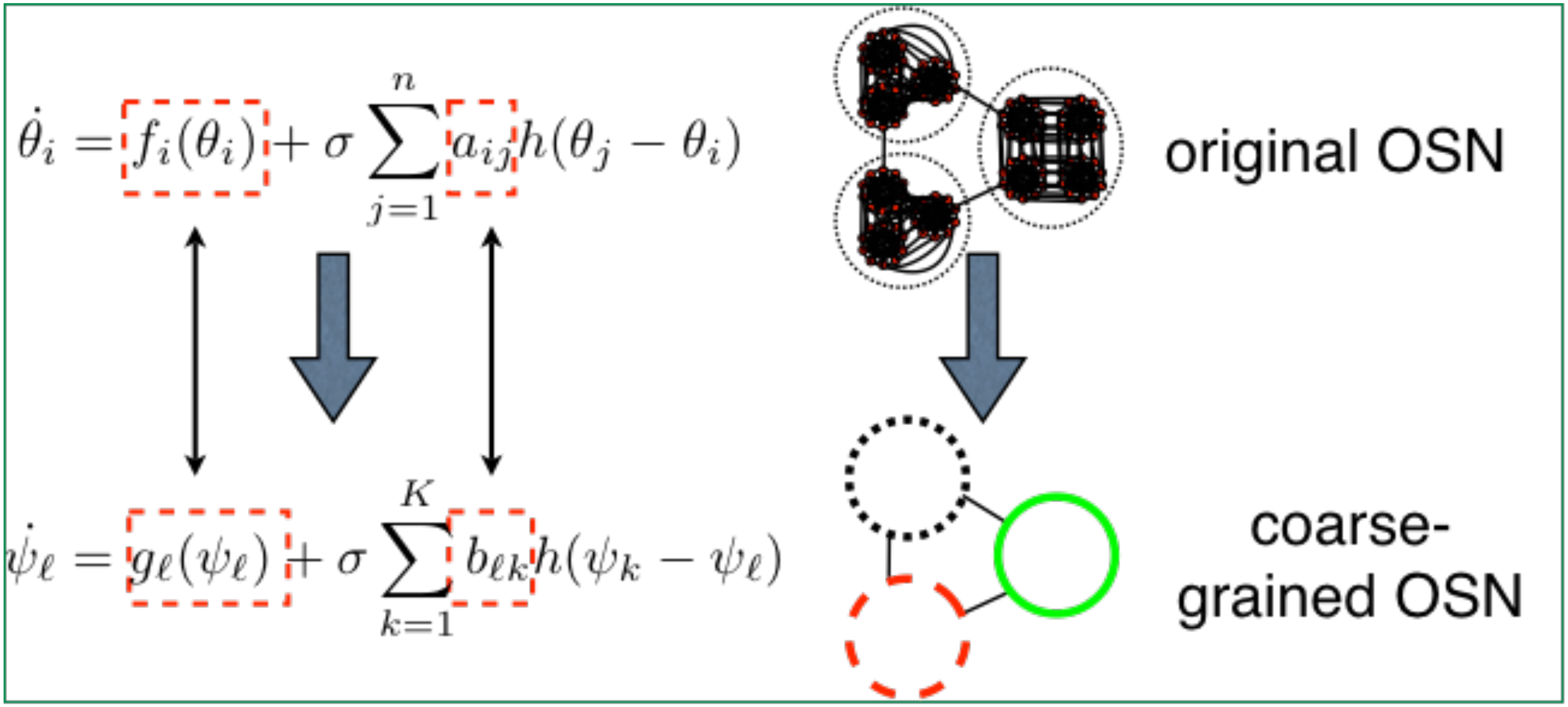
$$\dot{\phi}_\ell = \frac{1}{|C_\ell|} \sum_{i \in C_\ell} f_i(\theta_i) + \sigma \sum_{j=1}^{n} \frac{1}{|C_\ell|} \sum_{i \in C_\ell} a_{ij} h(\theta_j - \theta_i)$$

$\downarrow$ replacing each oscillator by its group average

$$\dot{\psi}_\ell = g_\ell(\psi_\ell) + \sigma \sum_{k=1}^{K} b_{\ell k} h(\psi_k - \psi_\ell)$$

average model for group ℓ

$$\left\{ \begin{array}{ll} g_\ell(\psi_\ell) \equiv \dfrac{1}{|C_\ell|} \displaystyle\sum_{i \in C_\ell} f_i(\psi_\ell) & \text{average dynamics of group } \ell \\ b_{\ell k} \equiv \dfrac{1}{|C_\ell|} \displaystyle\sum_{i \in C_\ell, j \in C_k} a_{ij} & \text{average \# edges from group } \ell \text{ to group } k \end{array} \right.$$

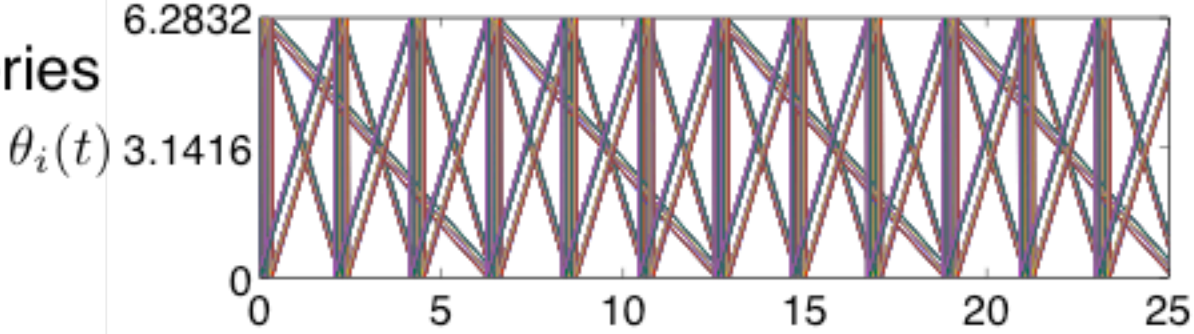


$$\dot{\theta}_i = \boxed{f_i(\theta_i)} + \sigma \sum_{j=1}^{n} \boxed{a_{ij}} h(\theta_j - \theta_i)$$

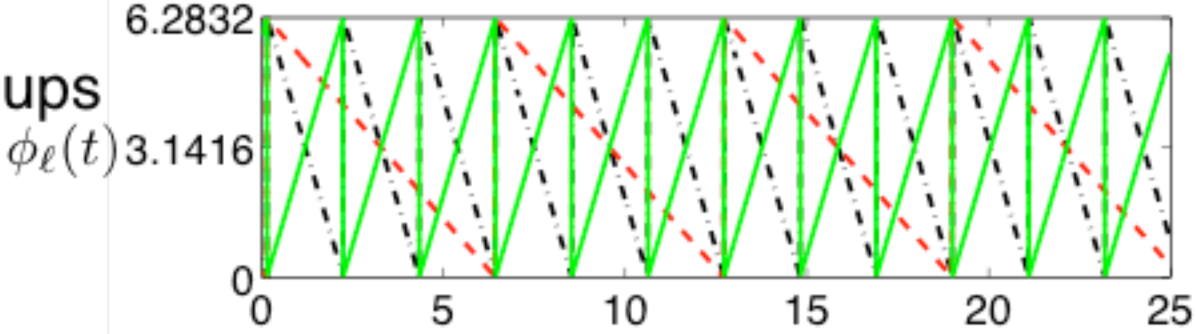
$$\dot{\psi}_\ell = \boxed{g_\ell(\psi_\ell)} + \sigma \sum_{k=1}^{K} \boxed{b_{\ell k}} h(\psi_k - \psi_\ell)$$

Validity of this Model Reduction

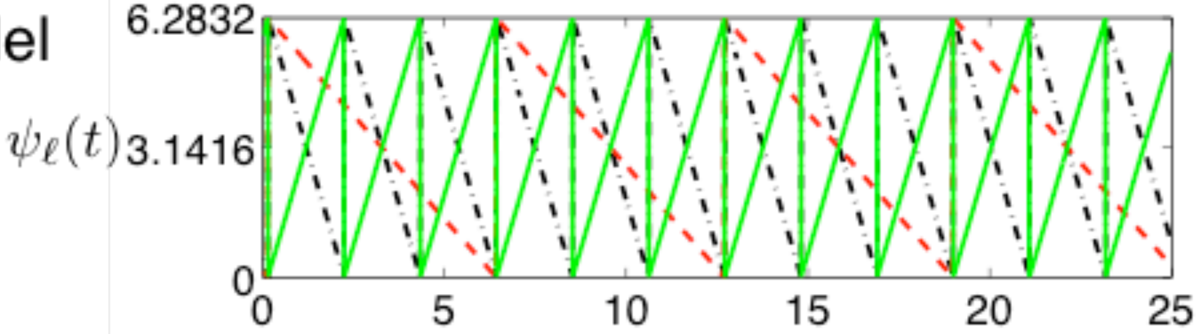
original time series



average by groups

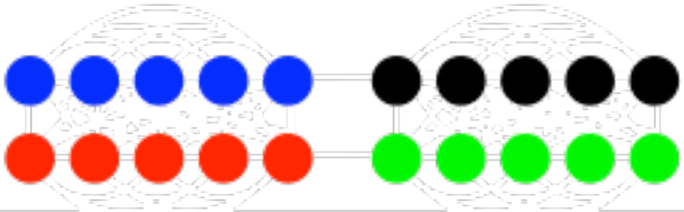


average model produced:

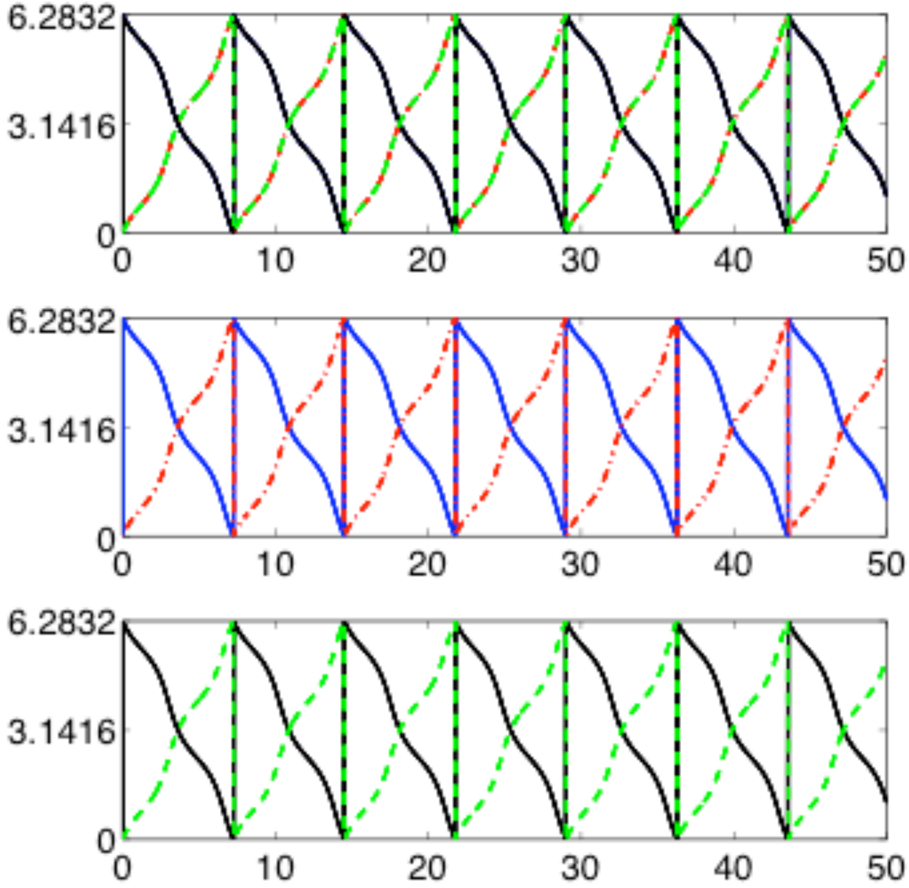


Clustering from Time Series?

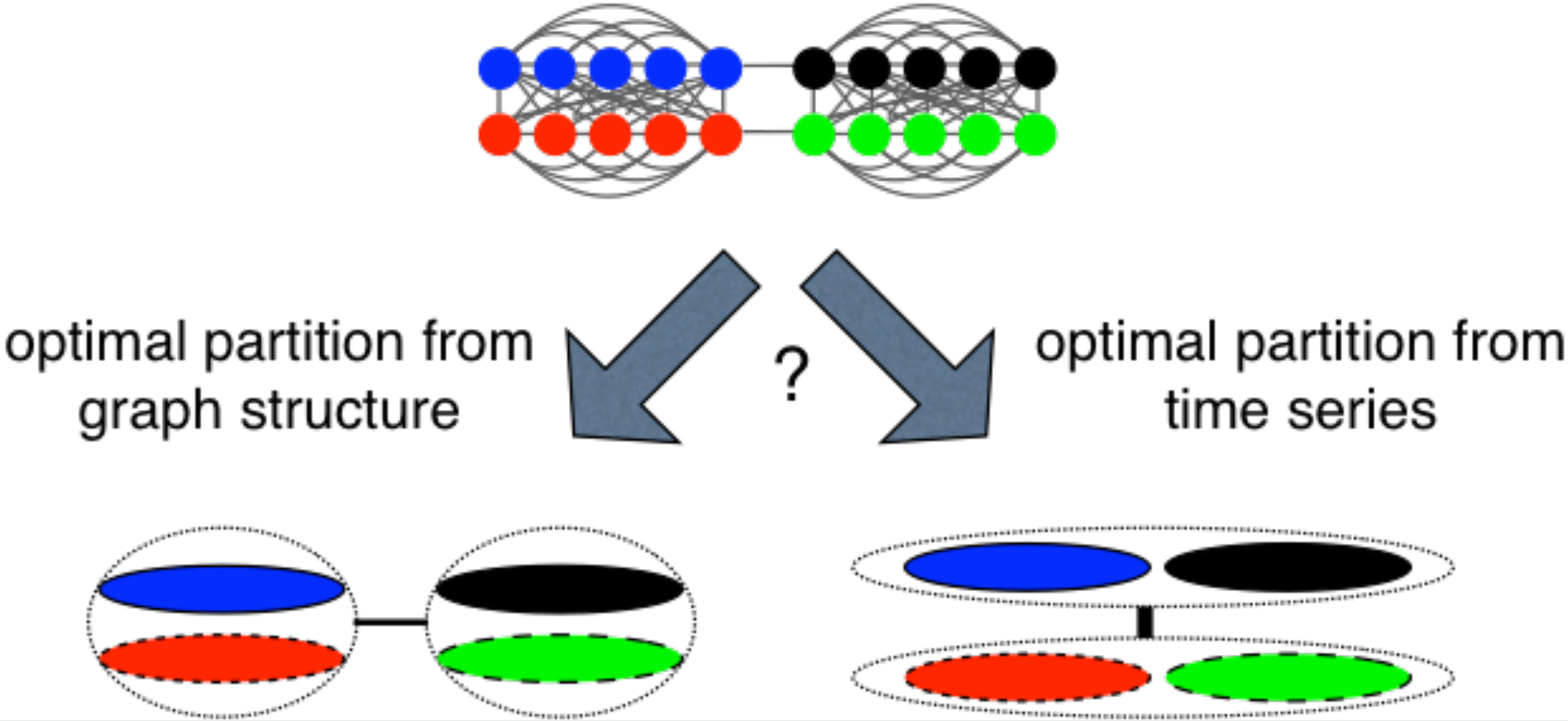
an OSN with
edges hidden



time series
from OSN



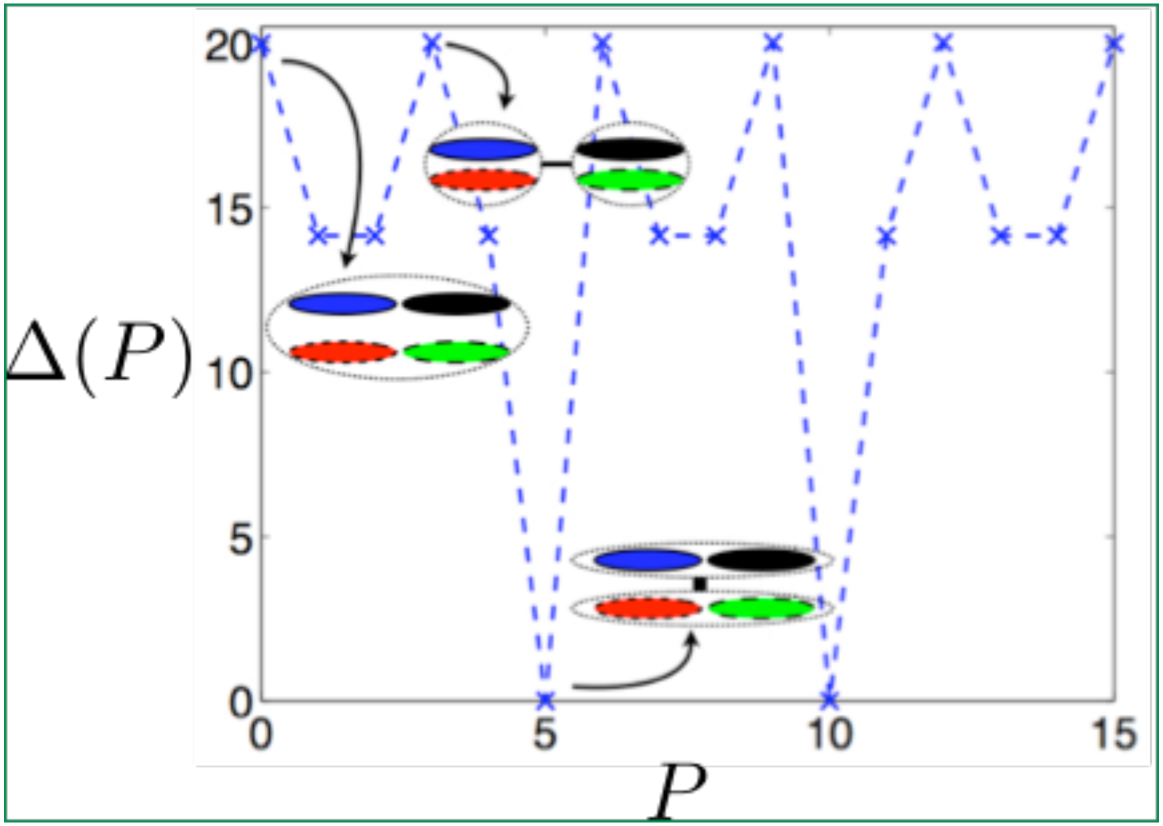
What is a Good Partition?



optimal partition from graph structure

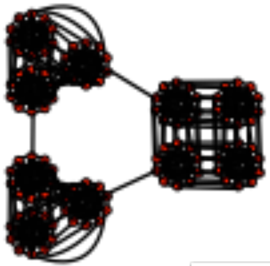
?

optimal partition from time series

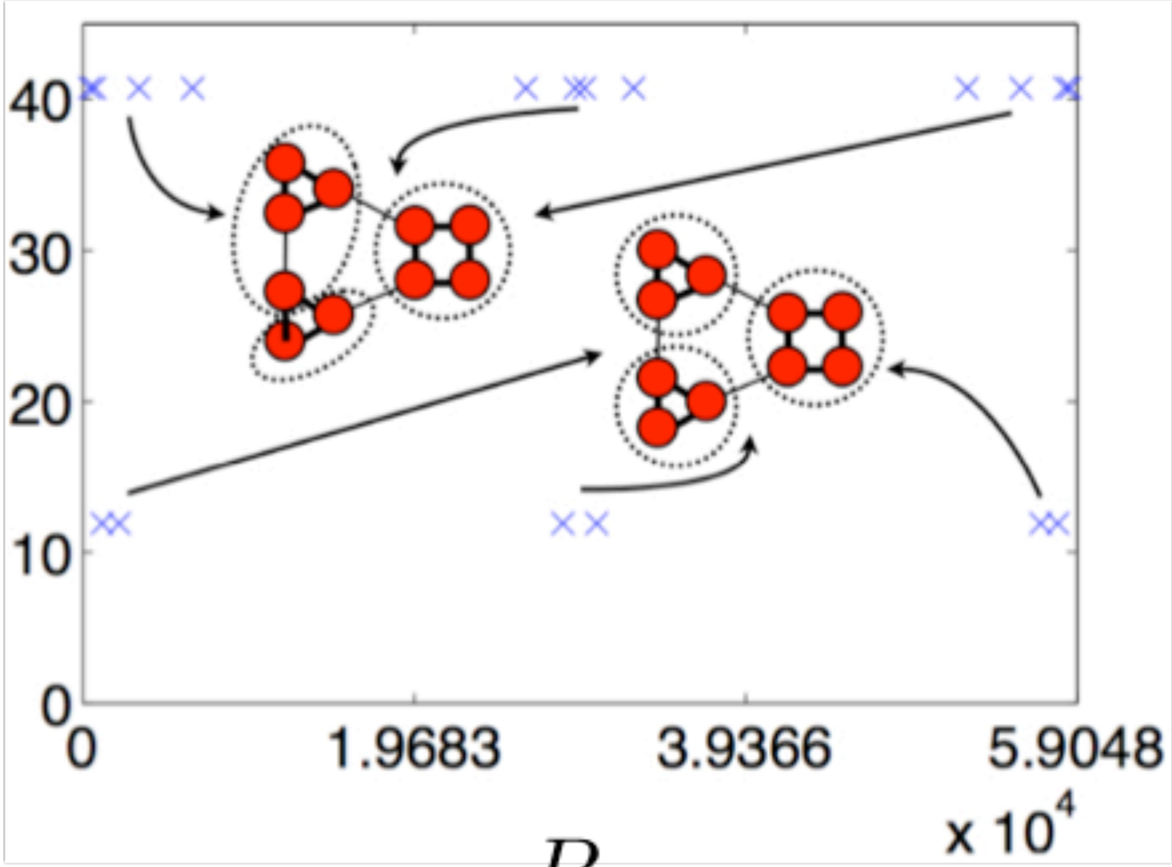

$\Delta(P)$
P
0
5
10
15
20
0
5
10
15

Another Example

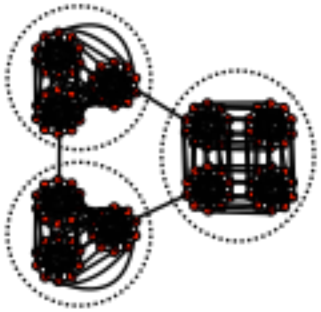
Partitions represented by
ternary vectors of length 10



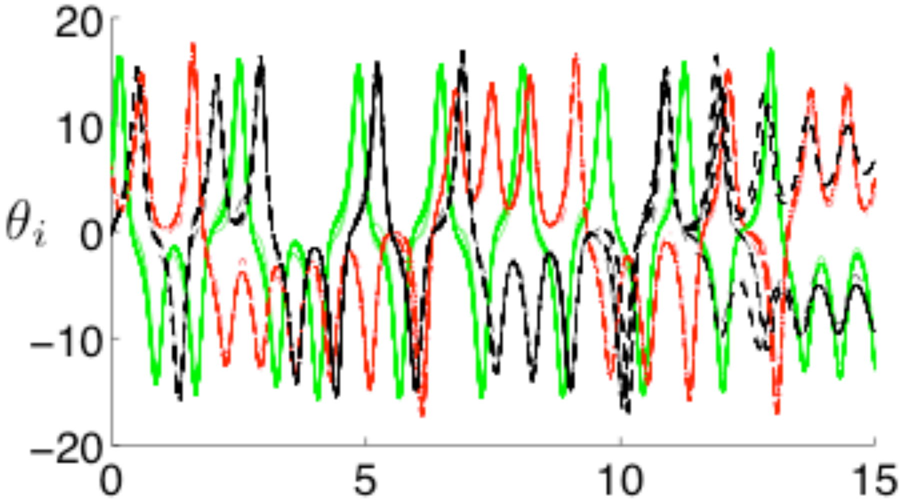
Results:

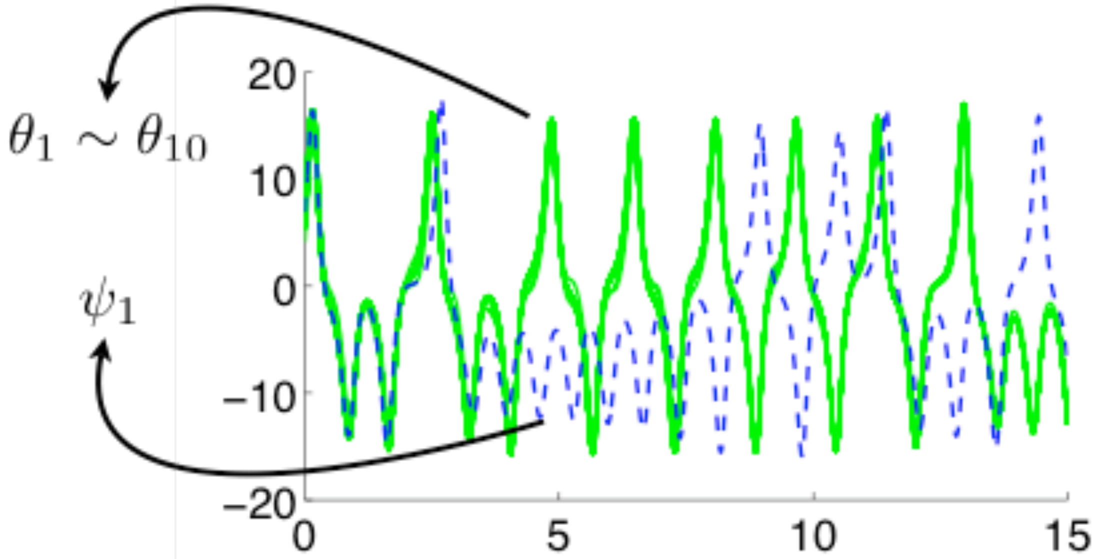


Model Reduction of Chaotic Oscillators



Lorenz oscillators





Is it a good model? What do we mean by ‘good’?

Conclusions, Two Themes here:

I. What is model reduction/dimension reduction?

-Fewer equations that somehow represent the whole.

-Perhaps Hierarchical modeling.

II. How do I know if I did a good job?

Two Themes here:

-Data is reproducible when shadowable.