Dynamics of Heterogeneous Voter Models

Sid Redner (physics.bu.edu/~redner)

Nonlinear Dynamics of Networks UMD April 5-9, 2010

T. Antal (BU→Harvard), N. Gibert (ENSTA), N. Masuda (Tokyo), M. Mobilia (BU→Leeds), V. Sood (BU→NBI), D. Volovik (BU)

NSF DMR0535503 & DMR0906504

The classic voter model

- 3 basic results

Voting on complex networks — T. Antal, V. Sood

- new conservation law
- two time-scale route to consensus
- short consensus time

Strategic voting (>2 states) — M. Mobilia, D. Volovik

- long time-scale switching

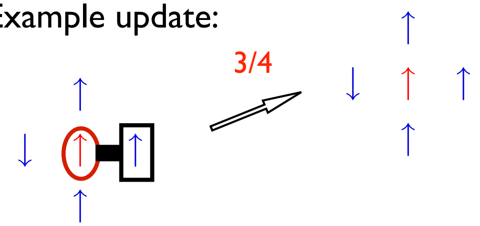
Partisan voting — N. Masuda, N. Gibert

- selfishness vs. collectiveness
- ultraslow evolution

Classic Voter Model

Clifford & Sudbury (1973)
Holley & Liggett (1975)

Example update:



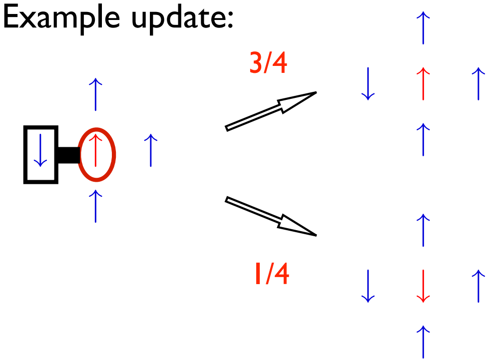
0. Binary voter variable at each site i
1. Pick a random voter
2. Assume state of randomly-selected neighbor

   individual has no self-confidence & adopts neighbor's state

Classic Voter Model

Clifford & Sudbury (1973)
Holley & Liggett (1975)

Example update:



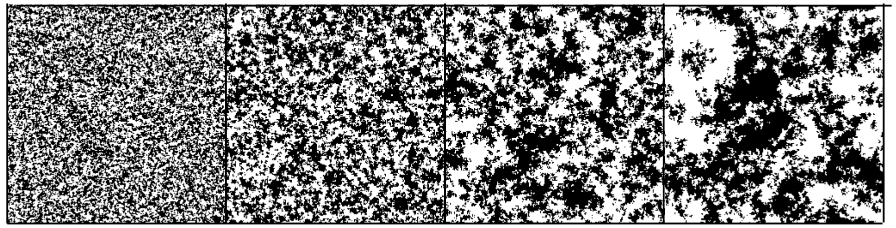
proportional rule

0. Binary voter variable at each site i
1. Pick a random voter
2. Assume state of randomly-selected neighbor
   individual has no self-confidence & adopts neighbor's state
3. Repeat 1 & 2 until consensus *necessarily* occurs in a finite system

Voter Model Evolution

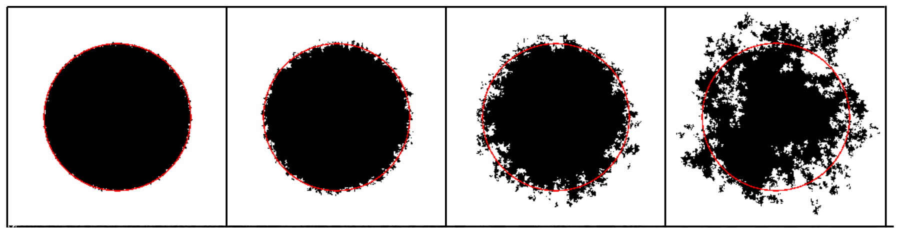
Dornic et al. (2001)

random initial condition:



t=4 t=16 t=64 t=256

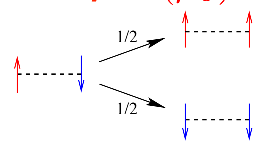
droplet initial condition:



Lattice Voter Model: 3 Basic Properties

1. Final State (Exit) Probability $\mathcal{E}(\rho_0) = \rho_0$

Evolution of a single active link:

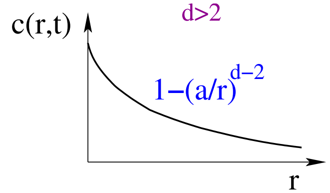


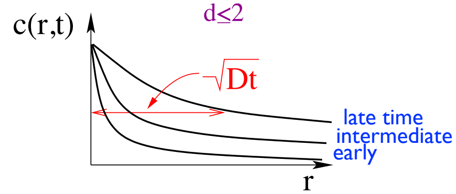
average magnetization conserved

2. Two-Spin Correlations

$$\frac{\partial c_2(\mathbf{r},t)}{\partial t} = \nabla^2 c_2(\mathbf{r},t)$$

$c_2(r\!=\!0,t) = 1$
$c_2(r\!>\!0,t\!=\!0) = 0$





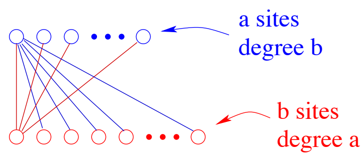
3. Consensus Time

dimension	consensus time
1	N^2
2	N ln N
>2	N

Voter Model on Complex Networks

C. Castellano, D. Vilon, A. Vespignani, EPL **63**, 153 (2003)
K. Suchecki, V. M. Eguiluz, M. San Miguel, EPL **69**, 228 (2005)
V. Sood, SR, PRL **94**, 178701 (2005); T. Antal, V. Sood, SR, PRE **77**, 041121 (2008)

illustrative example:
complete bipartite graph



pick site on **a** sublattice

pick ↓ on **a**

pick ↑ on ***b*** sublattice

$$dN_a = \frac{a}{a+b}\left[\frac{a-N_a}{a}\,\frac{N_b}{b} - \frac{N_a}{a}\,\frac{b-N_b}{b}\right]$$

$$dN_b = \frac{b}{a+b}\left[\frac{b-N_b}{b}\,\frac{N_a}{a} - \frac{N_b}{b}\,\frac{a-N_a}{a}\right]$$

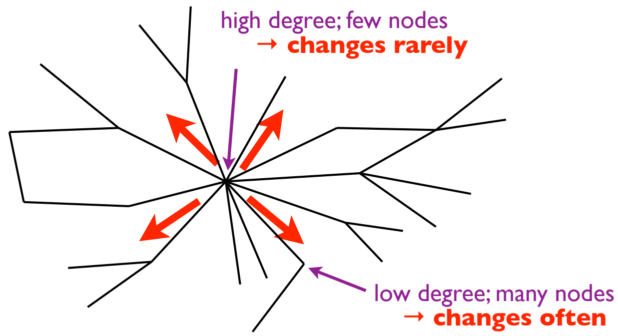
Subgraph densities: $\rho_a = N_a/a,\ \rho_b = N_b/b \quad dt = 1/(a+b)$

$$\rho_{a,b}(t) = \frac{1}{2}[\rho_{a,b}(0) - \rho_{b,a}(0)]\,e^{-2t} + \frac{1}{2}[\rho_a(0) + \rho_b(0)]$$

$$\rightarrow \frac{1}{2}[\rho_a(0) + \rho_b(0)]$$

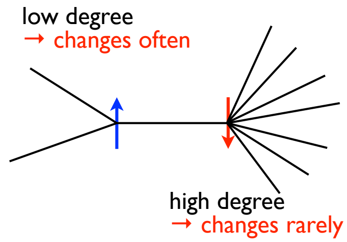
magnetization ***not*** conserved

Voter Model on Complex Networks



*"flow" from **high** degree to **low** degree*

New Conservation Law



to compensate different rates, consider:

degree-weighted 1st moment: $\omega = \frac{1}{\mu_1} \sum_k k\, n_k \rho_k$ *conserved!*

$\mu_1 = \text{av. degree}$

$n_k = \text{frac. nodes of degree } k$

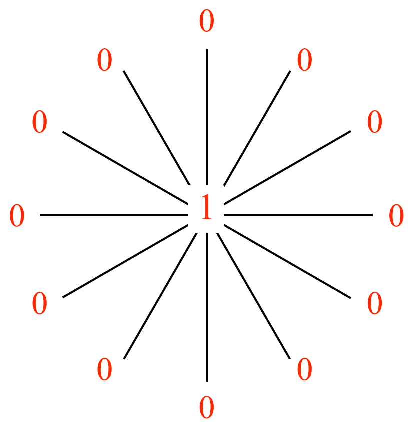
$\rho_k = \text{frac.} \uparrow \text{ on nodes of degree } k$

Exit Probability on Complex Graphs

$$\mathcal{E}(\omega) = \omega$$

Extreme case: star graph

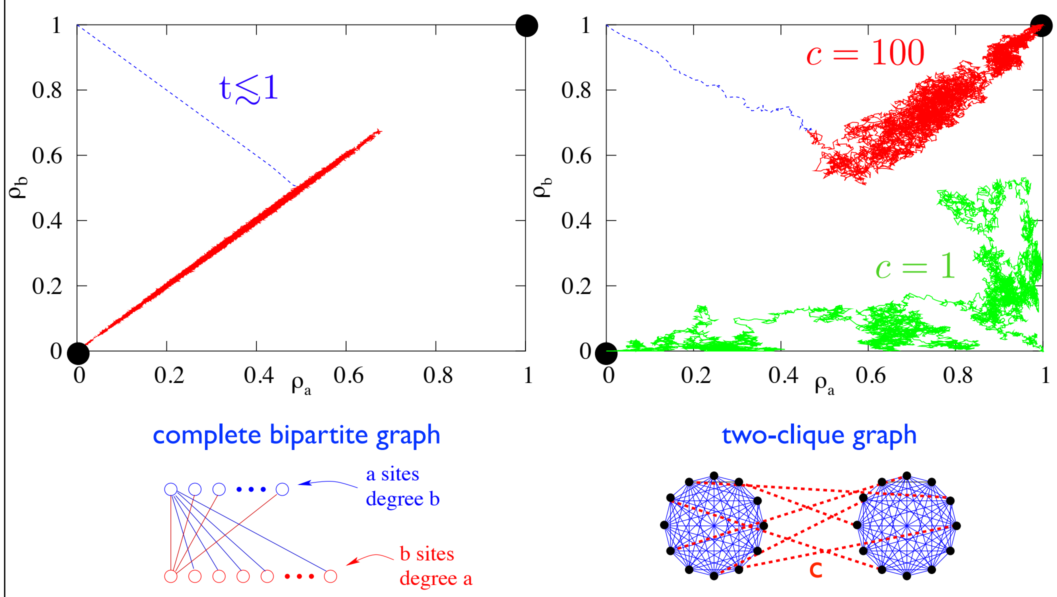
N nodes: degree 1
1 node: degree N



$$\omega = \frac{1}{\mu_1} \sum_k k\, n_k \rho_k = \frac{1}{2}$$

Final state: all 1 with prob. 1/2!

Route to Consensus on Complex Graphs



N=10000, C links/node

Consensus Time Evolution Equation

warmup: complete graph

$T(\rho) \equiv$ av. consensus time starting with density ρ

$$
\begin{aligned}
T(\rho) \quad = \quad & \mathcal{R}(\rho)[T(\rho+d\rho)+dt] \\
& +\mathcal{L}(\rho)[T(\rho-d\rho)+dt] \\
& +[1-\mathcal{R}(\rho)-\mathcal{L}(\rho)][T(\rho)+dt]
\end{aligned}
$$

Consensus Time Evolution Equation

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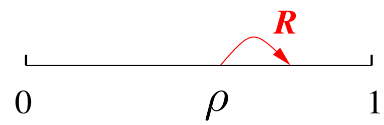

$$\begin{aligned} \mathcal{R}(\rho) &\equiv \mathrm{prob}(\downarrow\uparrow\rightarrow\uparrow\uparrow) \\ &= \rho(1-\rho) \end{aligned}$$

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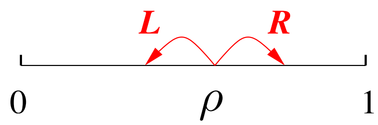

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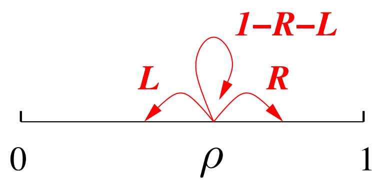
& + [1 - \mathcal{R}(\rho) - \mathcal{L}(\rho)][T(\rho) + dt]
\end{aligned}
$$



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\begin{aligned}
\mathcal{R}(\rho) &\equiv \mathrm{prob}(\downarrow\uparrow\rightarrow\uparrow\uparrow) \\
\mathcal{L}(\rho) &\equiv \mathrm{prob}(\uparrow\downarrow\rightarrow\downarrow\downarrow) \\
&= \rho(1-\rho)
\end{aligned}
$$

Consensus Time on Complete Graph

$$\begin{aligned} T(\rho) &= \mathcal{R}(\rho)[T(\rho+d\rho)+dt] \\ &\quad +\mathcal{L}(\rho)[T(\rho-d\rho)+dt] \\ &\quad +[1-\mathcal{R}(\rho)-\mathcal{L}(\rho)][T(\rho)+dt] \end{aligned}$$

continuum limit: $$T'' = -\frac{N}{\rho(1-\rho)}$$

solution:

$$T(\rho) = -N\left[\rho\ln\rho + (1-\rho)\ln(1-\rho)\right]$$

Consensus Time on Heterogeneous Networks

$$T(\{\rho_k\}) \equiv \text{av. consensus time starting with density } \rho_k \text{ on nodes of degree } k$$

$$\begin{aligned} T(\{\rho_k\}) = & \sum_k \mathcal{R}_k(\{\rho_k\})[T(\{\rho_k^+\}) + dt] \\ & + \sum_k \mathcal{L}_k(\{\rho_k\})[T(\{\rho_k^-\}) + dt] \\ & + \Big[1 - \sum_k [\mathcal{R}_k(\{\rho_k\}) + \mathcal{L}_k(\{\rho_k\})]\Big][T(\{\rho_k\}) + dt] \end{aligned}$$

$$\begin{aligned} \mathcal{R}_k(\{\rho_k\}) &= \text{prob}(\rho_k \to \rho_k^+) \\ &= \frac{1}{N} \sum_x{}' \frac{1}{k_x} \sum_y P(\downarrow, —, \uparrow) \\ &= n_k \omega (1 - \rho_k) \end{aligned}$$

$$\mathcal{L}_k(\{\rho_k\}) = n_k \rho_k (1 - \omega)$$

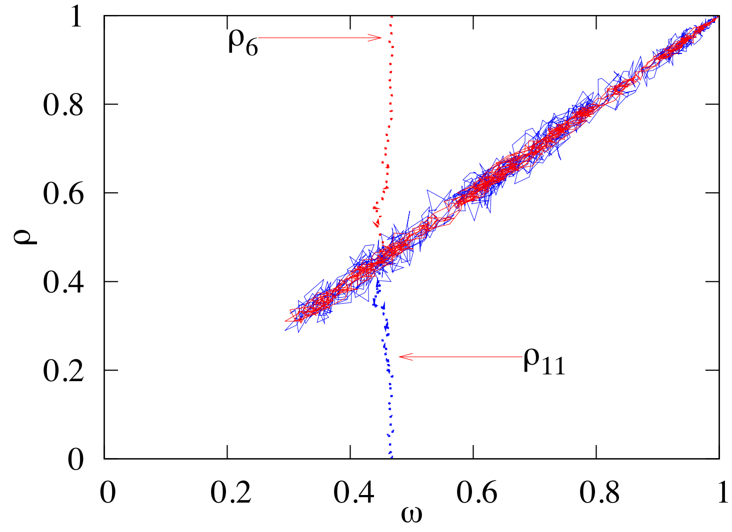
Consensus Time on Heterogeneous Networks

continuum limit:

$$\sum_k \left[(\omega - \rho_k)\frac{\partial T}{\partial \rho_k} + \frac{\omega + \rho_k - 2\omega\rho_k}{2Nn_k}\frac{\partial^2 T}{\partial \rho_k^2}\right] = -1$$

Molloy-Reed Scale-Free Network

$n_k \sim k^{-2.5}, \quad \mu_1 = 8$



Consensus Time on Heterogeneous Networks

continuum limit:

$$\sum_k \left[(\omega - \rho_k)\frac{\partial T}{\partial \rho_k} + \frac{\omega + \rho_k - 2\omega\rho_k}{2Nn_k}\frac{\partial^2 T}{\partial \rho_k^2}\right] = -1$$

now use $\rho_k \to \omega \quad \forall k$

and $\frac{\partial}{\partial \rho_k} = \frac{\partial \omega}{\partial \rho_k}\frac{\partial}{\partial \omega} = \frac{k n_k}{\mu_1}\frac{\partial}{\partial \omega}$

to give $$\frac{\partial^2 T}{\partial \omega^2} = -\frac{N\mu_1^2/\mu_2}{\omega(1-\omega)}$$ same as $$T'' = -\frac{N}{\rho(1-\rho)}$$

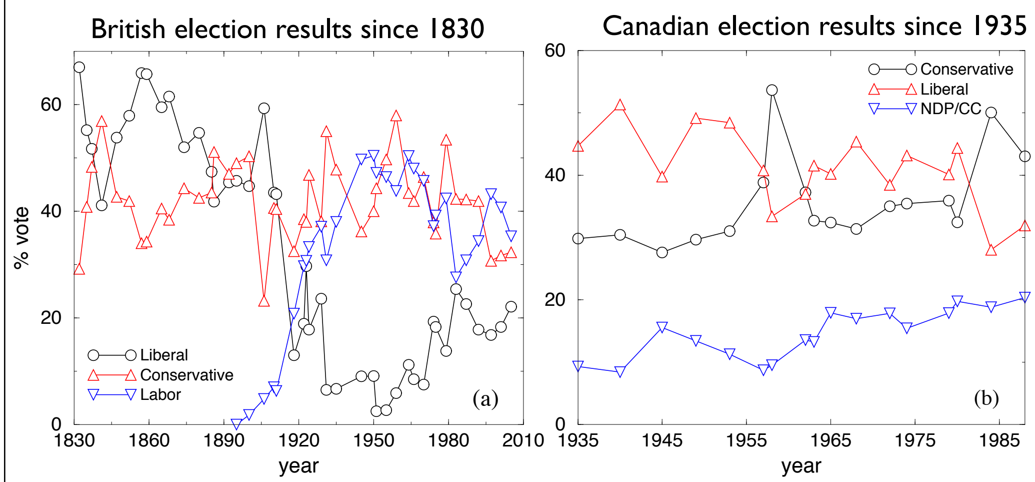
with *effective* size $N_{\rm eff} = N\,\mu_1^2/\mu_2$

Consensus Time for Power-Law Degree Distribution $n_k \sim k^{-\nu}$

$$T_N \propto N_{\text{eff}} = N\frac{\mu_1^2}{\mu_2} \sim \begin{cases} N & \nu > 3 \\ N/\ln N & \nu = 3 \\ N^{2(\nu-2)/(\nu-1)} & 2 < \nu < 3 \\ (\ln N)^2 & \nu = 2 \\ \mathcal{O}(1) & \nu < 2 \end{cases}$$

fast (<N)
consensus

Strategic Voting



Strategic Voter Model

D. Volovik, M. Mobilia, SR
EPL **85**, 48001 (2009)

randomly-selected voter changes to any other state equiprobably (rate T)

majority-minority interaction: minority *preferentially* changes to majority (rate r)

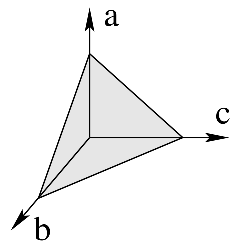
rate equations (A, B majority; c minority):

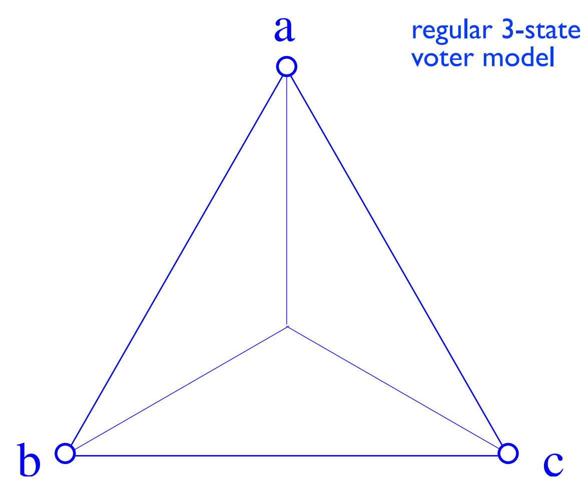
$$\dot{A} = T(B + c - 2A) + r\,Ac$$

$$\dot{B} = T(c + A - 2B) + r\,Bc$$

$$\dot{c} = T(A + B - 2c) - r\,(A + B)c$$

Phase Portrait





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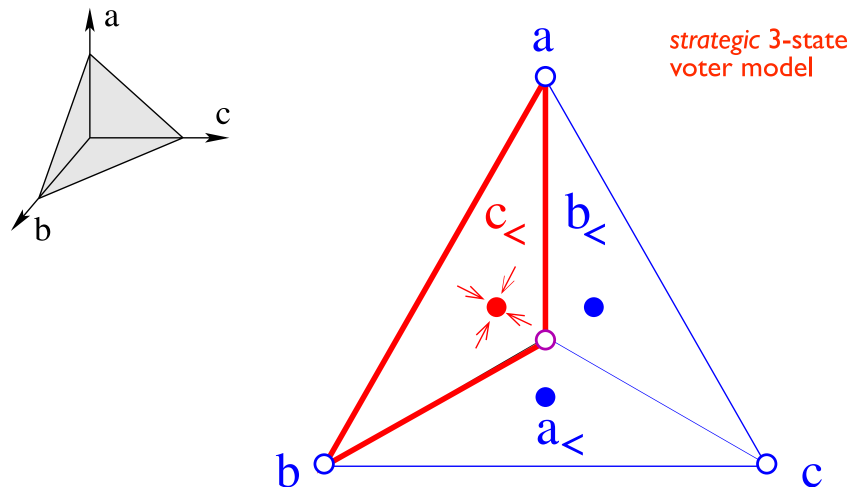
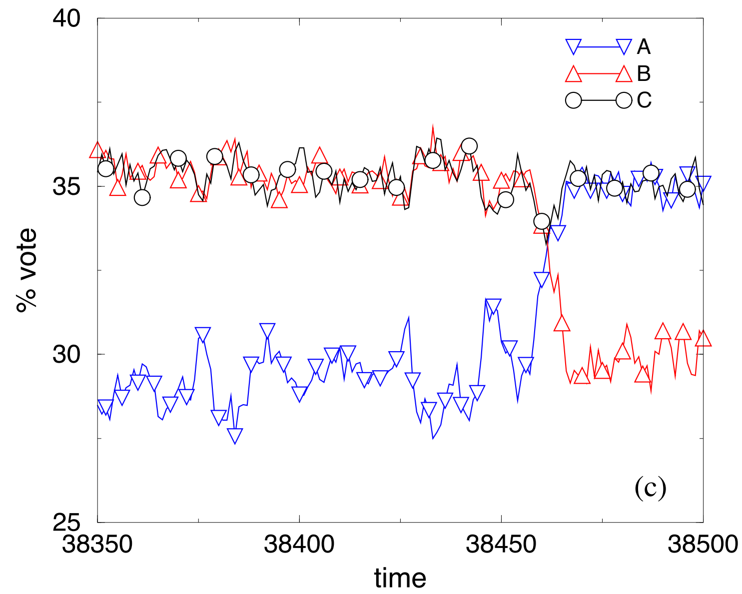
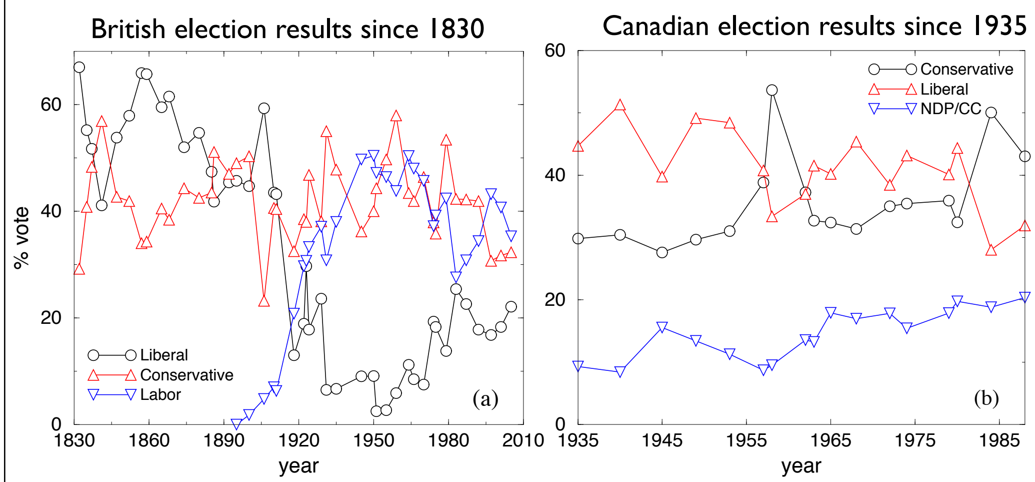


Slow Switching





British election results since 1830
Canadian election results since 1935
% vote
year
year
Liberal
Conservative
Labor
Conservative
Liberal
NDP/CC
(a)
(b)
0
20
40
60
1830
1860
1890
1920
1950
1980
2010
1935
1945
1955
1965
1975
1985

Partisan Voter Model

N. Masuda, N. Gibert, SR
arXiv:1003.0768

$\uparrow$ happy democrat, density D_h | $\downarrow$ sad democrat, density D_s | $\uparrow$ sad republican, density R_s | $\downarrow$ happy republican, density R_h

partisan voting update:

1. Pick voter, pick neighbor (as in usual voter model);

2a. If initial voter becomes *happy* by adopting neighboring state, change occurs with rate 1+ε;

$$\uparrow\downarrow \xrightarrow[1+\varepsilon]{} \downarrow\downarrow$$

2b. If initial voter becomes *unhappy* by adopting neighboring state, change occurs with rate 1-ε.

$$\downarrow\uparrow \xrightarrow[1-\varepsilon]{} \uparrow\uparrow$$

Partisan Voter Model: Mean-Field Limit

rate equations:

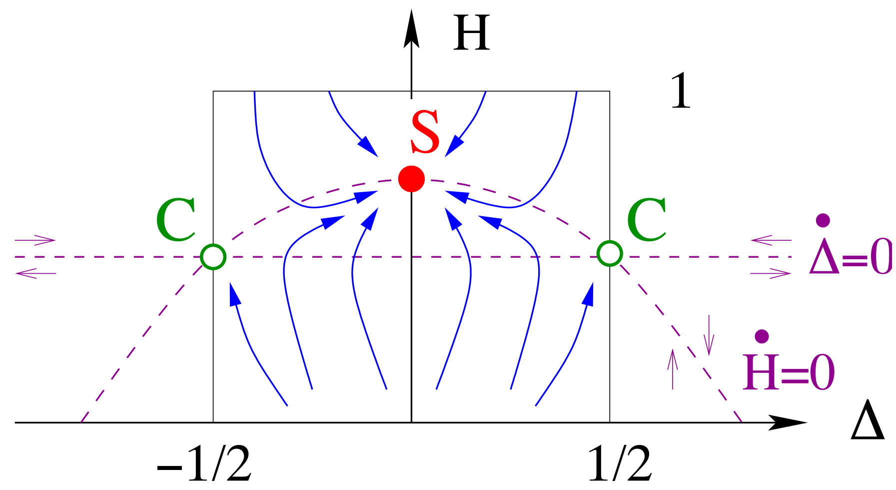
$$\dot{D}_h = \quad 2\epsilon\, D_h D_s + (1+\epsilon)\, D_s R_s \; - (1-\epsilon)\, D_h R_h$$
$$\dot{D}_s = -2\epsilon\, D_h D_s + (1-\epsilon)\, D_h R_h - (1+\epsilon)\, D_s R_s$$

$$\text{and } R \leftrightarrow D$$

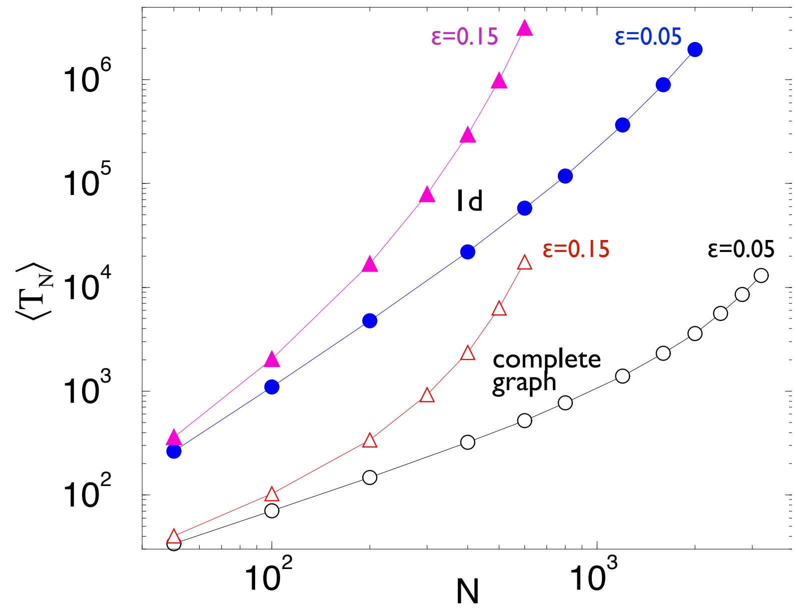
Symmetric Case: D=R=½

$H \equiv D_h + R_h$
$\quad = \text{density of happy voters}$

$\Delta \equiv D_h - R_h = D_h - (\frac{1}{2} - R_s) = \rho - \frac{1}{2}$
$\quad = \text{density democratic voters } -\frac{1}{2}$



Consensus Time on Finite Graphs



Summary & Outlook

Voter model:

paradigmatic, soluble, (but hopelessly naive)

Voter model on complex networks:

new conservation law
meandering route to consensus
fast consensus for broad degree distributions

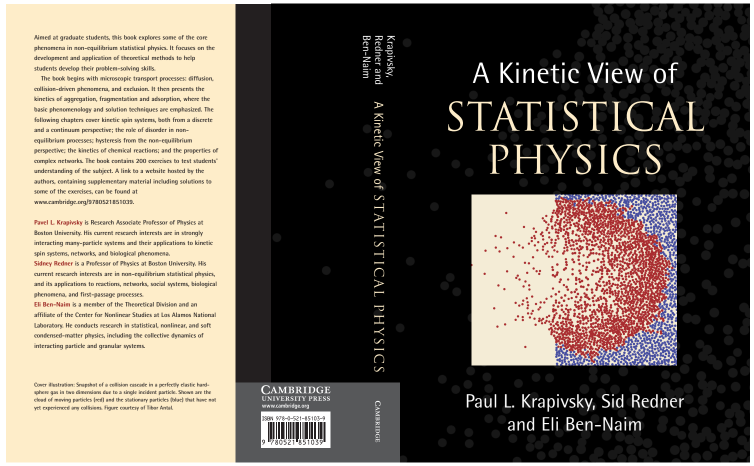
Extensions:

strategic voting → minority suppressed
partisan voting → selfishness forestalls consensus

Future:

"churn" rather than consensus
heterogeneity of real people
positive and negative social interactions → social balance

Crass Commercialism



to appear this October

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