UNIFIED PRINCIPLE OF ΣΔ MODULATION AS QUANTIZATION TECHNIQUE FOR OVERCOMPLETE EXPANSIONS

Thao Nguyen

Dept. of Electrical Engineering

City University of New York

DISCRETIZATION OF SIGNAL

$\{\boldsymbol{\varphi}_n\}_{n\in Z}$: generating family of vectors spanning input space

$$\mathbf{x} = \sum_{n\in Z} x_n \cdot \boldsymbol{\varphi}_n$$

$$\hat{\mathbf{x}} = \sum_{n\in Z} q_n \cdot \boldsymbol{\varphi}_n$$

where $q_n \in \underbrace{\{l_1, l_2, \cdots, l_N\}}_{\text{quantization levels}}$

error $\mathbf{x} - \hat{\mathbf{x}} = \sum_{n\in Z} (x_n - q_n) \cdot \boldsymbol{\varphi}_n$

USE OF REDUNDANCY IN ΣΔ MODULATION

Form vectors $$\mathbf{r}_k := \boldsymbol{\varphi}_k - \sum_{n \neq k} c_{n,k} \boldsymbol{\varphi}_n, \quad k \in Z$$

General notation $$\mathbf{r}_k := \sum_{n \in Z} d_{n,k} \boldsymbol{\varphi}_n, \quad k \in Z,$$ with $d_{k,k}=1$

$$D := \{d_{n,k}\}_{n,k \in Z}$$: redundancy operator

$$\mathbf{x} - \hat{\mathbf{x}} = \sum_{n \in Z} (x_n - q_n) \cdot \boldsymbol{\varphi}_n \quad \Leftrightarrow \quad \mathbf{x} - \hat{\mathbf{x}} = \sum_{k \in Z} u_k \cdot \mathbf{r}_k$$

where $$x_n - q_n = \sum_{k \in Z} d_{n,k} \cdot u_k$$

PROOF

$$\mathbf{x} - \hat{\mathbf{x}} = \sum_{n \in Z} \left(x_n - q_n \right) \cdot \boldsymbol{\varphi}_n$$

$$x_n - q_n = \sum_{k \in Z} d_{n,k} \cdot u_k \qquad\qquad \mathbf{r}_k = \sum_{n \in Z} d_{n,k} \cdot \boldsymbol{\varphi}_n$$

$$\mathbf{x} - \hat{\mathbf{x}} = \sum_{n \in Z} \underbrace{\left(\sum_{k \in Z} d_{n,k} \cdot u_k \right)}_{x_n - q_n} \cdot \boldsymbol{\varphi}_n = \sum_{k \in Z} u_k \cdot \underbrace{\left(\sum_{n \in Z} d_{n,k} \cdot \boldsymbol{\varphi}_n \right)}_{\mathbf{r}_k}$$

$$\boxed{\mathbf{x} - \hat{\mathbf{x}} = \sum_{k \in Z} u_k \cdot \mathbf{r}_k}$$

DIFFERENTIATION EXAMPLE

Form vectors $\mathbf{r}_k := \boldsymbol{\varphi}_k - \boldsymbol{\varphi}_{k+1}$

$$\mathbf{x} - \hat{\mathbf{x}} = \sum_{n \in Z} \left(x_n - q_n \right) \cdot \boldsymbol{\varphi}_n \qquad \Leftrightarrow \qquad \mathbf{x} - \hat{\mathbf{x}} = \sum_{k \in Z} u_k \cdot \mathbf{r}_k$$

where $x_n - q_n = u_n - u_{n-1}$

DIFFERENTIATION EXAMPLE

Form vectors $\mathbf{r}_k := \boldsymbol{\varphi}_k - \boldsymbol{\varphi}_{k+1}$

$$\mathbf{r}_k := \sum_{n \in Z} d_{n,k} \boldsymbol{\varphi}_n, \quad k \in Z,$$

where $d_{n,k} = d_{n-k}$ with $d_n := \delta_n - \delta_{n+1}$

$$D := \{d_{n-k}\}_{n,k \in Z}$$

particular case of “convolutional” or “shift-invariant” operator

PRINCIPLES OF ΣΔ MODULATION

Choose redundancy operator $D = \{d_{n,k}\}_{n,k\in Z}$ invertible and

such that $\mathbf{r}_k := \sum_{n\in Z} d_{n,k}\boldsymbol{\varphi}_n$ are "small"

Find quantized sequence $q_n \in \{l_1, l_2, \cdots, l_N\}$ so that equation

$$x_n - q_n = \sum_{k\in Z} d_{n,k}\cdot u_k$$

yields bounded and "small" solution in u_k

Hopefully, $\mathbf{x} - \hat{\mathbf{x}} = \sum_{n\in Z} u_n \cdot \mathbf{r}_n$ will be "small"

PRINCIPLES OF ΣΔ MODULATION

Choose redundancy operator $D = \{d_{n,k}\}_{n,k \in Z}$ invertible and

such that $\mathbf{r}_k := \sum_{n \in Z} d_{n,k} \boldsymbol{\varphi}_n$ are "small"

Hopefully, $\mathbf{x} - \hat{\mathbf{x}} = \sum_{n \in Z} u_n \cdot \mathbf{r}_n$ will be "small"

SPACE NORM

Main case of study: shift-invariant space of functions

$$\boldsymbol{\varphi}_n = \varphi(t - n\tau) \text{ where } \varphi(t) \text{ and } \tau \text{ are given}$$

- $$\left\|\mathbf{x}\right\|_2{}^2 = \int_{-\infty}^{\infty} \left|x(t)\right|^2 dt \quad \text{however} \quad \hat{\mathbf{x}} = \sum_{n \in Z} q_n \cdot \boldsymbol{\varphi}_n \notin L^2$$

  use of L2 norm requires special treatment, see [O.Yilmaz, 2004]

- $$\left\|\mathbf{x}\right\|_\infty = \sup_{t \in \mathbf{R}} \left|x(t)\right|$$

- $$\mathrm{MSE}(\mathbf{x}) = \lim_{T \to \infty} \tfrac{1}{T} \int_{-T}^{T} \left|x(t)\right|^2 dt$$

SPACE NORM

Main case of study: shift-invariant space of functions

$\boldsymbol{\varphi}_n = \varphi(t - n\tau)$ where $\varphi(t)$ and τ are given

-

- $\boxed{\|\mathbf{x}\|_\infty = \sup_{t \in \mathbf{R}} |x(t)|}$

- $\mathrm{MSE}(\mathbf{x}) = \lim_{T \to \infty} \frac{1}{T} \int_{-T}^{T} |x(t)|^2 dt$

SUP NORM BOUND

$$\mathbf{x} - \hat{\mathbf{x}} = \sum_{n \in Z} u_n \cdot \mathbf{r}_n$$

$$\Downarrow$$

$$\left| x(t) - \hat{x}(t) \right| \leq \left\| u \right\|_{\infty} \cdot \sum_{k \in Z} \left| r_k(t) \right|$$

[I.Daubechies & R.DeVore, 2003]

DIFFERENTIATION

$$D = \{d_{n-k}\}_{n,k \in Z} \quad \text{with} \quad d_n = \delta_n - \delta_{n-1}$$

$$\mathbf{r}_k := \boldsymbol{\varphi}_k - \boldsymbol{\varphi}_{k+1} \quad \text{with} \quad \boldsymbol{\varphi}_n = \varphi(t - n\tau)$$

$$\Downarrow$$

$$\sum_{k \in Z} |r_k(t)| \le \|\varphi'\|_{L^1}$$

$$\|\mathbf{x} - \hat{\mathbf{x}}\|_\infty \le \|u\|_\infty \cdot \|\varphi'\|_{L^1}$$

[I.Daubechies & R.DeVore, 2003]

APPLICATION TO BANDLIMITED INPUTS

Assumption : $x(t)$ lowpass of bandwidth $\Omega_0 = \frac{2\pi}{\tau_0}$

Then $x(t) := \sum_{n \in Z} x_n \cdot \varphi(t - n\tau)$

where $x_n = x(n\tau)$ and $\varphi(t) = \tau \operatorname{sinc}_{\tau_0}(t)$

with $\operatorname{sinc}_{\tau_0}(t) := \frac{\sin(\pi t/\tau_0)}{\pi t}$ and $\tau < \tau_0 := \frac{2\pi}{\Omega_0}$ $\rightarrow$ $R = \frac{\tau_0}{\tau} =$ redundancy

up to some relxation in
the frequency domain, see *

$$\|\mathbf{x} - \hat{\mathbf{x}}\|_\infty \leq \tau \cdot \|u\|_\infty \cdot \|\operatorname{sinc}_{\tau_0}{}'\|_{L^1}$$

[* I.Daubechies & R.DeVore, 2003]

m^{th} ORDER DIFFERENTIATION

$$D = \{d_{n-k}\}_{n,k\in Z} \quad \text{with} \quad d_n = (\delta_n - \delta_{n-1})^{(m)}$$

$$\mathbf{r}_k := \sum_{n\in Z} d_{n-k}\boldsymbol{\varphi}_n \quad \text{with} \quad \boldsymbol{\varphi}_n = \varphi(t - n\tau)$$

$$\Downarrow$$

$$\sum_{k\in Z} |r_k(t)| \le \tau^{m-1} \left\| \varphi^{(m)} \right\|_{L^1}$$

$$|x(t) - \hat{x}(t)| \le \|u\|_\infty \cdot \sum_{k\in Z} |r_k(t)|$$

[I.Daubechies & R.DeVore, 2003]

APPLICATION TO BANDLIMITED INPUTS

$$D = \{d_{n-k}\}_{n,k\in Z} \quad \text{with} \quad d_n = (\delta_n - \delta_{n-1})^{(m)}$$

$$\sum_{k\in Z} |r_k(t)| \leq \tau^{m-1} \left\| \mathrm{sinc}_{\tau_0}{}^{(m)} \right\|_{L^1}$$

$$\|\mathbf{x} - \hat{\mathbf{x}}\|_\infty \leq \tau^m \cdot \|u\|_\infty \cdot \left\| \mathrm{sinc}_{\tau_0}{}^{(m)} \right\|_{L^1}$$

[I.Daubechies & R.DeVore, 2003]

SPACE NORM

Main case of study: shift-invariant space of functions

$$\boldsymbol{\varphi}_n = \varphi(t - n\tau) \text{ where } \varphi(t) \text{ and } \tau \text{ are given}$$

-

- $$\|\mathbf{x}\|_\infty = \sup_{t\in\mathbf{R}} |x(t)|$$

- $$\boxed{\mathrm{MSE}(\mathbf{x}) = \lim_{T\to\infty} \tfrac{1}{T}\int_{-T}^{T} |x(t)|^2\, dt}$$

MSE ANALYSIS

Main case of study: shift-invariant space of functions

shift-invariant operator D

$$\boldsymbol{\varphi}_n = \varphi(t - n\tau) \text{ where } \varphi(t) \text{ and } \tau \text{ are given}$$

$$\Rightarrow \quad \mathbf{r}_k = r(t - k\tau) \quad \text{where } r(t) = \sum_{n\in Z} d_n\,\varphi(t - n\tau)$$

Theorem:

$$\mathbf{x} - \hat{\mathbf{x}} = \sum_{k\in Z} u_k \cdot \mathbf{r}_k \qquad \Rightarrow \qquad \boxed{\mathrm{MSE}(\mathbf{x} - \hat{\mathbf{x}}) = \sum_{k\in Z} a_k\, s_k}$$

where

$$a_k := \lim_{N\to\infty} \frac{1}{2N} \sum_{n=-N}^{N} u_n\, u_{n+k} \quad \text{and} \quad s_k := \frac{1}{\tau} \int_{\mathbf{R}} r(t)\, r(t - k\tau)\, dt$$

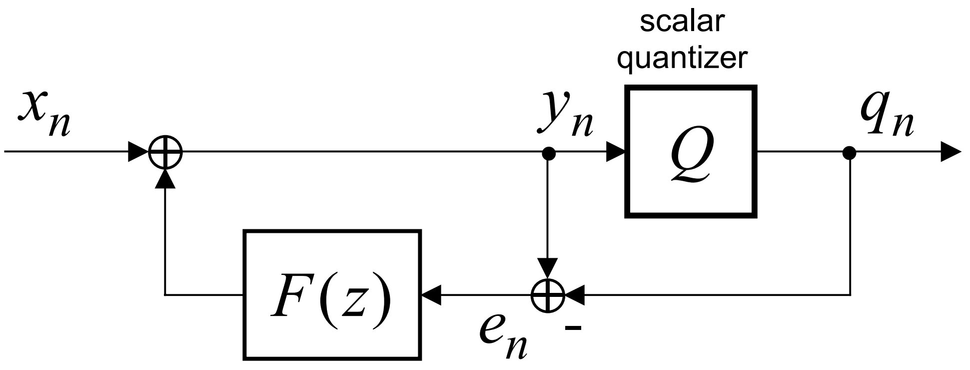
STATISTICAL MSE ANALYSIS

$$E\left\{\mathrm{MSE}(\mathbf{x}-\hat{\mathbf{x}})\right\} = \sum_{k\in Z} \overline{a}_k \, s_k$$

where

$\overline{a}_k := E\left\{u_n \, u_{n+k}\right\}$ and $s_k := \dfrac{1}{\tau}\displaystyle\int_{\mathbf{R}} r(t)\, r(t-k\tau)\, dt$

“WHITE NOISE” BEHAVIOR



$$u_n = g_n * e_n$$

$$E\{e_n\, e_{n+k}\} \xrightarrow[\text{large \#bits}]{} \sigma_e^2\, \delta_k$$

$$\Rightarrow \bar{a}_k := E\{u_n\, u_{n+k}\} \xrightarrow[\text{large \#bits}]{} \bar{a}_0\, \delta_k$$

“WHITE NOISE” MODEL

$$E\left\{\mathrm{MSE}(\mathbf{x}-\hat{\mathbf{x}})\right\} = \frac{\overline{a}_0}{\tau}\left\|\mathbf{r}\right\|_2^{\,2}$$

$$\overline{a}_k := E\left\{u_n\, u_{n+k}\right\} = \overline{a}_0\, \delta_k$$

$$s_0 = \frac{1}{\tau}\int_{\mathbf{R}} \left|r(t)\right|^2 dt = \frac{1}{\tau}\left\|\mathbf{r}\right\|_2^{\,2}$$

FIR DESIGN OF D
UNDER "WHITE NOISE" MODEL

Assume constraint

$$D = \{d_{n-k}\}_{n,k\in Z} \text{ with } \{d_n\}_{n\in Z} = \{\underbrace{d_0}_{=1}, d_1, \cdots, d_m\}$$

$$r(t) = \sum_{n\in Z} d_n\,\varphi(t - n\tau) \quad \Rightarrow \quad \mathbf{r} = \boldsymbol{\varphi}_0 + d_1\boldsymbol{\varphi}_1 + \cdots + d_m\boldsymbol{\varphi}_m$$

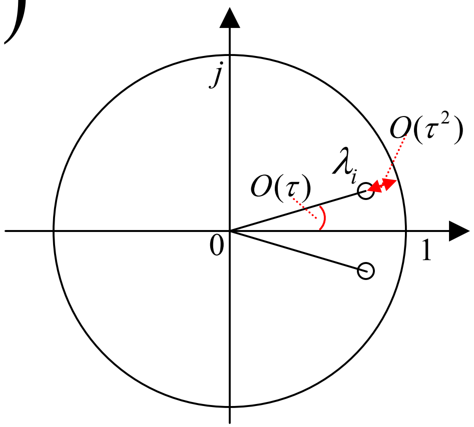
$$E\{\mathrm{MSE}(\mathbf{x} - \hat{\mathbf{x}})\} = \frac{\overline{a}_0}{\tau}\|\mathbf{r}\|_2^{\,2}$$

is iminimized when $d_1, \cdots, d_m$ are chosen such that

$$d_1\boldsymbol{\varphi}_1 + \cdots + d_m\boldsymbol{\varphi}_m = -\mathrm{proj}_{\langle \boldsymbol{\varphi}_1, \ldots, \boldsymbol{\varphi}_m \rangle}(\boldsymbol{\varphi}_0)$$

BANDLIMITED CASE

$$\varphi(t) = \tau \,\mathrm{sinc}_{\tau_0}(t)$$

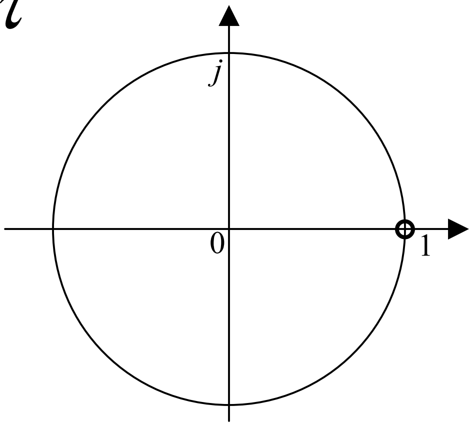


White noise model optimization

$$\Rightarrow D(z) = \prod_{i=1}^{m} (1 - \lambda_i z^{-1})$$

$$E\{\mathrm{MSE}(\mathbf{x} - \hat{\mathbf{x}})\} \approx \alpha\, \tau^{2m+1}$$

Differentiation operator



$$D(z) = (1 - z^{-1})^m$$

$$E\{\mathrm{MSE}(\mathbf{x} - \hat{\mathbf{x}})\} \approx \beta\, \tau^{2m+1}$$

$$\|\mathbf{x} - \hat{\mathbf{x}}\|_\infty^2 \le \gamma\, \tau^{2m}$$

PRINCIPLES OF ΣΔ MODULATION

Choose redundancy operator $D = \{d_{n,k}\}_{n,k\in Z}$ invertible and

such that $\mathbf{r}_k := \sum_{n\in Z} d_{n,k}\boldsymbol{\varphi}_n$ are "small"

Find quantized sequence $q_n \in \{l_1, l_2, \cdots, l_N\}$ so that equation

$$x_n - q_n = \sum_{k\in Z} d_{n,k} \cdot u_k$$

yields bounded and "small" solution in u_k

Hopefully, $\mathbf{x} - \hat{\mathbf{x}} = \sum_{n\in Z} u_n \cdot \mathbf{r}_n$ will be "small"

PRINCIPLES OF ΣΔ MODULATION

Find quantized sequence $q_n \in \{l_1, l_2, \cdots, l_N\}$ so that equation

$$x_n - q_n = \sum_{k \in Z} d_{n,k} \cdot u_k$$

yields bounded and "small" solution in u_k

CAUSAL AND TIME-INVARIANT FRAMEWORK

- $D = \{d_{n,k}\}_{n,k \in Z}$ such that $d_{n,k} = d_{n-k}$ with $d_n = 0, \forall n < 0$ and $d_0 = 1$
- q_n must be decided at instant n

Find quantized sequence $q_n \in \{l_1, l_2, \cdots, l_N\}$ so that equation

$$x_n - q_n = d_n * u_n$$

yields bounded and "small" solution in u_k

CAUSAL AND TIME-INVARIANT FRAMEWORK

- $D = \{d_{n,k}\}_{n,k \in Z}$ such that $d_{n,k} = d_{n-k}$ with $d_n = 0, \forall n < 0$ and $d_0 = 1$
- q_n must be decided at instant n

Find quantized sequence $q_n \in \{l_1, l_2, \cdots, l_N\}$ so that equation

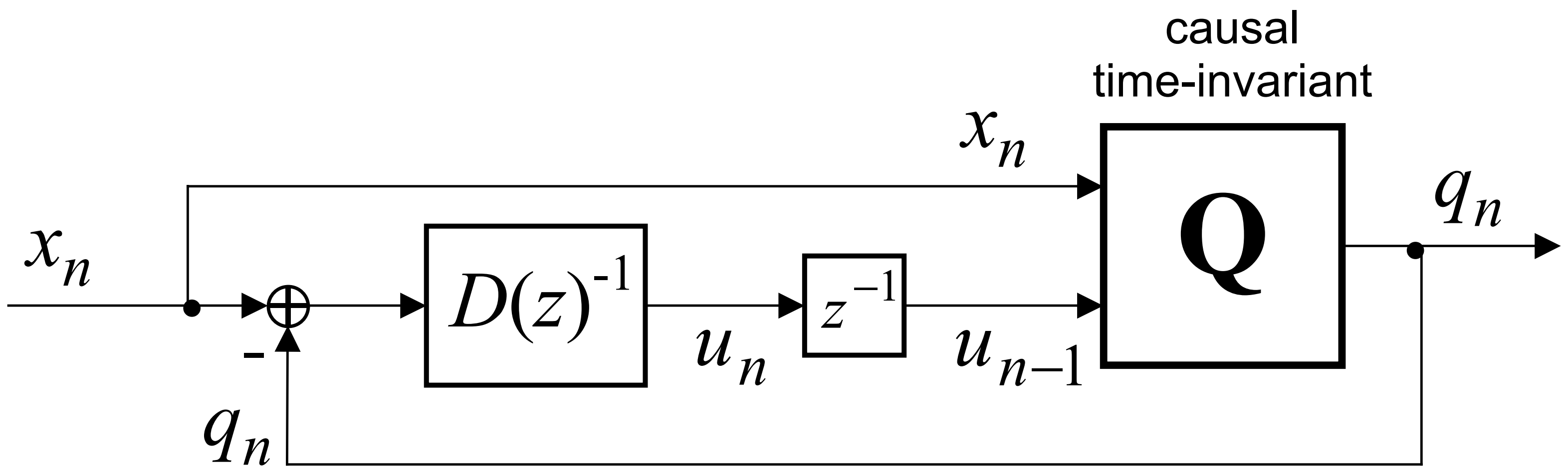
$$u_n = -\sum_{k>0} d_k \cdot u_{n-k} + (x_n - q_n)$$

yields bounded and "small" solution in u_k

$$q_n = \mathbf{Q}(x_n, x_{n-1}, \cdots; u_{n-1}, u_{n-2}, \cdots) \in \{l_1, l_2, \cdots, l_N\}$$

CAUSAL AND TIME-INVARIANT RESOLUTION

dynamical system

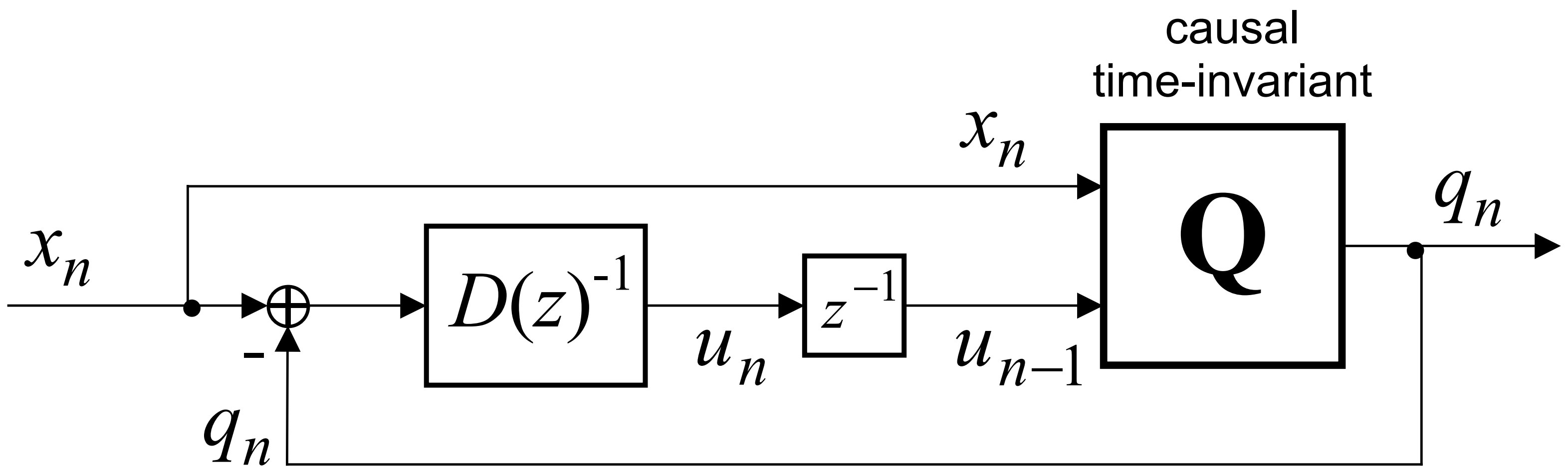


$$u_n = -\sum_{k>0} d_k \cdot u_{n-k} + (x_n - q_n)$$

$$q_n = \mathbf{Q}(x_n, x_{n-1}, \cdots; u_{n-1}, u_{n-2}, \cdots)$$

STABLE ONE-BIT SCHEME

[Daubechies & DeVore (2003)]

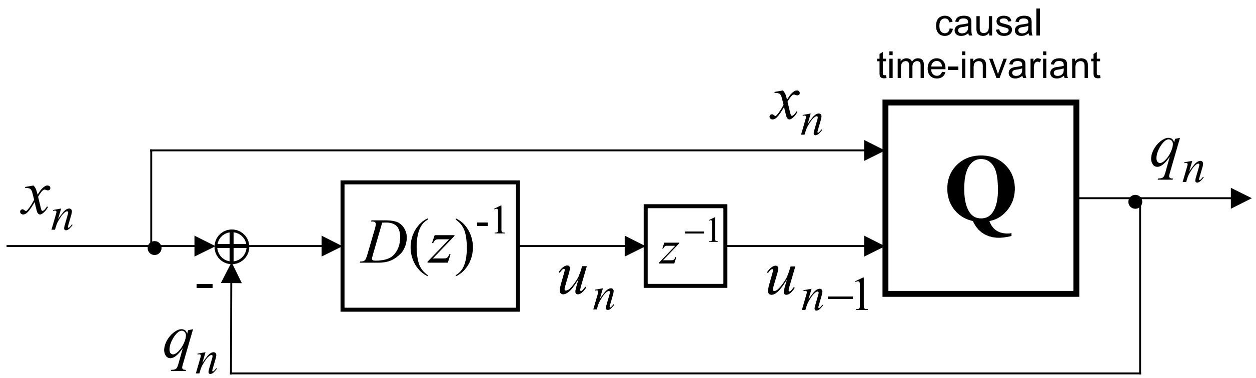


$D = \{d_{n-k}\}_{n,k\in Z}$ such that $d_n = (\delta_n - \delta_{n-1})^{(m)}$ $\quad \left(D(z) = (1 - z^{-1})^m\right)$

$$|x_n| \le \frac{\Delta}{2} - \varepsilon$$

$$q_n = \mathbf{Q}(x_n; u_{n-1}, u_{n-2}, \cdots, u_{n-m}) \in \left\{+\frac{\Delta}{2}, -\frac{\Delta}{2}\right\}$$

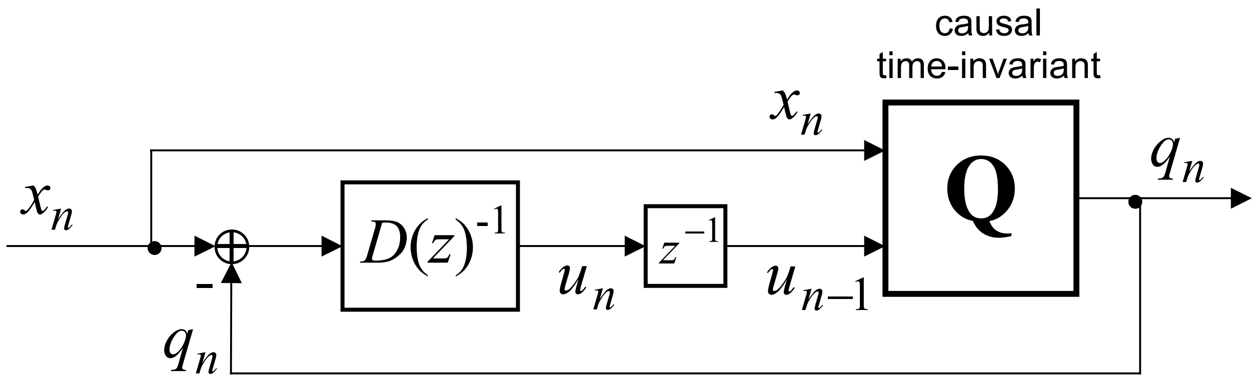
RESTRICTED SCHEME



restriction: $\mathbf{Q}(x_n, x_{n-1}, \cdots; u_{n-1}, u_{n-2}, \cdots) = Q(x_n + c_n * u_n)$

where c_n is strictly causal and Q is a **scalar** quantizer

RESTRICTED SCHEME



restriction: $\mathbf{Q}(x_n, x_{n-1}, \cdots; u_{n-1}, u_{n-2}, \cdots) = Q(\underbrace{x_n + c_n * u_n}_{y_n})$

$$q_n = Q(y_n)$$

$$x_n - q_n = d_n * u_n$$

$$y_n = x_n + c_n * u_n$$

RESTRICTED SCHEME

Define $$e_n := y_n - q_n$$

Perform change of state variable

$$e_n = (d_n + c_n) * u_n$$

$$q_n = Q(y_n)$$

$$y_n - q_n = (d_n + c_n) * u_n$$

$$y_n = x_n + c_n * u_n$$

RESTRICTED SCHEME

Define $$e_n := y_n - q_n$$

Perform change of state variable

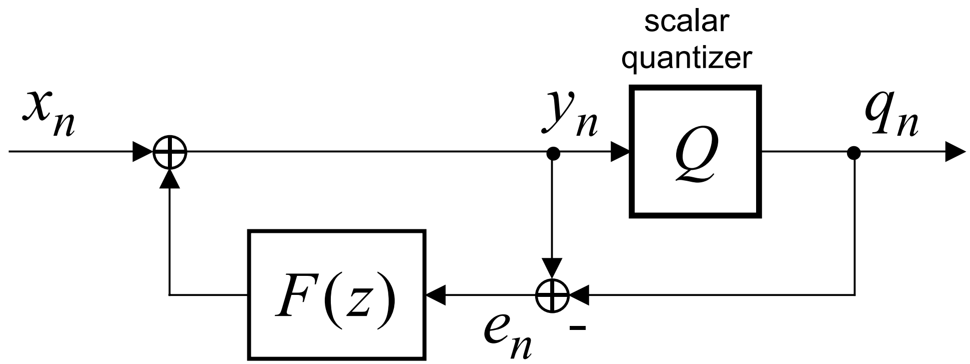
$$e_n = (d_n + c_n) * u_n$$

$$u_n = g_n * e_n$$

where g_n such that $G(z) = \frac{1}{D(z) + C(z)}$

$$\begin{aligned} q_n &= Q(y_n) \\ y_n - q_n &= (d_n + c_n) * u_n \\ y_n &= x_n + c_n * u_n \end{aligned} \quad \Leftrightarrow \quad \begin{aligned} q_n &= Q(y_n) \\ e_n &= y_n - q_n \\ y_n &= x_n + \underbrace{c_n * g_n}_{f_n} * e_n \end{aligned}$$

ERROR DIFFUSION



$$u_n = g_n * e_n$$

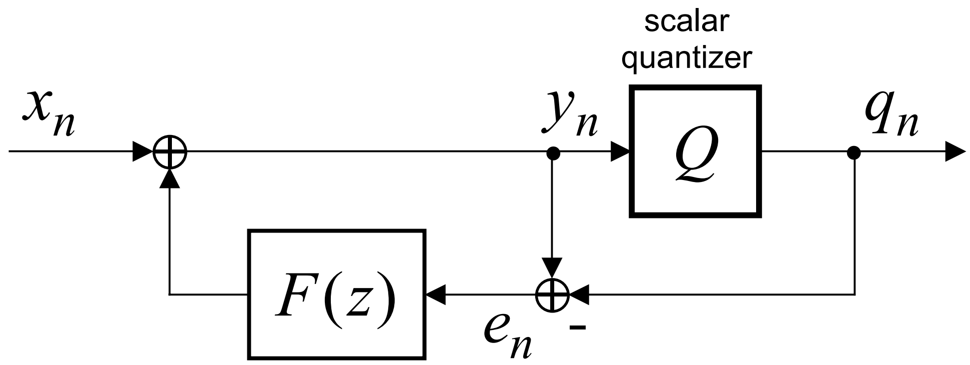
where $G(z) = \dfrac{1}{D(z) + C(z)}$ and $F(z) = C(z)\,G(z)$

$$q_n = Q(y_n)$$

$$e_n = y_n - q_n$$

$$y_n = x_n + \underbrace{c_n * g_n}_{f_n} * e_n$$

STABLE ERROR DIFFUSION



$$u_n = g_n * e_n$$

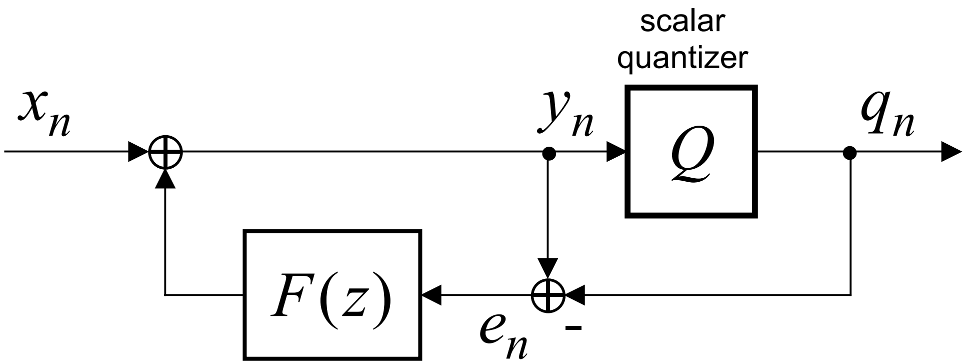
where $G(z) = \dfrac{1}{D(z) + C(z)}$ and $F(z) = C(z)G(z)$

$$q_n = Q(y_n)$$

$$e_n = y_n - q_n$$

$$\|y\|_\infty \le \|x\|_\infty + \|f\|_1 \cdot \|e\|_\infty \quad \Leftarrow \quad y_n = x_n + f_n * e_n$$

STABLE ERROR DIFFUSION



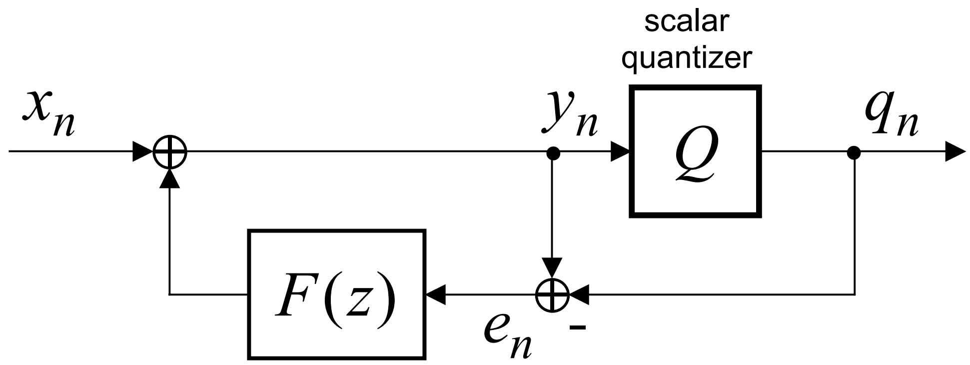
$$u_n = g_n * e_n$$

where $G(z) = \dfrac{1}{D(z) + C(z)}$ and $F(z) = C(z)\,G(z)$

- choose $C(z)$ so that both $G(z)$ and $F(z)$ are stable
- choose uniform quantizer Q of step size Δ $\Rightarrow$ $\|e\|_\infty \le \dfrac{\Delta}{2}$

$$\|y\|_\infty \le \|x\|_\infty + \|f\|_1 \cdot \frac{\Delta}{2} \quad \Rightarrow \quad \text{finite quantizer } Q$$

STABLE 1-BIT ERROR DIFFUSION



$$u_n = g_n * e_n$$

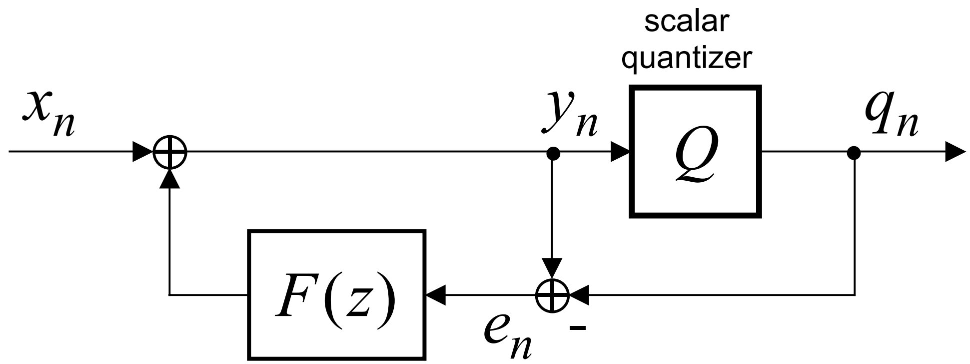
where $G(z) = \dfrac{1}{D(z)+C(z)}$ and $F(z) = C(z)G(z)$

- choose $C(z)$ so that both $G(z)$ and $F(z)$ are stable
- choose uniform quantizer Q of step size Δ $\quad \Rightarrow \quad \|e\|_\infty \le \dfrac{\Delta}{2}$
- design f_n so that

$$\|y\|_\infty \le \|x\|_\infty + \|f\|_1 \cdot \frac{\Delta}{2} \le \Delta \quad \Rightarrow$$

1-bit quantizer Q

STABLE 1-BIT ERROR DIFFUSION



$$u_n = g_n * e_n$$

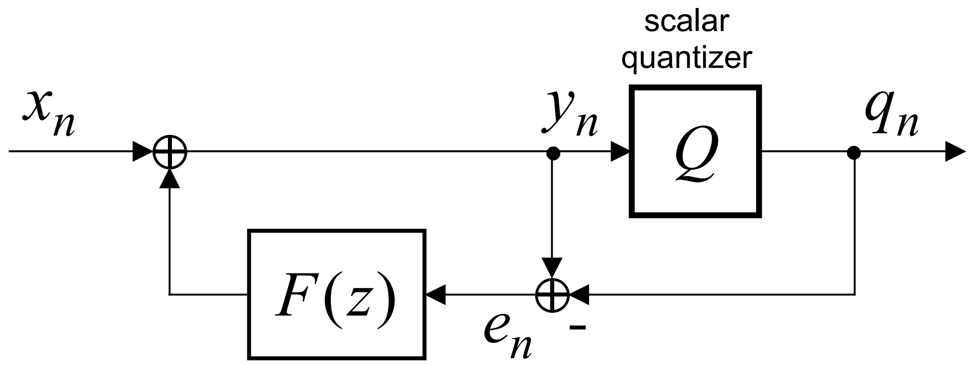
where $G(z) = \dfrac{1}{D(z) + C(z)}$ and $F(z) = C(z)\,G(z)$

$\Rightarrow$ g_n causal with $g_0 = 1$ and $\boxed{F(z) = 1 - G(z)\,D(z)}$

- design $\boxed{g_n}$ so that

$$\|x\|_\infty + \|f\|_1 \cdot \frac{\Delta}{2} \le \Delta \quad \Rightarrow \quad \text{1-bit quantizer } Q$$

CASE OF DIFFERENTIATOR



$$D(z) = (1 - z^{-1})^m$$

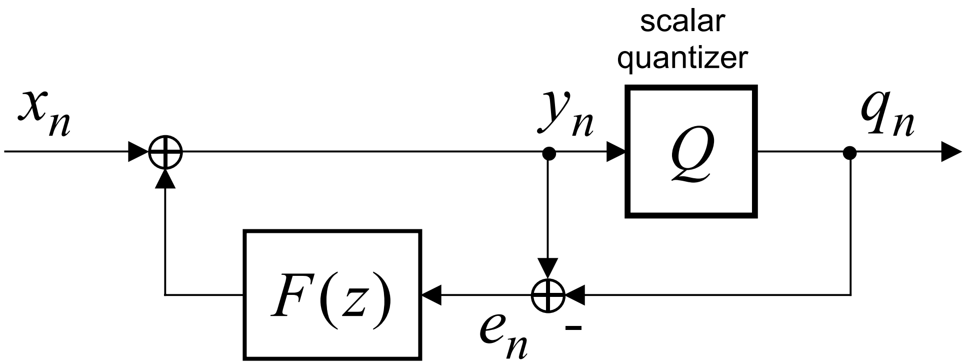
$$F(z) = 1 - G(z)\,D(z)$$

choose $G(z) = 1$ then $F(z) = 1 - D(z)$

$$\|f_1\| = 2^m - 1$$

$$\|x\|_\infty + \|f\|_1 \cdot \frac{\Delta}{2} \le \Delta \quad \Leftrightarrow \quad \begin{cases} m = 1 \\ \|x\|_\infty \le \dfrac{\Delta}{2} \end{cases}$$

SINGLE-LOOP SINGLE-BIT CASE



$$D(z) = 1 - z^{-1}$$

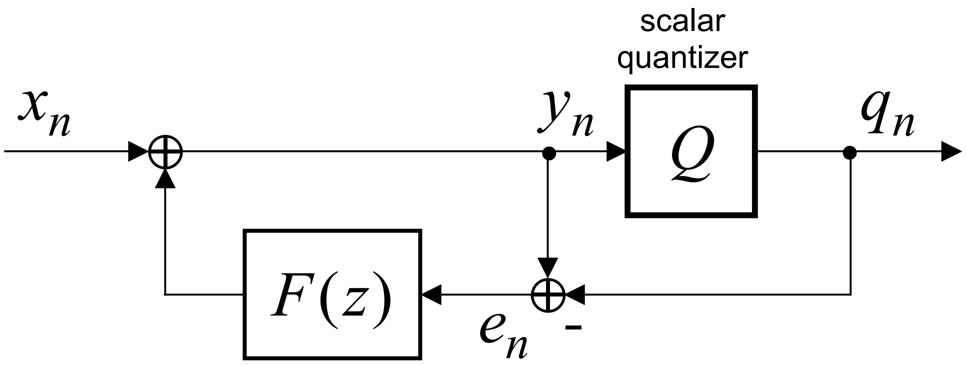
$$F(z) = 1 - G(z)\,D(z)$$

choose $G(z) = 1$ then $F(z) = z^{-1}$

$$\|f_1\| = 1$$

$$\|x\|_\infty + \|f\|_1 \cdot \frac{\Delta}{2} \le \Delta \quad \Leftrightarrow \quad \begin{cases} m = 1 \\ \|x\|_\infty \le \dfrac{\Delta}{2} \end{cases}$$

CASE OF DIFFERENTIATOR WITH $m>1$

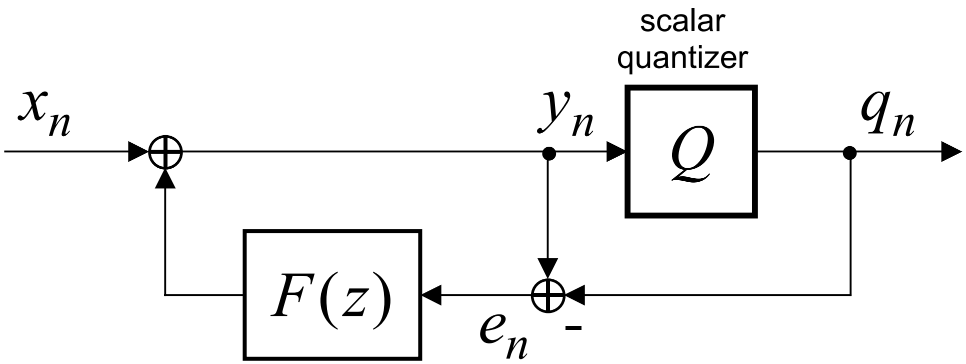


$$D(z) = (1 - z^{-1})^m$$

$$F(z) = 1 - G(z)\,D(z)$$

$$\|x\|_\infty + \|f\|_1 \cdot \frac{\Delta}{2} \le \Delta \quad \Rightarrow \quad G(z) \ne 1$$

DIFFERENTIATOR WITH FIR $G(z)$



$$D(z) = (1 - z^{-1})^m$$

$$F(z) = 1 - G(z)\,D(z)$$

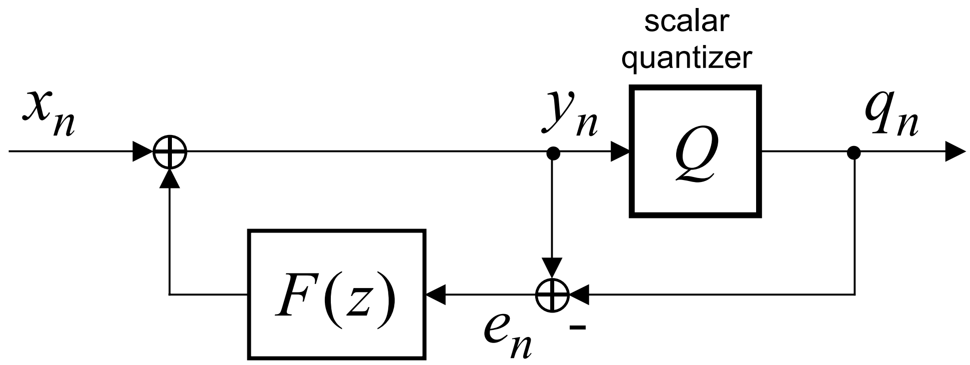
Assumption: $\|x\|_\infty \le \frac{\Delta}{2} - \varepsilon$

There exists FIR filter $G(z)$ such that $\|x\|_\infty + \|f\|_1 \cdot \frac{\Delta}{2} \le \Delta$

$$\text{length}(g_n) \approx 6m^2$$

[S.Gunturk, 2003]

DIFFERENTIATOR WITH IIR $G(z)$



$$D(z) = (1 - z^{-1})^m$$

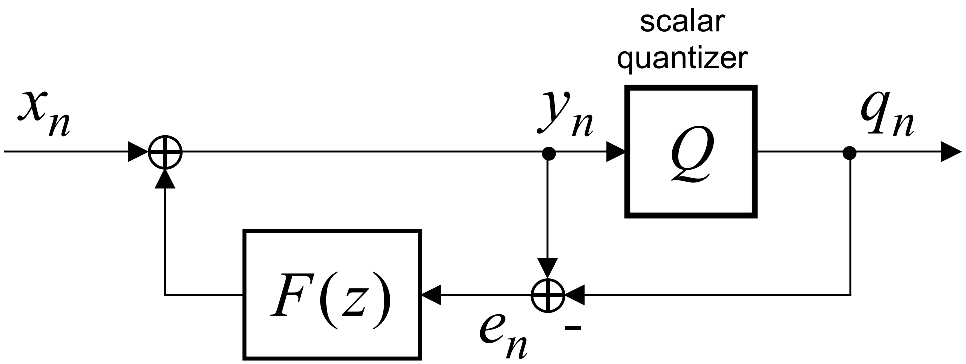
$$F(z) = 1 - G(z)\,D(z)$$

with $G(z)$ of the type $G(z) = \dfrac{1}{1 + a_1 z^{-1} + \cdots + a_m z^{-m}}$

- no existing analytical method to guarantee $\|x\|_\infty + \|f\|_1 \cdot \dfrac{\Delta}{2} \leq \Delta$
- in practice, however, most popular 1-bit scheme !
- empirical design method established by R. Schreier (1993)

that leads to (overloaded) stability and high performances

DIFFERENTIATOR WITH IIR $G(z)$: 2nd ORDER



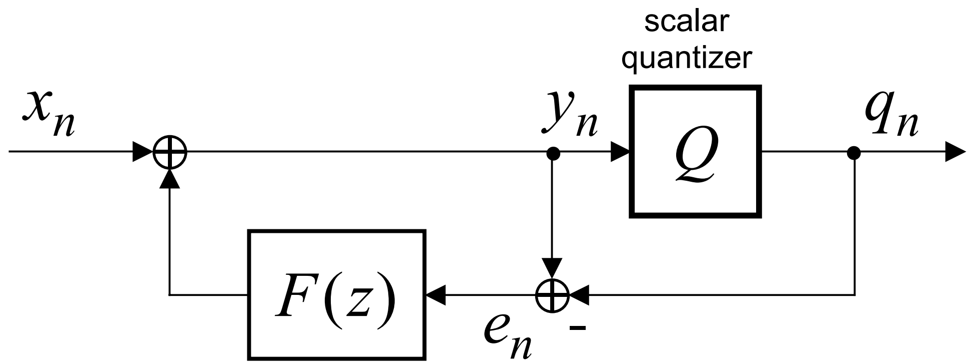
$$D(z) = (1 - z^{-1})^2$$

$$F(z) = 1 - G(z)\,D(z)$$

with $G(z)$ of the type $G(z) = \dfrac{1}{1 + a_1 z^{-1} + a_2 z^{-2}}$

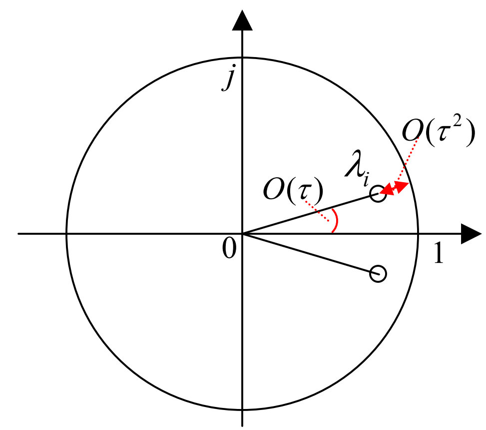
Rigorous proof of (overloaded) stability by [O.Yilmaz, 2002]

LOWPASS ΣΔ WITH IIR $G(z)$



$$D(z) = \prod_{i=1}^{m} (1 - \lambda_i z^{-1})$$

$$F(z) = 1 - G(z)\,D(z)$$



with $G(z)$ of the type $G(z) = \dfrac{1}{1 + a_1 z^{-1} + \cdots + a_m z^{-m}}$

- no existing analytical method to guarantee $\|x\|_\infty + \|f\|_1 \cdot \dfrac{\Delta}{2} \leq \Delta$
- in practice, however, most popular 1-bit scheme !
- empirical design method established by R. Schreier (1993)

that leads to (overloaded) stability and high performances

PRINCIPLES OF ΣΔ MODULATION

Choose redundancy operator $D = \{d_{n,k}\}_{n,k \in Z}$ invertible and

such that $\mathbf{r}_k := \sum_{n \neq k} d_{n,k} \boldsymbol{\varphi}_n$ are "small"

Find quantized sequence $q_n \in \{l_1, l_2, \cdots, l_N\}$ so that equation

$$x_n - q_n = \sum_{k \in Z} d_{n,k} \cdot u_k$$

yields bounded and "small" solution in u_k

Hopefully, $\mathbf{x} - \hat{\mathbf{x}} = \sum_{n \in Z} u_n \cdot \mathbf{r}_n$ will be "small"

VARIOUS APPLICATIONS

- Bandpass $\Sigma\Delta$ modulation
- Multi-channel $\Sigma\Delta$ modulation
- 2D $\Sigma\Delta$ modulation (image halftoning, time-frequency)
- Finite dimensional space $\Sigma\Delta$ modulation

BANDPASS ΣΔ MODULATION

$$D = \{d_{n-k}\}_{n,k\in Z} \quad \text{with} \quad d_n = (\delta_n - \delta_{n-1})^{(m)}$$

$$\mathbf{r}_k := \sum_{n\in Z} d_{n-k}\boldsymbol{\varphi}_n \quad \text{with} \quad \boldsymbol{\varphi}_n = \varphi(t - n\tau)$$

$$\Downarrow \qquad \varphi(t) = \cos(\Omega_0 t)\,\psi(t)$$

lowpass

$$\sum_{k\in Z} |r_k(t)| \le \tau^{m-1} \left\| \varphi^{(m)} \right\|_{L^1}$$

$$|x(t) - \hat{x}(t)| \le \|u\|_\infty \cdot \sum_{k\in Z} |r_k(t)|$$

BANDPASS ΣΔ MODULATION

$D = \{d_{n-k}\}_{n,k\in Z}$ with $d_n = \left(\delta_n - e^{j\omega_0}\delta_{n-1}\right)^{(m)} * \left(\delta_n - e^{-j\omega_0}\delta_{n-1}\right)^{(m)}$

and $\omega_0 = \Omega_0 \tau$

$$\mathbf{r}_k := \sum_{n\in Z} d_{n-k}\boldsymbol{\varphi}_n \quad \text{with} \quad \boldsymbol{\varphi}_n = \varphi(t - n\tau)$$

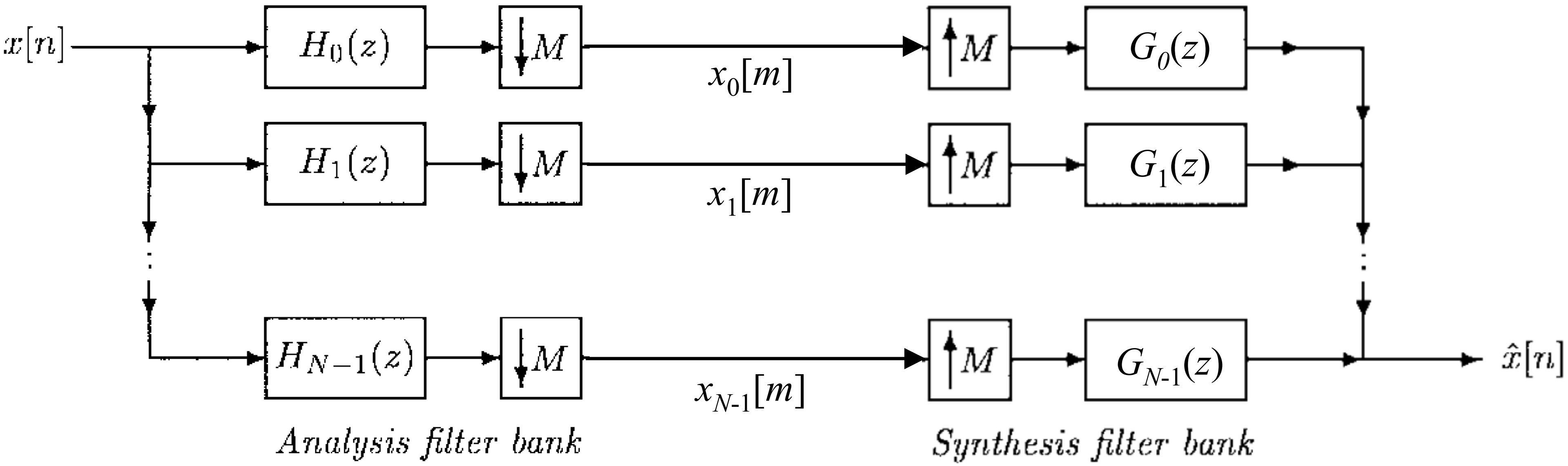
$$\Downarrow \qquad \varphi(t) = \cos(\Omega_0 t)\,\psi(t)$$

lowpass

$$\sum_{k\in Z} |r_k(t)| \le (2\tau)^{m-1} \left\|\psi^{(m)}\right\|_{L^1}$$

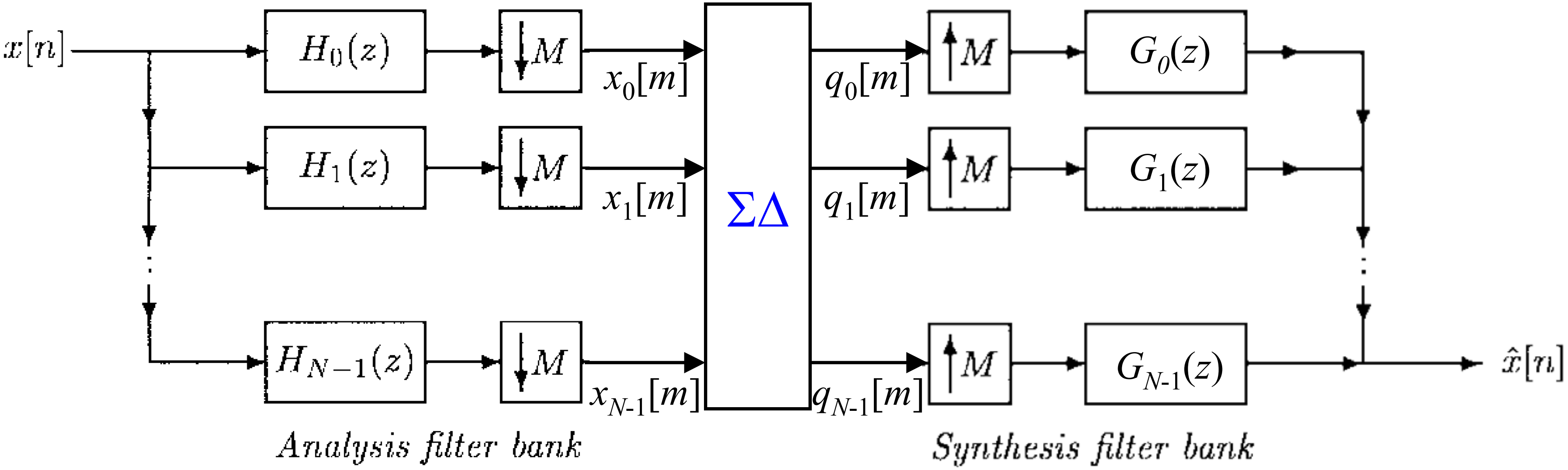
$$|x(t) - \hat{x}(t)| \le \|u\|_\infty \cdot \sum_{k\in Z} |r_k(t)|$$

MULTI-CHANNEL ΣΔ MODULATION



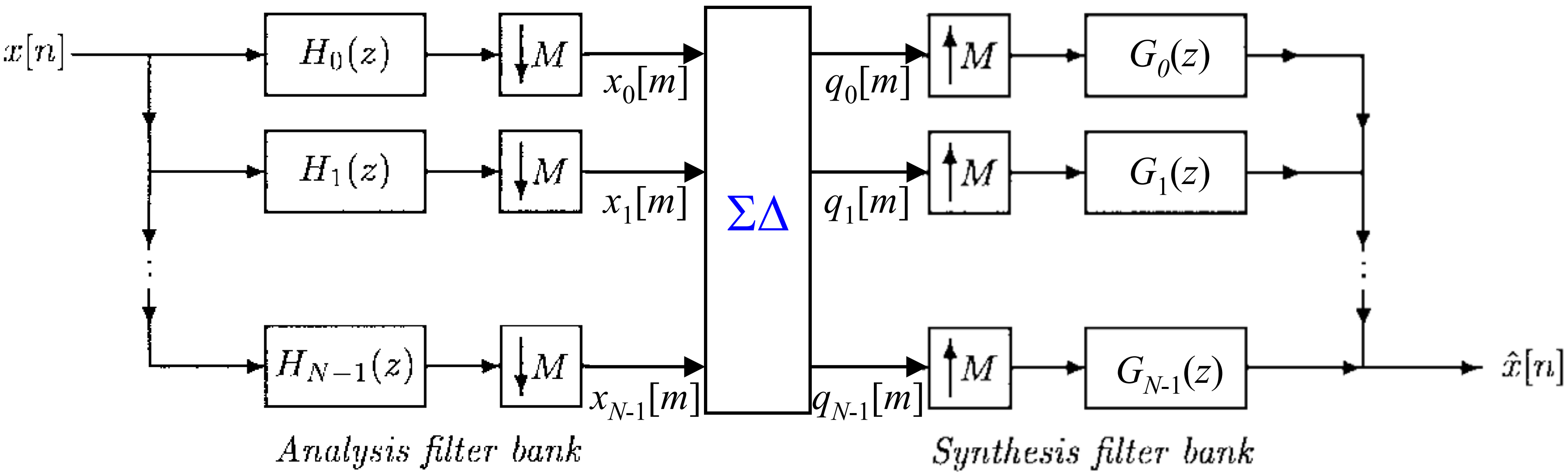
oversampling when $N>M$

MULTI-CHANNEL ΣΔ MODULATION

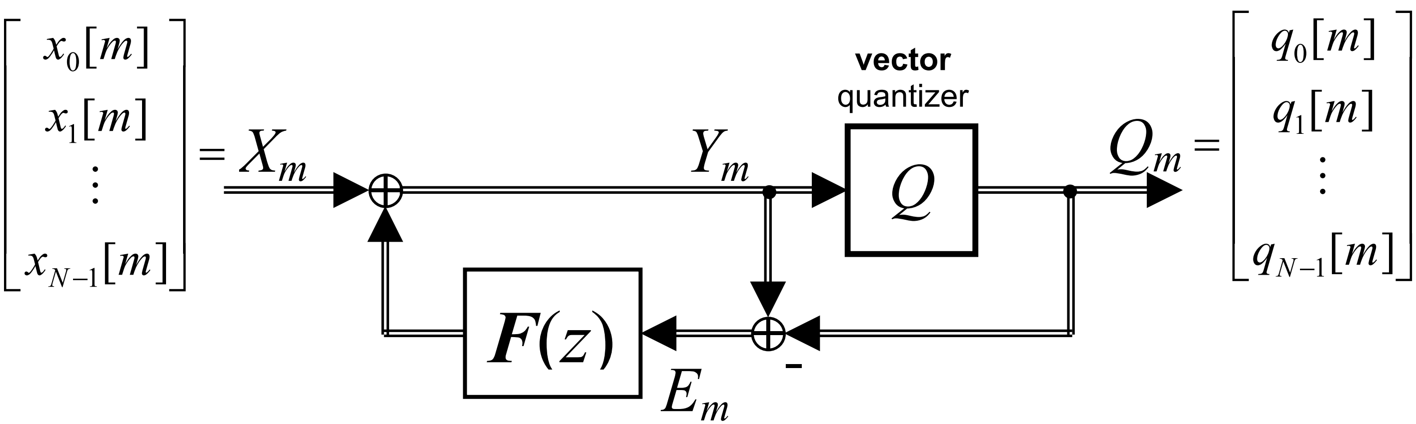


$N > M$

MULTI-CHANNEL ΣΔ MODULATION

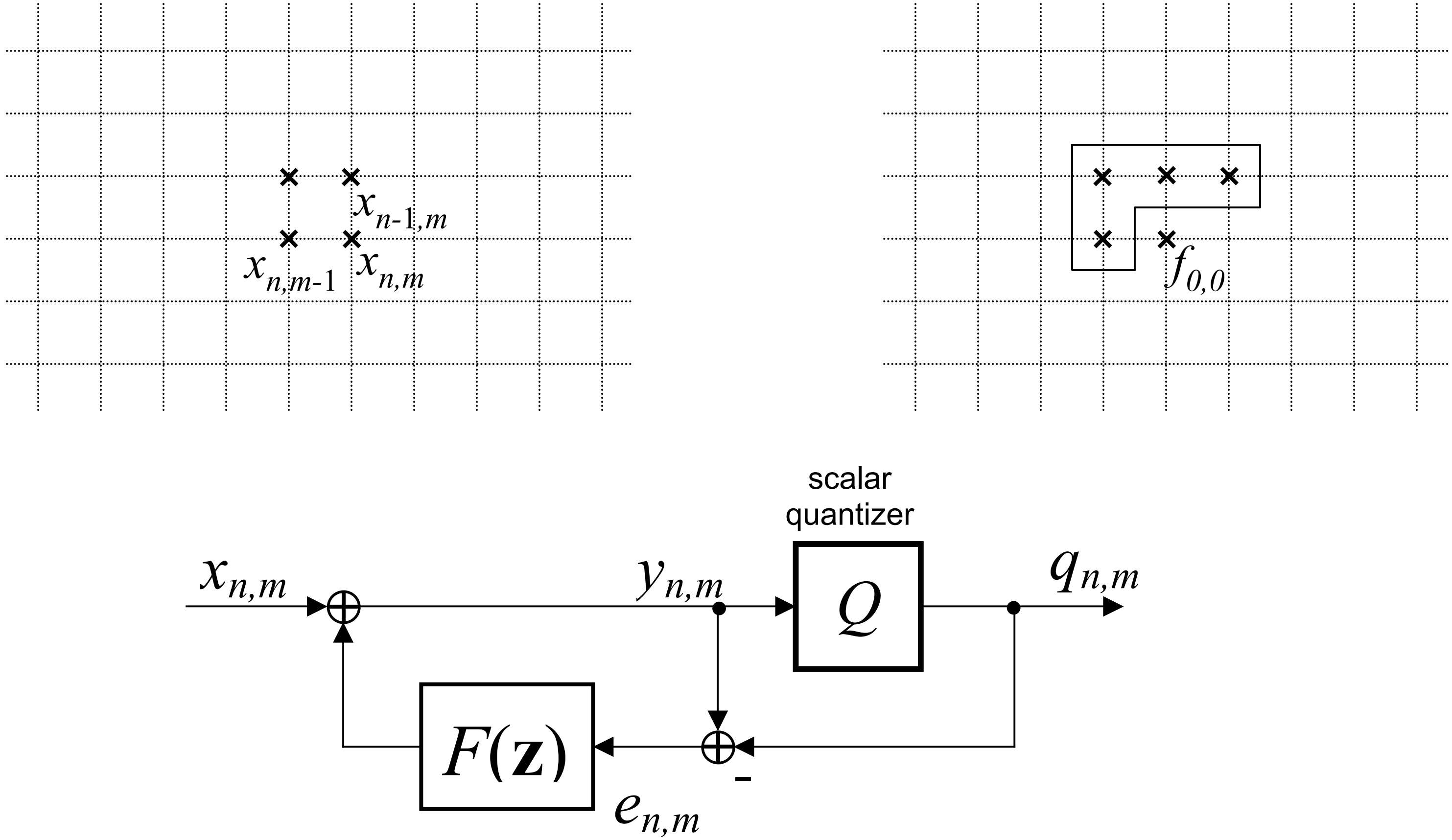


$$N > M$$



[H.Bolcskei, 2001]

MULTI-DIMENSIONAL ΣΔ MODULATION



[Image halftoning: C.W.Wu]

[Time-frequency analysis: O.Yilmax, 2004]

INFINITE DIMENSIONAL SPACE

$$\mathbf{x} = \sum_{n \in Z} x_n \cdot \boldsymbol{\varphi}_n \qquad \hat{\mathbf{x}} = \sum_{n \in Z} q_n \cdot \boldsymbol{\varphi}_n$$

$$\mathbf{r}_k := \sum_{n \in Z} d_{n-k} \cdot \boldsymbol{\varphi}_n$$

$$x_n - q_n = \sum_{k \in Z} d_{n-k} \cdot u_k$$

$$\mathbf{x} - \hat{\mathbf{x}} = \sum_{n \in Z} \left(x_n - q_n \right) \cdot \boldsymbol{\varphi}_n \qquad \Leftrightarrow \qquad \mathbf{x} - \hat{\mathbf{x}} = \sum_{k \in Z} u_k \cdot \mathbf{r}_k$$

FINITE DIMENSIONAL SPACE

$$\mathbf{x} = \sum_{n=1}^{N} x_n \cdot \boldsymbol{\varphi}_n \qquad \hat{\mathbf{x}} = \sum_{n=1}^{N} q_n \cdot \boldsymbol{\varphi}_n$$

$$\mathbf{r}_k := \sum_{n=1}^{N} d_{n-k} \cdot \boldsymbol{\varphi}_n, \quad k = 1, \cdots, N$$

$$x_n - q_n = \sum_{k=1}^{N} d_{n-k} \cdot u_k, \quad n = 1, \cdots, N$$

$$\mathbf{x} - \hat{\mathbf{x}} = \sum_{n=1}^{N} \left(x_n - q_n\right) \cdot \boldsymbol{\varphi}_n \quad \Leftrightarrow \quad \mathbf{x} - \hat{\mathbf{x}} = \sum_{n=1}^{N} u_k \cdot \mathbf{r}_k + B$$

boundary term

[J.Benedetto, O.Yilmaz & A.Powell, 2005]

PRINCIPLES OF ΣΔ MODULATION

Choose redundancy operator $D = \{d_{n,k}\}_{n,k\in Z}$ invertible and

such that $\mathbf{r}_k := \sum_{n\in Z} d_{n,k}\boldsymbol{\varphi}_n$ are “small”

Find quantized sequence $q_n \in \{l_1, l_2, \cdots, l_N\}$ so that equation

$$x_n - q_n = \sum_{k\in Z} d_{n,k} \cdot u_k$$

yields bounded and “small” solution in u_k

Hopefully, $\mathbf{x} - \hat{\mathbf{x}} = \sum_{n\in Z} u_n \cdot \mathbf{r}_n$ will be “small”