Inverse problems and deconvolution

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Agenda

- General inverse problems with noise
- Different approaches
  - SVD
  - Galerkin+Thresholding
  - Vaguelettes-wavelets
  - Wavelets-vaguelettes
- WaveVD
- Conditions, minimax, adaptation
- Deconvolution
  - Regular case
  - 'Box-car' case

Johnstone, Kerkyacharian, P, Raimondo : Wavelet Deconvolution, JRSS B (2004), ***66****, pp 1-27.*

Model :

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  - $Y(g) = (Af, g) + \varepsilon \xi(g)$
  - $g \in H, \xi(g) \sim N(0, \|g\|^2)$

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- The model reduces (putting $y_k = Y(\psi_k)$ and $f = \sum_k \theta_k\phi_k$):

$$y_k = (A\sum_l \theta_l\phi_l, \psi_k) + \varepsilon\xi(\psi_k) = b_k\theta_k + \varepsilon\xi_k, \; k \in \mathbb{N}$$

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- With standard λ_k for example :

$$\begin{aligned} \lambda_k &= \frac{1}{1 + (k/w)^\alpha} \, w > 0, \; \alpha > 0 \; (\text{Tikhonov} - \text{Phillips}) \\ \lambda_k &= (1 - (k/w)^\alpha)_+ \, w > 0, \; \alpha > 0 \; (\text{Pinsker}) \end{aligned}$$

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- *Difficulty :* L_2 *norm is necessary and the spaces of regularity are glued to the basis* ϕ_k

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- General inverse problems ✓
- Different appraoches
  - SVD ✓
  - Galerkin-thresholding
  - Vaguelettes-wavelets
  - Wavelets-vaguelettes

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\Lambda \longrightarrow \hat{\beta}_\Lambda := (\hat{\beta}_\lambda)_{\lambda \in \Lambda}, \qquad \Lambda' &\longrightarrow Y_{\Lambda'} := (y_\lambda)_{\lambda \in \Lambda'} \\
Y_{\Lambda'} &= A_{\Lambda,\Lambda'} \hat{\beta}_\Lambda
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- *Difficulty : inverse the system*

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- General Inverse problems ✓
- Different appraoches
  - SVD ✓
  - Galerkin-thresholding ✓
  - Vaguelettes-wavelets (*Abramovich, Silverman)*
  - Wavelets-vaguelettes (*Donoho)*

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- $\beta_\lambda = (f, \psi_\lambda) = \sum f_i \psi_\lambda^i$
- A (SVD) $\Longrightarrow$ there exists a b.o.n (g_i), $A^* g_i = b_i e_i,\ b_i \neq 0$ and (e_i) b.o.n.

$$Y(g_i) = (Af, g_i) + \varepsilon \xi_i = (f, A^* g_i) + \varepsilon \xi_i = b_i f_i + \varepsilon \xi_i$$

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- What kind of thresholding, conditions, resultats ?

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- $\sigma_j = 2^{j\nu}$

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- $$\begin{aligned} f &\in B_{(2\nu+1)\alpha+\frac{1}{p}-\frac{1}{2},p,\infty} \text{ et} \\ \sup_{t>0} &\ t^{-p(1-\alpha)} \sum_{jk} 2^{j[(\nu+\frac{1}{2})p-1]} I\{|\beta_{jk}| \geq 2^{j\nu} t\} < \infty. \end{aligned}$$

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These results are minimax

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- Different approaches ✓
- WaveVD ✓
- Conditions, minimax, adaptation ✓
- Deconvolution
  - Regular case
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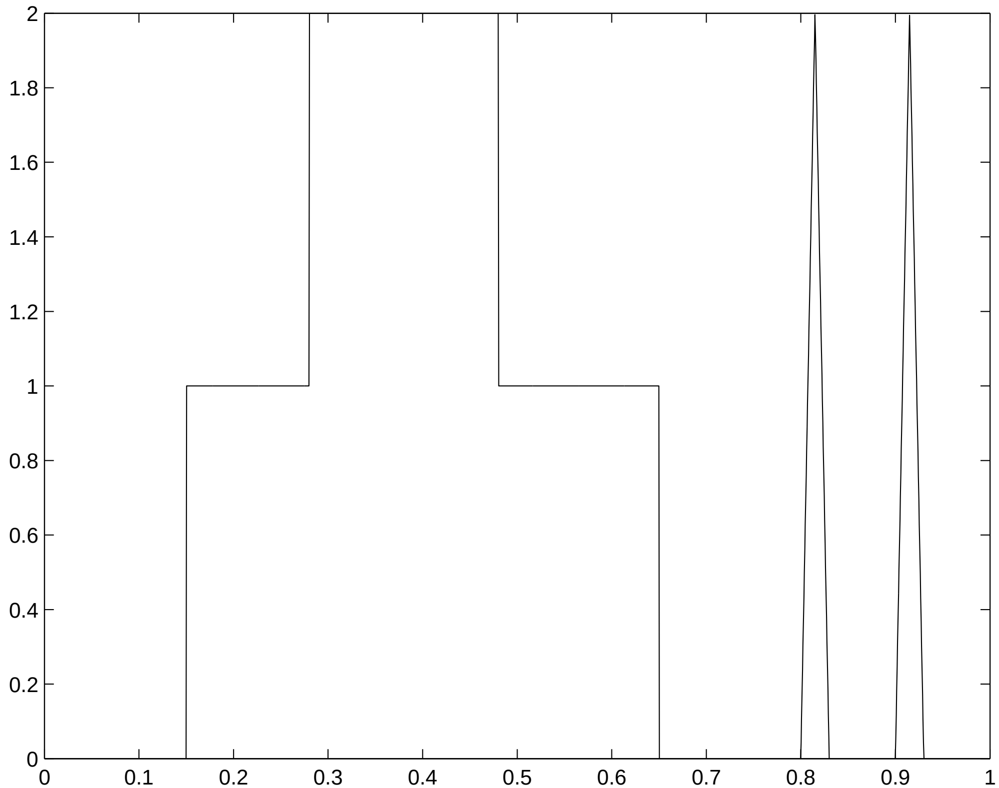

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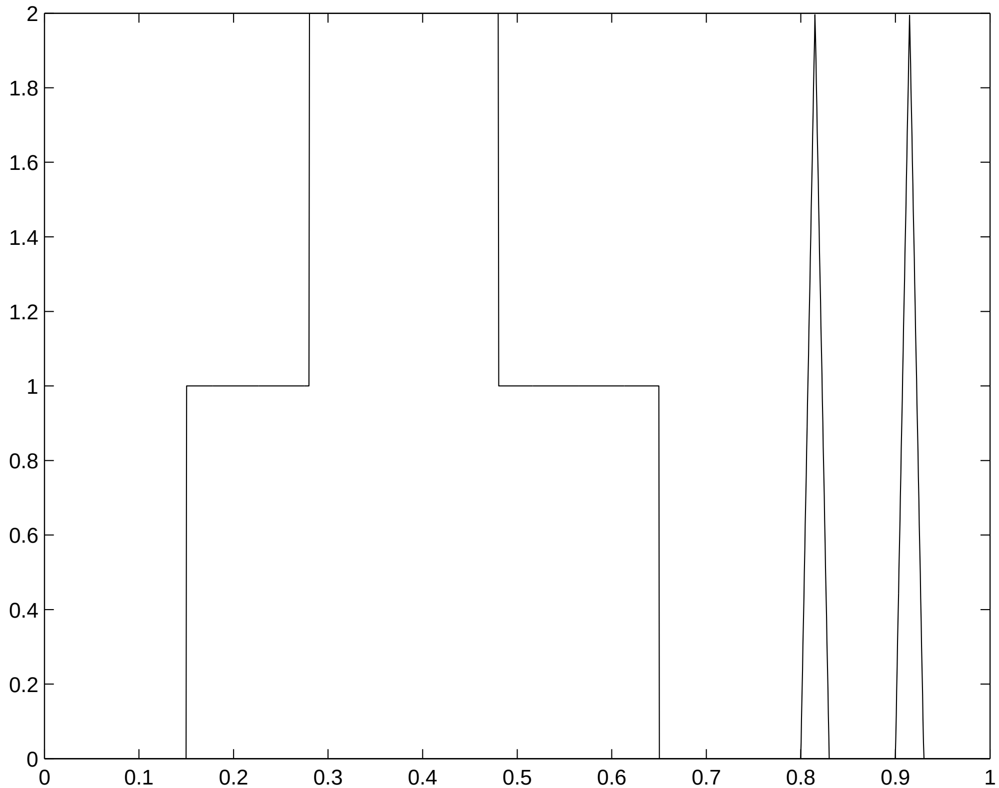

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Signal after convolution and noise

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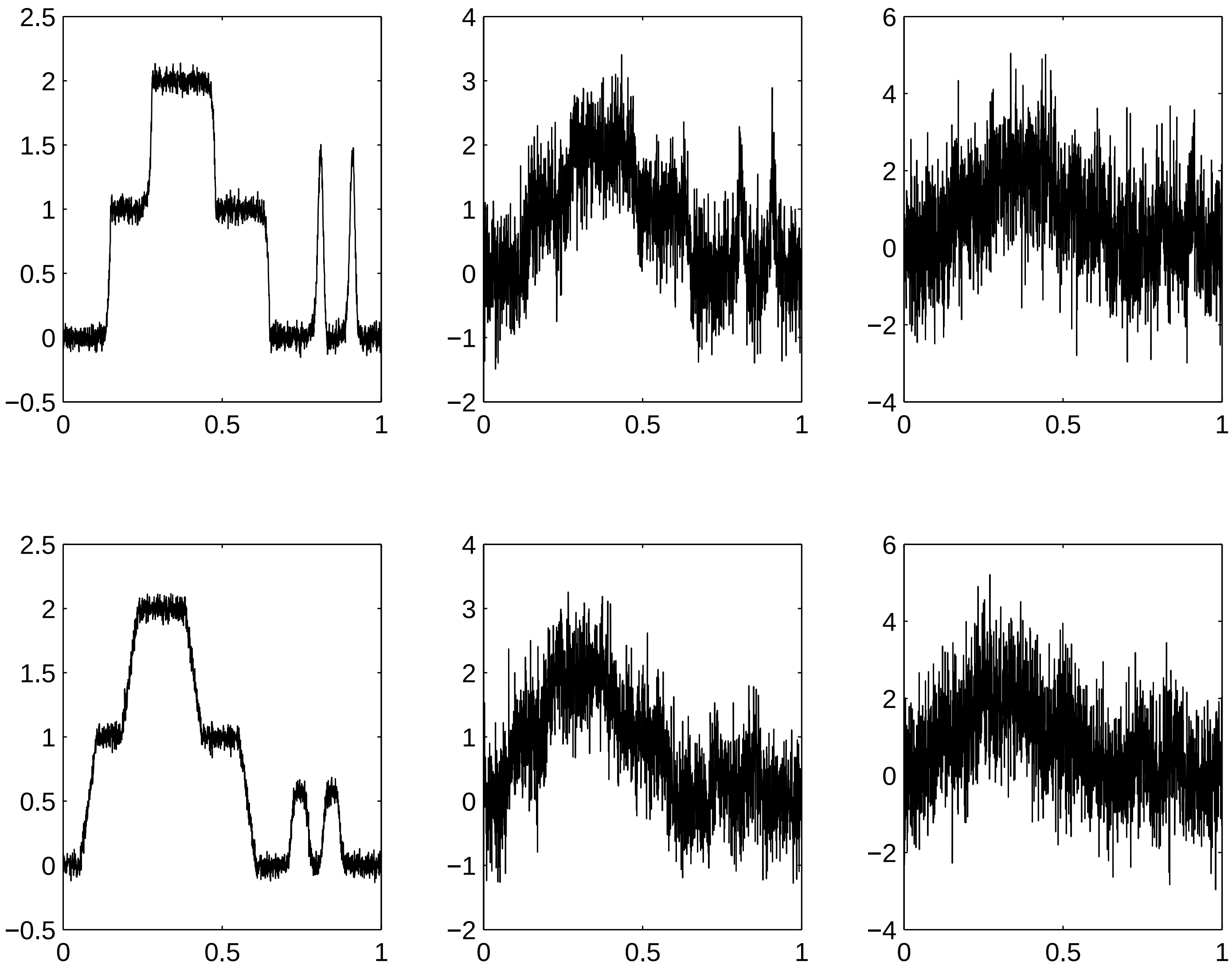

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- 3 types of noise
  - $0.05, \quad 0.25, \quad 0.5$

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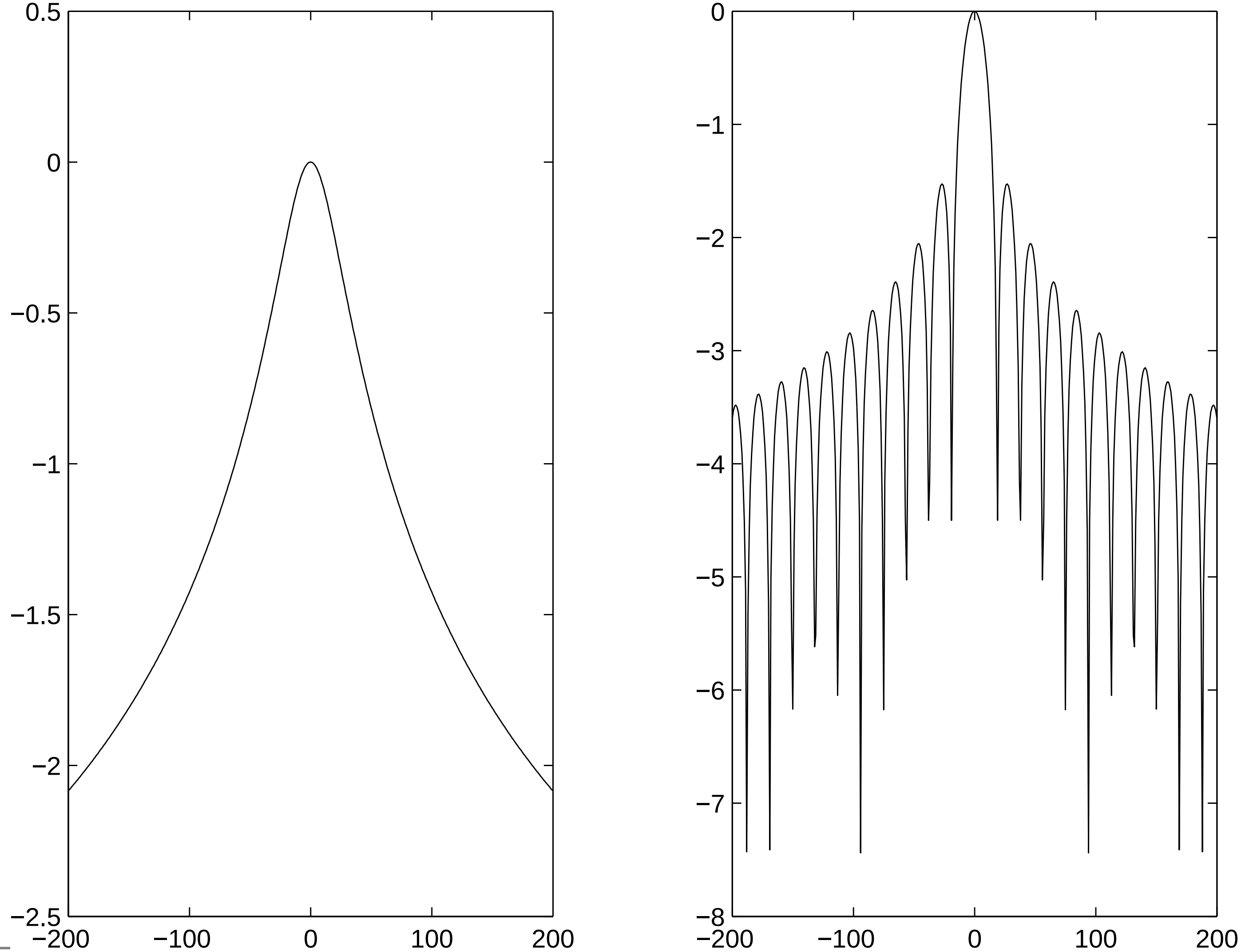
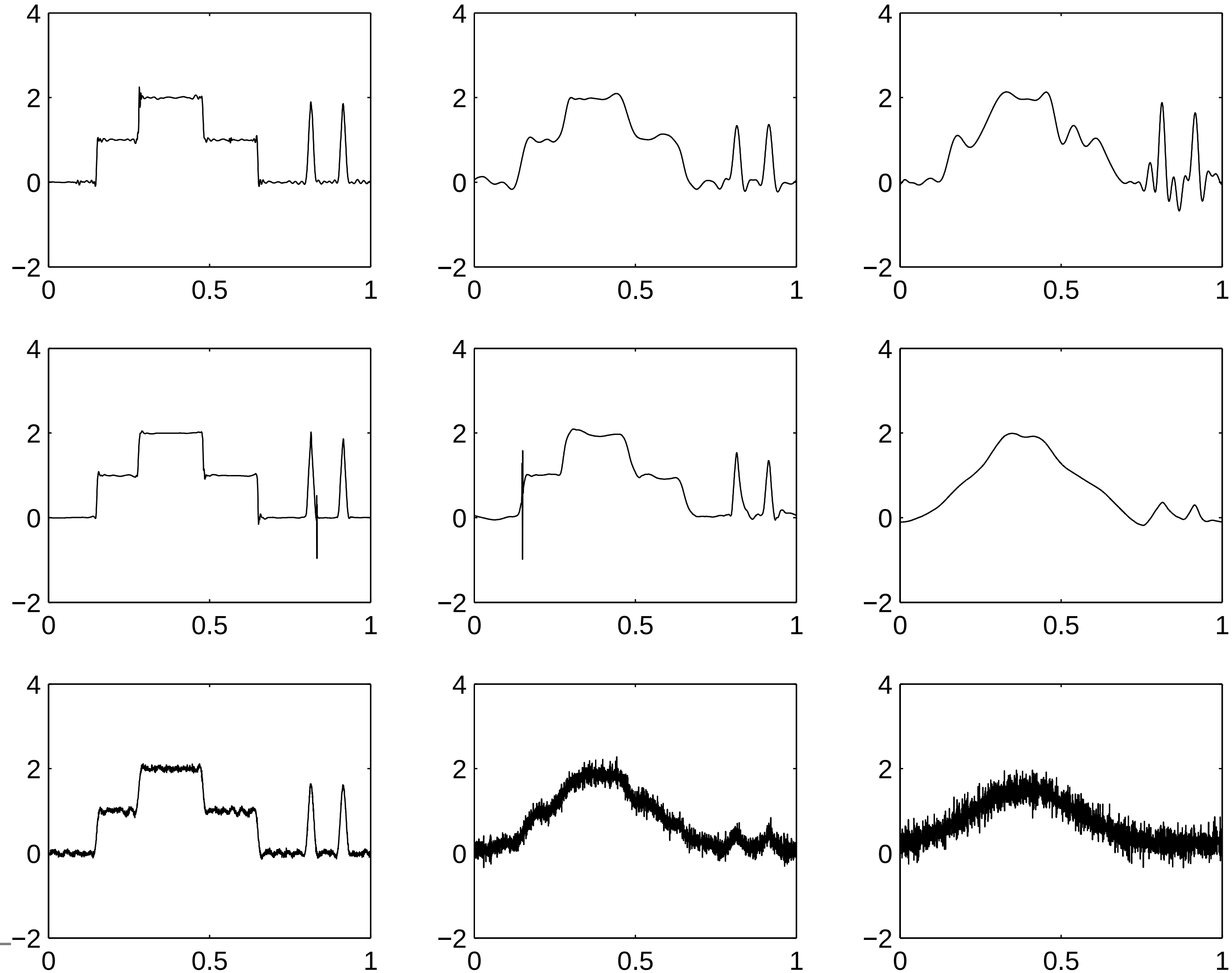
- Mais : α Badly Approximable implies $2^{-j}\sum_{i\in C_j}[\frac{1}{b_i(\alpha)}]^2 \leq 2^{2j\nu},\ \nu = 3/2$

Log-spectre



numerical results : Box-car



Algorithmes

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- Kolaczyk : algorithme de calcul pour l'ondelette de Meyer périodisée
- Donoho- Raimondo : Cycle spinning
- http://www.usyd.edu.au:8000/u/marcr/.

Diophantine approximations

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$$a = [a_0; a_1, a_2, \ldots] = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \ldots}}} \qquad (1)$$

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- Property of *best approximation* : for $n \geq 1$,

$$\inf_{1 \leq k \leq q_n} ||ka|| = |q_n a - p_n| = ||q_n a||,$$

$\|x\|$ is the distance between x and the closest integer

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- BUT : The set of BA's is of Lebesgue measure 0.

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- Same result up to a logarithmic factor.

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- *Abramovitch, Silverman*

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- Minimax et adaptation, en norme L_p.

Maxisets

$$f = \sum_{i \in \mathbb{N}} \theta_i \psi_i$$
$$\hat{f}_n = \sum_{i \le \Lambda_n} \hat{\theta}_i^{\,n} 1_{(|\hat{\theta}_i^{\,n}| \ge \sigma_i \kappa c(n))} \psi_i.$$

$$E_f^n |\hat{\theta}_i^{\,n} - \theta_i|^{2p} \le C \sigma_i^{2p} c(n)^{2p},$$

$$P_n\left(|\hat{\theta}_i^{\,n} - \theta_i| \ge \kappa \sigma_i c(n)/2 \right) \le C c(n)^{2p} \wedge c(n)^4,$$

$\Lambda_n \uparrow \infty$, $c(n) \downarrow 0$

Conditions on the basis

$\mathcal{E} = \{\psi_i;\ i \in \mathbb{N}\}$ **Unconditionnal and p-democratic**

i) $|\theta_i| \leq |\theta'_i|$ for all i, implies

$$\|\sum_i \theta_i \psi_i\|_p \leq K \|\sum_i \theta'_i \psi_i\|_p .$$

ii)

$$\begin{aligned} \frac{1}{C_1} \inf_{i \in \Lambda} |\frac{\theta_i}{\sigma_i}| (\mu_\sigma\{\Lambda\})^{1/p} &\leq \|\sum_{i \in \Lambda} \theta_i \psi_i\|_p \\ &\leq C_1 \sup_{i \in \Lambda} |\frac{\theta_i}{\sigma_i}| (\mu_\sigma\{\Lambda\})^{1/p} \end{aligned}$$

$$\boxed{\mu_\sigma\{i\} = \|\sigma_i \psi_i\|_p^p}$$