PVM finite volume methods. Application to geophysical flows.

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Outline

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   - Model problem
   - Path-conservative numerical schemes
   - Roe linearization
2. PVM methods
   - Some examples
   - Numerical tests
3. Applications
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   - Tsunamis generated by submarine landslides
4. Conclusions

Model problem

Lets consider the system

$$w_t + F(w)_x + B(w) \cdot w_x = S(w)H_x, \tag{1}$$

where

- $w(x,t)$ takes values on an open convex set $\mathcal{O} \subset \mathbb{R}^N$,
- F is a regular function from $\mathcal{O}$ to $\mathbb{R}^N$,
- B is a regular matrix function from $\mathcal{O}$ to $\mathcal{M}_{N\times N}(\mathbb{R})$,
- S is a function from $\mathcal{O}$ to $\mathbb{R}^N$, and
- H is a function from $\mathbb{R}$ to $\mathbb{R}$.

By adding to (1) the equation $H_t = 0$, the system (1) can be rewritten under the form

$$W_t + \mathcal{A}(W) \cdot W_x = 0, \tag{2}$$

where

W is the augmented vector

$$W = \begin{bmatrix} w \\ H \end{bmatrix} \in \Omega = \mathcal{O} \times \mathbb{R} \subset \mathbb{R}^{N+1}$$

and

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where

$\mathcal{A}(W)$ is the matrix whose block structure is given by:

$$\mathcal{A}(W) = \left[\begin{array}{c|c} A(w) & -S(w) \\ \hline 0 & 0 \end{array}\right],$$

where

$$A(w) = J(w) + B(w), \quad \text{being } J(w) = \frac{\partial F}{\partial w}(w).$$

Difficulties

Main difficulties

- Non conservative products $\mathcal{A}(W) \cdot W_x$. Solutions may develop discontinuities and the concept of weak solution in the sense of distributions cannot be used. The theory introduced by DLM 1995 is used here to define the weak solutions of the system. This theory allows one to give a sense to the non conservative terms of the system as Borel measures provided a prescribed family of paths in the space of states.
- Derivation of numerical schemes for non-conservative systems: Path-conservative numerical schemes (Parés 2006).
- The eigenstructure of systems like two-layer Shallow-Water system or two-phase flow model of Pitman Le are not explicitly known: PVM methods.

Path-conservative schemes

We consider here path-conservative numerical schemes in the sense defined in Parés 2006, that is, numerical schemes of the general form:

$$W_i^{n+1} = W_i^n - \frac{\Delta t}{\Delta x}\left(\mathcal{D}^+_{i-1/2} + \mathcal{D}^-_{i+1/2}\right), \tag{3}$$

where:

- Δx and Δt are, for simplicity, assumed to be constant;
- W_i^n is the approximation provided by the numerical scheme of the cell average of the exact solution at the i-th cell, $I_i = [x_{i-1/2}, x_{i+1/2}]$ at the n-th time level $t^n = n\Delta t$, and
- $$\mathcal{D}^\pm_{i+1/2} = \mathcal{D}^\pm\left(W_i^n, W_{i+1}^n\right),$$

  where $\mathcal{D}^-$ and $\mathcal{D}^+$ are two Lipschitz continuous functions from $\Omega \times \Omega$ to Ω satisfying:

  $$\mathcal{D}^\pm(W, W) = 0, \quad \forall W \in \Omega, \tag{4}$$

  and
- for every $W_L, W_R \in \Omega$,

$$\mathcal{D}^-(W_L, W_R) + \mathcal{D}^+(W_L, W_R) = \int_0^1 \mathcal{A}\big(\Phi(s; W_L, W_R)\big)\frac{\partial \Phi}{\partial s}(s; W_L, W_R)\, ds,$$

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Convergence

- In Castro, LeFloch, Muñoz and Parés, 2008 and Parés-Muñoz, 2009 has been proved that the numerical solutions provided by finite difference/volumes path-conservative numerical scheme converge to functions which solve a perturbed system in which an error source-term appear on the right hand side (which is a measure supported on the discontinuities). This problem is common to any numerical scheme that introduces numerical diffusion.
- In Muñoz-Parés, 2010 is shown that in certain situations this error vanishes for finite difference/volumes methods: this is the case of systems of balance laws.

Roe linearization I

PVM numerical schemes are defined using a generalized Roe matrix for (2) as defined by Toumi 1992:
Given a family of paths $\Phi = [\Phi_w, \Phi_H]^T$, a function $\mathcal{A}_\Phi : \Omega \times \Omega \mapsto \mathcal{M}_{(N+1)\times(N+1)}(\mathbb{R})$ is called a Roe linearization if it verifies the following properties:

- for any $W_L, W_R \in \Omega$, $\mathcal{A}_\Phi(W_L, W_R)$ has $N+1$ distinct real eigenvalues,
- for every $W \in \Omega$,
$$\mathcal{A}_\Phi(W, W) = \mathcal{A}(W); \qquad (4)$$
- for any $W_L, W_R \in \Omega$,
$$\mathcal{A}_\Phi(W_L, W_R) \cdot (W_R - W_L) = \int_0^1 \mathcal{A}(\Phi(s; W_L, W_R)) \frac{\partial \Phi}{\partial s}(s; W_L, W_R)\, ds. \qquad (5)$$

Roe-based schemes I

The following Roe linearizations $\mathcal{A}_\Phi(W_L, W_R)$ for system (1) are considered (Parés-Castro 2004):

$$\mathcal{A}_\Phi(W_L, W_R) = \left[\begin{array}{c|c} A_\Phi(w_L, w_R) & -S_\Phi(w_L, w_R) \\ \hline 0 & 0 \end{array}\right],$$

where

$$A_\Phi(w_L, w_R) = J(w_L, w_R) + B_\Phi(w_L, w_R).$$

Here, $J(w_L, w_R)$ is a Roe matrix of the Jacobian of the flux F in the usual sense:

$$J(w_L, w_R) \cdot (w_R - w_L) = F(w_R) - F(w_L);$$

$$B_\Phi(w_L, w_R) \cdot (w_R - w_L) = \int_0^1 B(\Phi_w(s; W_L, W_R)) \frac{\partial \Phi_w}{\partial s}(s; W_L, W_R)\, ds;$$

$$S_\Phi(w_L, w_R)(H_R - H_L) = \int_0^1 S(\Phi_w(s; W_L, W_R)) \frac{\partial \Phi_H}{\partial s}(s; W_L, W_R)\, ds.$$

It can be easily shown that, the resulting matrix is a Roe linearization provided it has $N + 1$ different real eigenvalues.

Roe-based schemes II

Once the Roe linearization has been chosen, a numerical scheme can be defined by

$$W_i^{n+1} = W_i - \frac{\Delta t}{\Delta x}\left(\mathcal{D}^+_{i-1/2} + \mathcal{D}^-_{i+1/2}\right),$$

where

$$\mathcal{D}^\pm_{i+1/2} = \widehat{\mathcal{A}}^\pm_\Phi(W_i^n, W_{i+1}^n)\cdot(W_{i+1}^n - W_i^n),$$

being

$$\mathcal{A}_\Phi(W_L, W_R) = \widehat{\mathcal{A}}^+_\Phi(W_L, W_R) + \widehat{\mathcal{A}}^-_\Phi(W_L, W_R)$$

is any decomposition of the Roe linearization of the form:

$$\widehat{\mathcal{A}}^\pm_\Phi(W_L, W_R) = \frac{1}{2}\left(\mathcal{A}_\Phi(W_L, W_R) \pm \mathcal{Q}_\Phi(W_L, W_R)\right),$$

where $\mathcal{Q}_\Phi(W_L, W_R)$ can be interpreted as a numerical viscosity matrix.

Roe-based schemes III

The numerical scheme in the unknowns w can be written as follows:

$$w_i^{n+1} = w_i^n - \frac{\Delta t}{\Delta x}\left(D^+_{i-1/2} + D^-_{i+1/2}\right), \tag{6}$$

being

$$\begin{aligned} D^{\pm}_{i+1/2} = \quad & \frac{1}{2}\Big(F(w_{i+1}) - F(w_i) + B_{i+1/2}(w_{i+1} - w_i) - S_{i+1/2}(H_{i+1} - H_i) \\ & \pm Q_{i+1/2}(w_{i+1} - w_i - A^{-1}_{i+1/2}S_{i+1/2}(H_{i+1} - H_i))\Big), \end{aligned} \tag{7}$$

being

- $B_{i+1/2} = B_\Phi(W_i, W_{i+1})$,
- $S_{i+1/2} = S_\Phi(W_i, W_{i+1})$,
- $A_{i+1/2} = A_\Phi(W_i, W_{i+1})$ and
- $Q_{i+1/2} = Q_\Phi(W_i, W_{i+1})$ a numerical viscosity matrix obtained from $\mathcal{Q}_\Phi(W_i, W_{i+1})$

Different numerical schemes can be obtained for different definitions of $Q_{i+1/2}$

Roe-based schemes IV

- Roe scheme corresponds to the choice

$$Q_\Phi(W_L, W_R) = |A_\Phi(W_L, W_R)|,$$

- Lax-Friedrichs scheme:

$$Q_\Phi(W_L, W_R) = \frac{\Delta x}{\Delta t} Id,$$

  being *Id* the identity matrix.

- Lax-Wendroff scheme:

$$Q_\Phi(W_L, W_R) = \frac{\Delta t}{\Delta x} A_\Phi^2(W_L, W_R),$$

- FORCE and GFORCE schemes are presented in the bibliography as a convex combination of Lax-Friedrichs and Lax-Wendroff scheme:

$$Q_\Phi(W_L, W_R) = (1-\omega)\frac{\Delta x}{\Delta t} Id + \omega \frac{\Delta t}{\Delta x} A_\Phi^2(W_L, W_R),$$

  with $\omega = 0.5$ and $\omega = \frac{1}{1+\alpha}$, respectively, being α the CFL parameter.

PVM methods I

We propose a class of finite volume methods defined by

$$Q_{i+1/2} = P_l(A_{i+1/2}),$$

being $P_l(x)$ a polinomial of degree l,

$$P_l(x) = \sum_{j=0}^{l} \alpha_j \, x^j,$$

and $A_{i+1/2} = A_\Phi(w_i, w_{i+1})$ a Roe matrix. That is, $Q_{i+1/2}$ can be seen as a Polynomial Viscosity Matrix (PVM).

See also: P. Degond, P.F. Peyrard, G. Russo, Ph. Villedieu. *Polynomial upwind schemes for hyperbolic systems.* C. R. Acad. Sci. Paris 1 328, 479-483, 1999.

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$Q_{i+1/2}$ has the same eigenvectors than $A_{i+1/2}$ and if $\lambda_{i+1/2}$ is an eigenvalue of $A_{i+1/2}$, then $P_l(\lambda_{i+1/2})$ is an eigenvalue of $Q_{i+1/2}$

PVM methods II

- If
$$\alpha\frac{\Delta x}{\Delta t} \geq P_l(\lambda_{i+1/2}) \geq |\lambda_{i+1/2}|,\ \alpha \in (0,1),\ i = 1,\cdots,N,$$
then the numerical scheme is linearly L^∞-stable. Therefore, a sufficient condition to ensure that the numerical scheme is linearly L^∞-stable is that
$$\alpha\frac{\Delta x}{\Delta t} \geq P_l(x) \geq |x| \quad \forall x \in [\lambda_{1,i+1/2}, \lambda_{N,i+1/2}]. \tag{8}$$

- Let us consider the following notation: PVM-$l(S_0,\cdots,S_k)$.
    - In practice, the parameters $S_0,\cdots,S_k$ will be related to the approximations of some wave speeds.

Upwind methods

A PVM method is said to be upwind if

$$P_l(A_\Phi) = \begin{cases} A_\Phi & \text{if } \lambda_1 > 0 \\ -A_\Phi & \text{if } \lambda_N < 0, \end{cases}$$

and it will be denoted as PVM-lU.

PVM-(N-1)U($\lambda_1, \cdots, \lambda_N$) or Roe method

$$P_{N-1}(\lambda_j) = |\lambda_j|, \quad j = 1, \cdots, N.$$

$$Q_\Phi(w_L, w_R) = |A_\Phi(w_L, w_R)| = \sum_{j=0}^{N-1} \alpha_j A_\Phi^j(w_L, w_R),$$

where $\alpha_j, j = 0, \cdots, N-1$ are the solution of the following linear system:

$$\begin{pmatrix} 1 & \lambda_1 & \dots & \lambda_1^{N-1} \\ 1 & \lambda_2 & \dots & \lambda_2^{N-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \lambda_N & \dots & \lambda_N^{N-1} \end{pmatrix} \begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_{N-1} \end{pmatrix} = \begin{pmatrix} |\lambda_1| \\ |\lambda_2| \\ \vdots \\ |\lambda_N| \end{pmatrix},$$

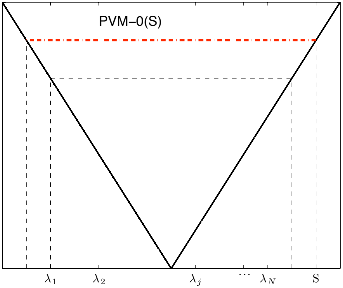
$\lambda_1, \cdots, \lambda_N$ are the eigenvalues of the matrix $A_\Phi(w_L, w_R)$.

PVM-0(S_0) methods: Rusanov, Lax-Friedrichs and modified Lax-Friedrichs schemes

$$P_0(x) = S_0,$$

$$S_0 \in \{S_{Rus}, S_{LF}, S_{LF}^{mod}\}, \quad S_{Rus} = \max_j |\lambda_{j,i+1/2}|, \; S_{LF} = \frac{\Delta x}{\Delta t} \text{ and } S_{LF}^{mod} = \alpha \frac{\Delta x}{\Delta t}.$$

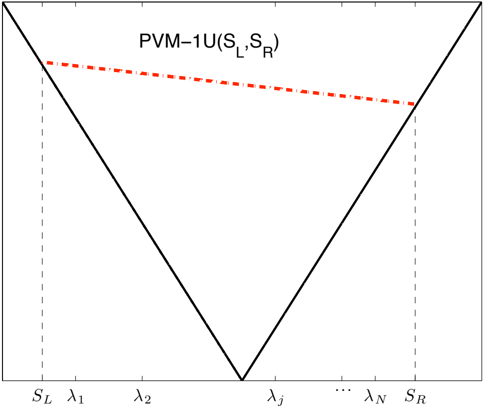
- Rusanov scheme corresponds to the choice $S_0 = S_{Rus}$,
- Lax-Friedrichs with $S_0 = S_{LF}$
- modified Lax-Friedrichs with $S_0 = S_{LF}^{mod}$.



PVM-1U(S_L, S_R) or HLL method

$$P_1(x) = \alpha_0 + \alpha_1\, x \quad \text{such as } P_1(S_L) = |S_L|,\; P_1(S_R) = |S_R|,$$

where S_L (respectively S_R) is an approximation of the minimum (respectively maximum) wave speed.



PVM-1U(S_L, S_R) or HLL method

Remarks

- The usual HLL scheme coincides with PVM-1U(S_L, S_R) in the case of conservative systems.

Let us suppose that the system is conservative. Then, the conservative flux associated to PVM-1U(S_L,S_R) is $\phi_{i+1/2} = D^-_{i+1/2} + F(w_i)$. Taking into account that

$$\alpha_0 = \frac{S_R|S_L| - S_L|S_R|}{S_R - S_L}, \quad \alpha_1 = \frac{|S_R| - |S_L|}{S_R - S_L},$$

then

$$\begin{aligned}
\phi_{i+1/2} &= \frac{F(w_i)(S_R + |S_R| - S_L - |S_L|) + F(w_{i+1})(S_R - |S_R| - S_L + |S_L|)}{2S_R - 2S_L} \\
&\quad - \frac{(S_R|S_L| - S_L|S_R|)(w_{i+1} - w_i)}{2S_R - 2S_L} \\
&= \frac{S_R^+ F(w_i) - S_L^- F(w_{i+1}) + (S_R^+ S_L^-)(w_{i+1} - w_i)}{S_R^+ - S_L^-}
\end{aligned}$$

which is a compact definition of the HLL flux, being $S_R^+ = \max(S_R, 0)$ and $S_L^- = \min(S_L, 0)$.

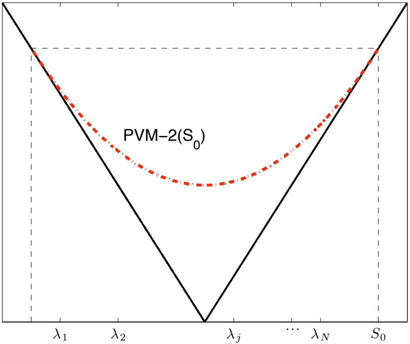
PVM-2(S_0) methods or FORCE type methods

$$P_2(x) = \alpha_0 + \alpha_2 x^2, \text{ such as } P_2(S_0) = S_0,\ P_2'(S_0) = 1,\ S_0 \in \{S_{Rus}, S_{LF}, S_{LF}^{mod}\}.$$

Remarks

- If $S_0 = S_{LF}$ then we obtain FORCE method.
- GFORCE scheme can be obtained by imposing

$$P_2(S_{LF}^{mod}) = S_{LF}^{mod}, \quad P_2'(S_{LF}^{mod}) = \frac{2\alpha}{1+\alpha},$$



PVM-2U(S_M, S_m) method

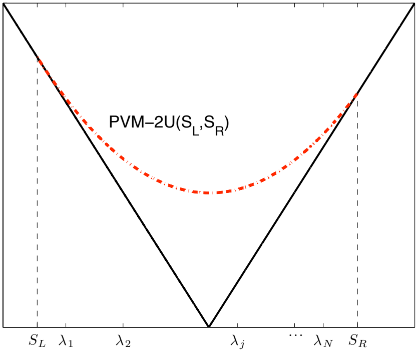
$$P_2(x) = \alpha_0 + \alpha_1 x + \alpha_2 x^2,$$

such as

$$P_2(S_m) = |S_m|,\ P_2(S_M) = |S_M|,\ P_2'(S_M) = \mathrm{sgn}(S_M),$$

where

$$S_M = \begin{cases} \lambda_{1,i+1/2} & \text{if } |\lambda_{1,i+1/2}| \geq |\lambda_{N,i+1/2}|, \\ \lambda_{N,i+1/2} & \text{if } |\lambda_{1,i+1/2}| < |\lambda_{N,i+1/2}|. \end{cases} \quad S_m = \begin{cases} \lambda_{N,i+1/2} & \text{if } |\lambda_{1,i+1/2}| \geq |\lambda_{N,i+1/2}| \\ \lambda_{1,i+1/2} & \text{if } |\lambda_{1,i+1/2}| < |\lambda_{N,i+1/2}| \end{cases}$$



PVM-2U(S_L, S_R, S_{int}) method or CFP method

$$P_2(x) = \alpha_0 + \alpha_1 x + \alpha_2 x^2,$$

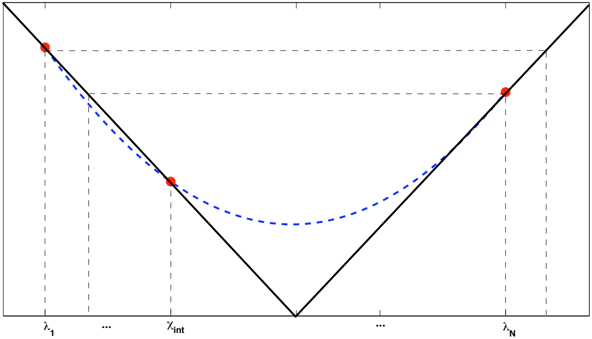
$$\begin{pmatrix} 1 & S_L & S_L^2 \\ 1 & S_R & S_R^2 \\ 1 & S_{int} & S_{int}^2 \end{pmatrix} \begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} |S_L| \\ |S_R| \\ |S_{int}| \end{pmatrix},$$

S_L (respectively S_R) is an approximation of the minimum (respectively maximum) wave speed and

$$S_{int} = \mathcal{S}_{ext} \max(|\lambda_{2,i+1/2}|, \ldots, |\lambda_{N-1,i+1/2}|),$$

$$\mathcal{S}_{ext} = \begin{cases} \mathrm{sgn}(S_L + S_R), & \text{if } (S_L + S_R) \neq 0, \\ 1, & \text{otherwise.} \end{cases}$$

PVM-2U(S_L, S_R, S_{int}) method or CFP method

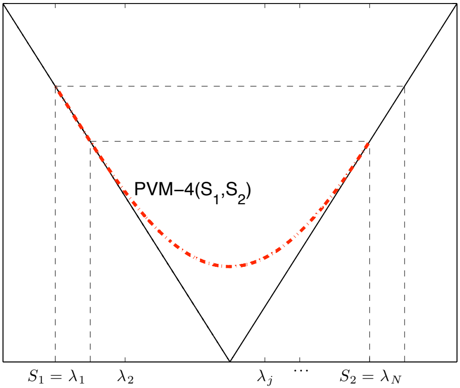


PVM-4(S_M, S_I) and PVM-4(S_0) methods

$$P_4(x) = \alpha_0 + \alpha_2 x^2 + \alpha_4 x^4,$$

$$P_4(S_M) = |S_M|, \quad P_4(S_I) = S_I, \quad P_4'(S_I) = 1,$$

$$S_I = \begin{cases} \max\limits_{2\leq j\leq N} (|\lambda_{j,i+1/2}|) & \text{if } |\lambda_{1,i+1/2}| \geq |\lambda_{N,i+1/2}|, \\ \max\limits_{1\leq j\leq (N-1)} (|\lambda_{j,i+1/2}|) & \text{if } |\lambda_{1,i+1/2}| < |\lambda_{N,i+1/2}|. \end{cases}$$

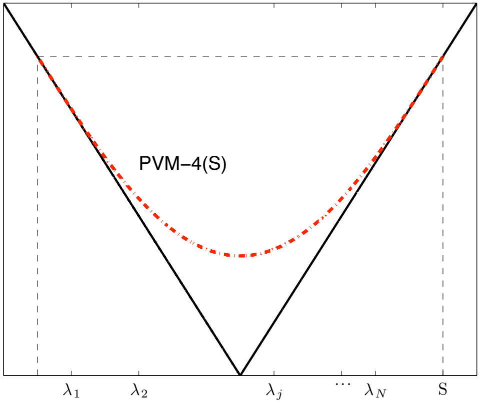


PVM-4(S_M, S_I) and PVM-4(S_0) methods

$$P_4(x) = \alpha_0 + \alpha_2 x^2 + \alpha_4 x^4,$$

$$P_4(S_M) = |S_M|, \quad P_4(S_I) = S_I, \quad P_4'(S_I) = 1,$$

PVM-4(S_0): $S_I = S_M = S_0$.



Extension to high order and/or higher dimensions

Extension to high order and/or higher dimensions

- M. Castro, J.M. Gallardo and C. Parés.
  High order finite volume schemes based on reconstruction of states for solving hyperbolic systems with nonconservative products. Applications to shallow water systems. Math. Comp. 75: 1103-1134, 2006.
- M. Castro, J.M. Gallardo, J.A. López and C. Parés.
  Well-balanced high order extensions of Godunov's method for semilinear balance laws. SIAM J. Num. Anal., 46(2): 1012-1039, 2008.
- M. Castro, E.D. Fernández, A. Ferreiro, J.A. García and C. Parés.
  High order extensions of Roe schemes for two dimensional nonconservative hyperbolic systems. J. Sci. Comput., 39: 67-114, 2009.

High performance implementation

CPU implementation

M. Castro, J.A. García, J.M. González and C. Parés.
A parallel 2d finite volume scheme for solving systems of balance laws with nonconservative products: application to shallow flows. Comp. Meth. Appl. Mech. Eng. 196, 2788-2815, 2006.

M. Castro, J.A. García, J.M. González and C. Parés.
Solving shallow-water systems in 2D domains using finite volume methods and multimedia SSE instructions. J. Comput. App. Math., 221: 16-32, 2008.

GPU implementation

M. Lastra, J. M. Mantas, C. Ureña, M. J. Castro, J. A. García-Rodríguez.
Simulation of shallow-water systems using graphics processing units. Math. Comput. Simul. 80, 598618, 2009.

M. de la Asunción, J. M. Mantas, M. J. Castro.
Simulation of one-layer shallow water systems on multicore and CUDA architectures. J. Supercomput., 2009, (DOI: 10.1007/s11227-010-0406-2).

Two-fluid flow model of Pitman and Le

$$\begin{cases} \dfrac{\partial h_f}{\partial t} + \dfrac{\partial q_f}{\partial x} = 0, \\[2ex] \dfrac{\partial q_f}{\partial t} + \dfrac{\partial}{\partial x}\left(\dfrac{q_f^2}{h_f} + \dfrac{g}{2}h_f^2\right) + gh_f\dfrac{\partial h_s}{\partial x} = -gh_f\dfrac{db}{dx}, \\[2ex] \dfrac{\partial h_s}{\partial t} + \dfrac{\partial q_s}{\partial x} = 0, \\[2ex] \dfrac{\partial q_s}{\partial t} + \dfrac{\partial}{\partial x}\left(\dfrac{q_s^2}{h_s} + \dfrac{g}{2}h_s^2 + g\dfrac{1-r}{2}h_sh_f\right) + rgh_s\dfrac{\partial h_f}{\partial x} = -gh_s\dfrac{db}{dx}. \end{cases}$$

- index s (f respectively) makes reference to the solid (fluid respectively) phase.
- $b(x)$ represents the fixed bottom topography,
- r is the ratio of densities between the solid and fluid phase.
- The unknowns h_s and h_f are related to the total height of the granular fluid h and the solid fraction ψ by

$$h_s = \psi h, \quad \text{and} \quad h_f = (1-\psi)h.$$

- The unknowns q_s and q_f represent the mass-flow of each phase and are related with the velocities of each phase by $q_s = u_s h_s$ and $q_f = u_f h_f$.

Numerical example

Let us consider a rectangular channel in the domain $[-0.9, 1.0]$ with topography

$$b(x) = \begin{cases} 0.25(\cos(\pi(x-0.5)/0.1)+1) & \text{if } |x-0.5| < 0.1, \\ 0 & \text{otherwise.} \end{cases}$$

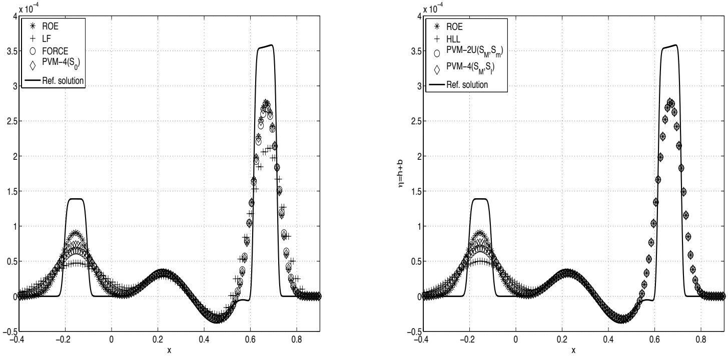
As initial condition we set $u_s(x,0) = u_f(x,0) = 0$ and

$$h(x,0) = \begin{cases} 1 + 10^{-3} & \text{if } -0.6 < x < -0.5 \\ 1 - b(x) & \text{otherwise,} \end{cases}$$

$$\psi(x,0) = \begin{cases} 0.6 - 10^{-3} & \text{if } -0.6 < x < -0.5 \\ 0.6 & \text{otherwise.} \end{cases}$$

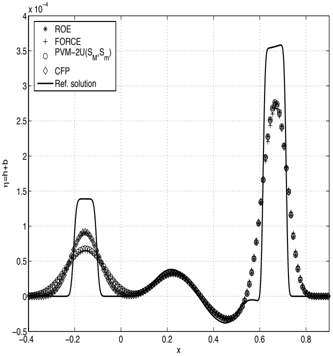
- Free boundary conditions are set,
- $T = 1.25$,
- $\Delta x = 0.01$,
- CFL=0.9,
- first order aproximation of the eigenvalues are used,
- a reference solution computed with Roe scheme for $\Delta x = 1/3200$.

Free surface $\eta = h + b$.



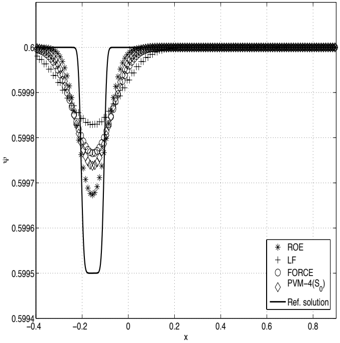
(a) PVM-0,2,4(S_0) (b) PVM-1U,2U,4U(S_M,S_I)

Free surface $\eta = h + b$.

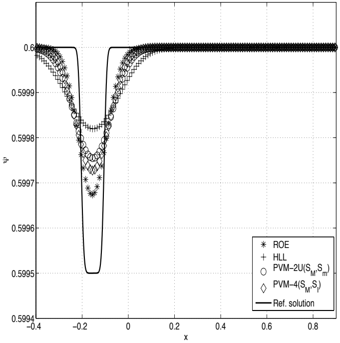


(c) CFP

Solid volume fraction ψ.

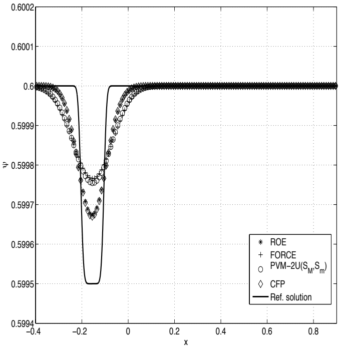


(d) PVM-0,2,4(S_0)



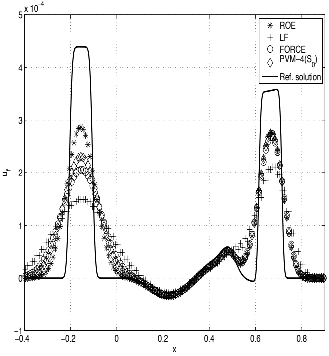
(e) PVM-1U,2U,4U(S_M,S_I)

Solid volume fraction ψ.

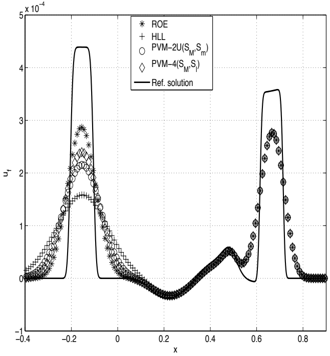


(f) CFP

Phase velocity u_f.

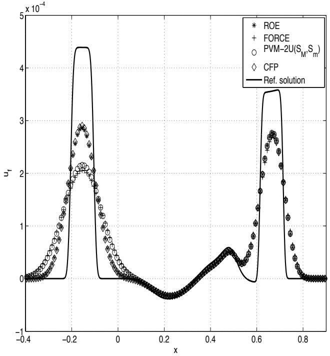


(g) PVM-0,2,4(S_0)



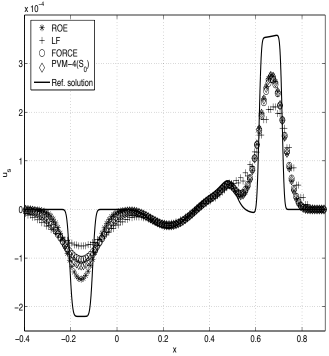
(h) PVM-1U,2U,4U(S_M,S_I)

Phase velocity u_f.

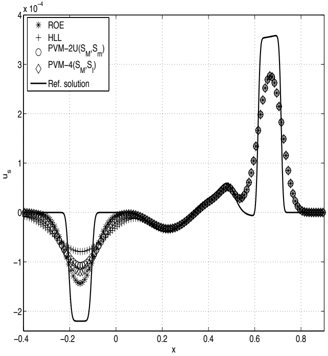


(i) CFP

Phase velocity u_s.

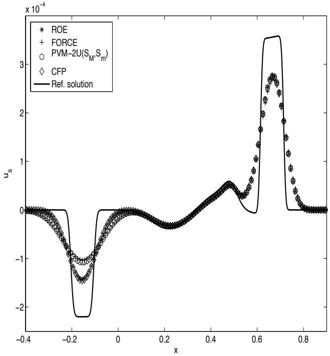


(j) PVM-0,2,4(S_0)



(k) PVM-1U,2U,4U(S_M,S_I)

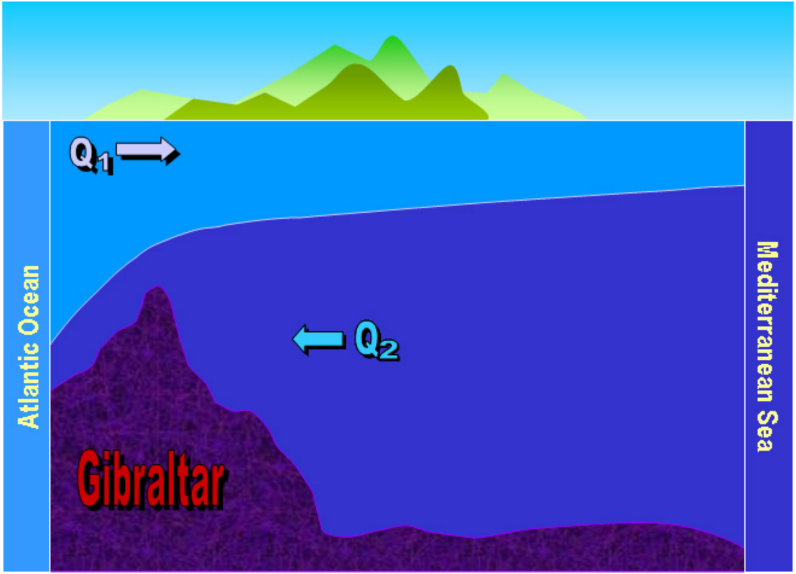
Phase velocity u_s.



(l) CFP

Tidal forcing at the Strait of Gibraltar

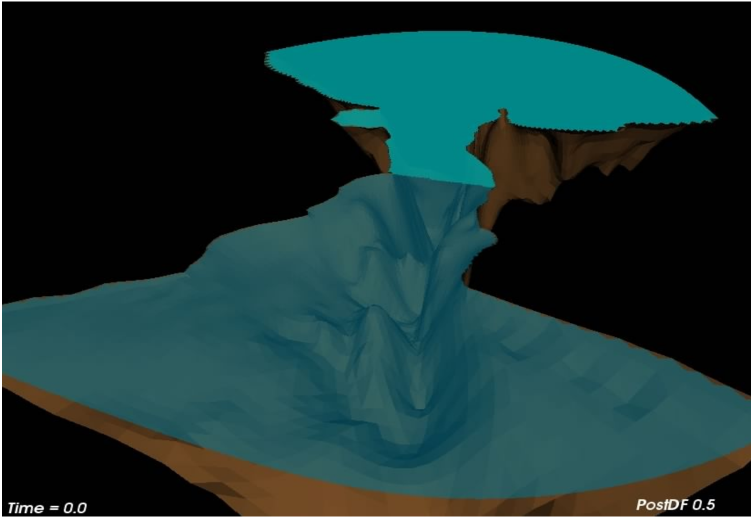
Tidal forcing at the Strait of Gibraltar



Lock-Exchange Experiment

- The final stationary state represents the secular exchange.
- Maximal flow independent of the computational domain (approx. 0.85 Sv.)

Lock-Exchange Experiment II



Tidal Experiment

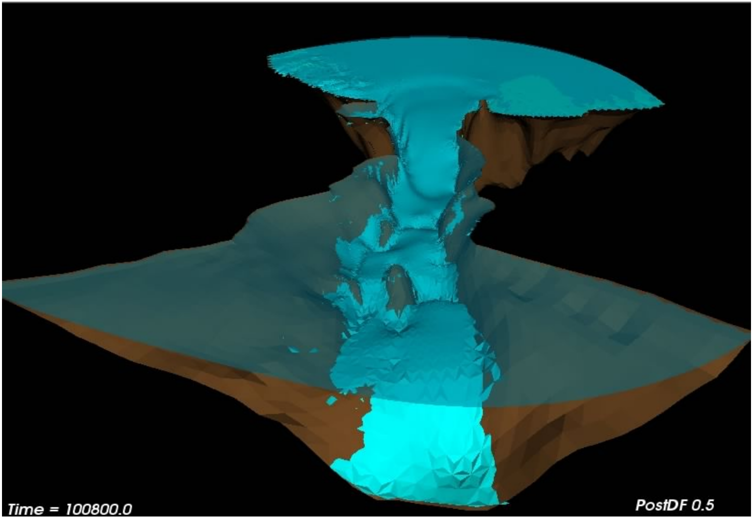
The model is forced at the open boundaries with boundary conditions that simulate the four main tidal components to be considered (M2, S2, O1 and K1):

$$h_1(x_B,t) + h_2(x_B,t) = \bar{h}_B + \sum_{n=1}^{4} Z_n(x_B) cos(\alpha_n t - \phi_n(x_B)).$$

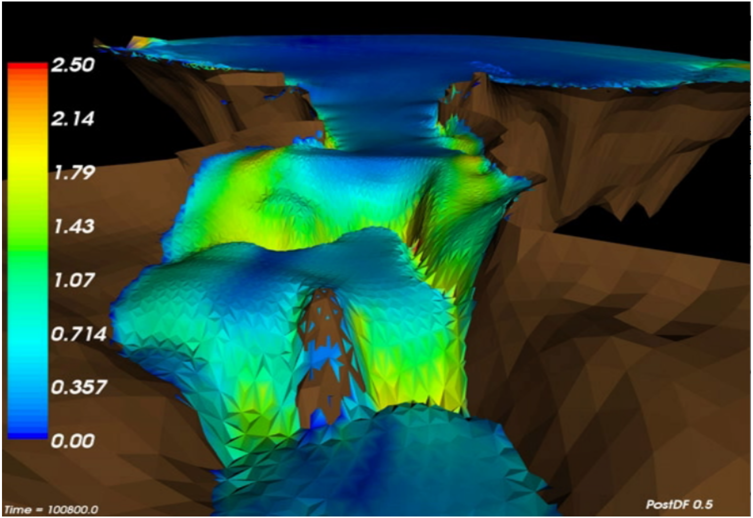
- x_B represents a point of the open boundaries;
- $Z_n(x_B)$ and $\phi_n(x_B)$ are the prescribed surface elevation amplitudes and phases of the n-th tidal constituent at the boundary sections;
- α_n its frequency;
- $\bar{h}_B$ the total depth of the water column corresponding to the steady state solution at this boundary.

Tidal data from FES2004 (Lyard F., Lefevre F., Letellier T., Francis O., 2006, *Modelling the global ocean tides: modern insights from FES2004*, Ocean Dynamics).

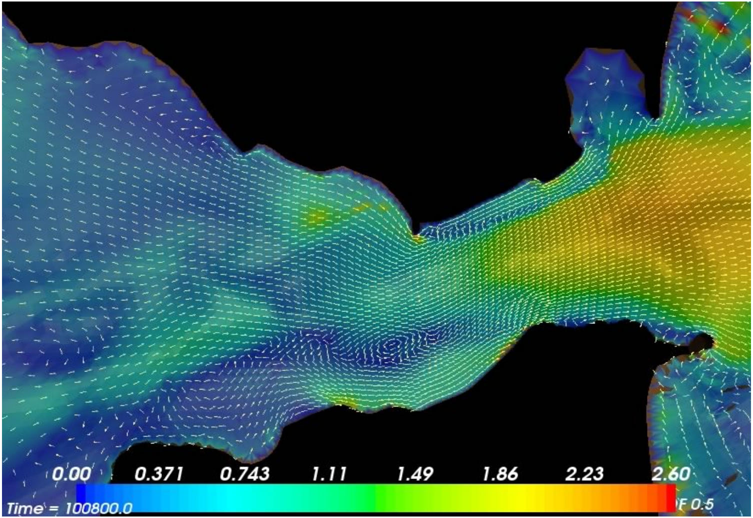
Tidal Experiment Animations I



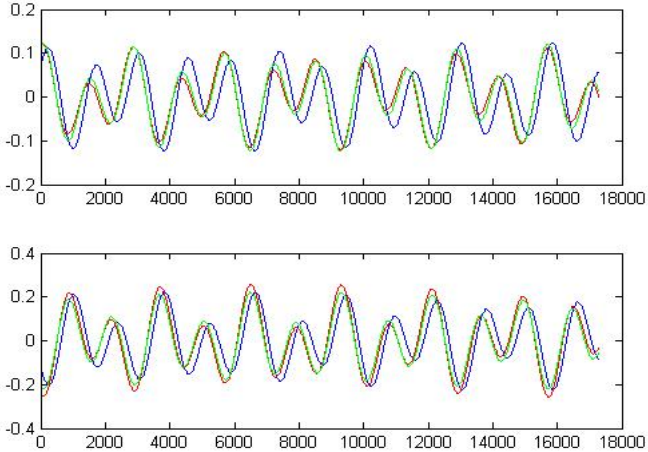
Tidal Experiment Animations II



Tidal Experiment Animations III



Subinertial signals at Camarinal sill



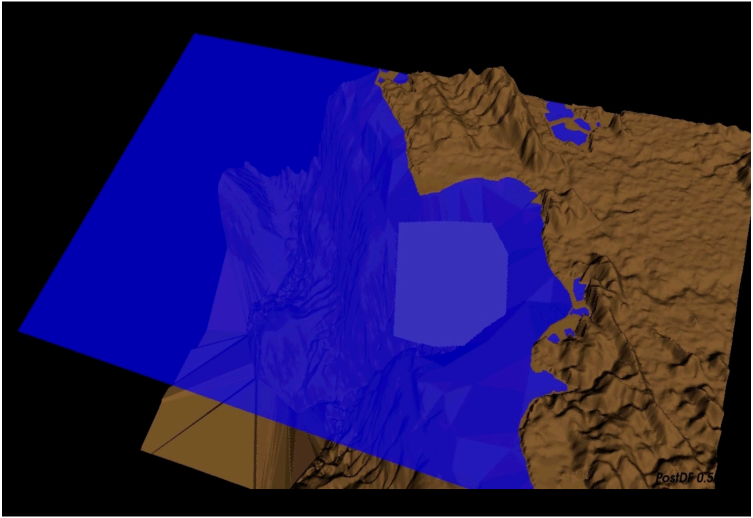
An aerial photograph of the zone

2D Two-layer Savage-Hutter shallow-water model

E. Fernández Nieto, F. Bouchut, D. Bresch, M.J. Castro, A. Mangeney. *A new Savage-Hutter type model for submarine avalanches and generated tsunami*. J. Comp. Phys., 227: 7720-7754, 2008.

F. Bouchut, M. Westdickenberg. *Gravity driven shallow water models for arbitrary topography*. Comm. in Math. Sci. 2: 359-389, 2004

Tsunamis generated by submarine landslides



Conclusions

Conclusions

- PVM methods are defined in terms of viscosity matrices computed by a suitable evaluation of Roe matrices.
- They only need some information about the eigenvalues of the system to be defined, and no spectral decomposition of Roe matrices is needed.
- They are faster than Roe method.
- They include upwind and centered schemes such as: Lax-Friedrichs, Rusanov, HLL, FORCE or GFORCE method.
- Some new solvers are also proposed.
- Their extension to high order or/and 2D problems is straight forward.
- Application to real problems have been performed