Development of a Coastal Inundation Model using a Triangular Discontinuous Galerkin Method

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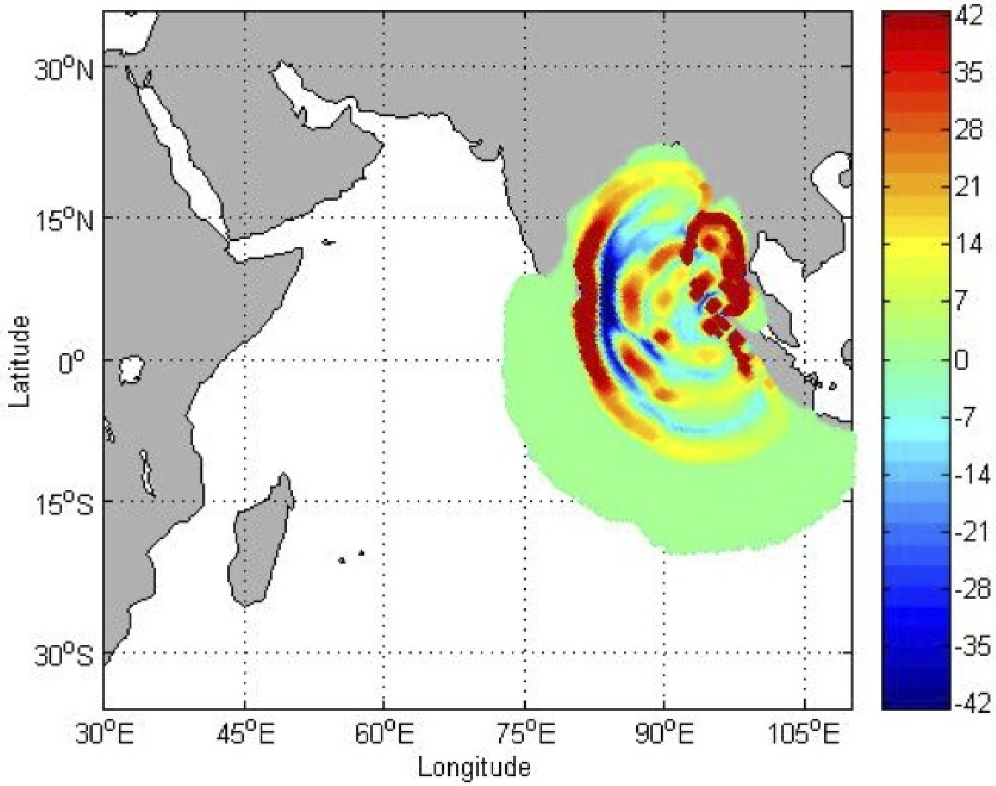
Motivation

Numerical modeling of storm surges and tsunamis



Alevras 2009

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Outline

- Discontinuous Galerkin Method applied to SWE
- Coastal Ocean Modeling
- Adaptive Mesh Refinement

Shallow Water Equations

$$\frac{\partial q}{\partial t} + \nabla \cdot F(q) = S(q), \textit{ where } \; q = (\phi, U)^T$$

$$F(q) = \begin{pmatrix} U \\ \frac{U \otimes U}{\phi} + \frac{1}{2}\left(\phi^2 - \phi_b^2\right) I_2 \end{pmatrix}$$

$$S(q) = -\begin{pmatrix} 0 \\ f(k \times U) - \phi_s \nabla \phi_b - \frac{\tau}{\rho H} + \gamma U \end{pmatrix}$$

where,

$$\phi = g(h_s + h_b)$$

$$U = \phi \bar{u}$$

h_s- free surface height, h_b- bathymetry
$g-$ gravitational acceleration
$f = f_0 + \beta(y - y_m) -$ Coriolis parameter
$\tau-$ wind stress, $\gamma-$ bottom friction

Discontinuous Galerkin Method

The domain Ω is decomposed into N_e conforming elements.

$$\Omega = \bigcup_{e=1}^{N_e} \Omega_e$$

For the operators, a non−singular mapping

$$x = \Psi\left(\xi\right)$$

transforms the physical coordinate system $X = \left(x, y\right)^T$ to local reference coordinate system $\xi = \left(\xi, \eta\right)^T$.

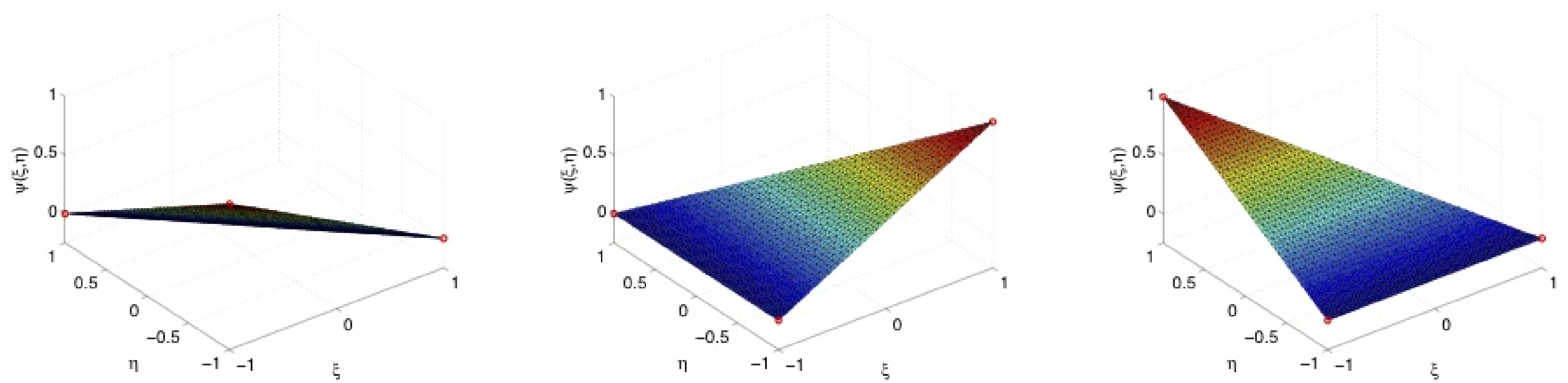
Discontinuous Galerkin Method

The local elementwise solution is approximated by Nth order polynomial in ξ by

$$q_N(\xi) = \sum_{i=1}^{M_n} \psi_i(\xi)\, q_N(\xi_i)$$

where $M_n = \frac{1}{2}(N+1)(N+2)$ is the number of interpolation points and $\psi_i(\xi)$ the associated Lagrange polynomials.

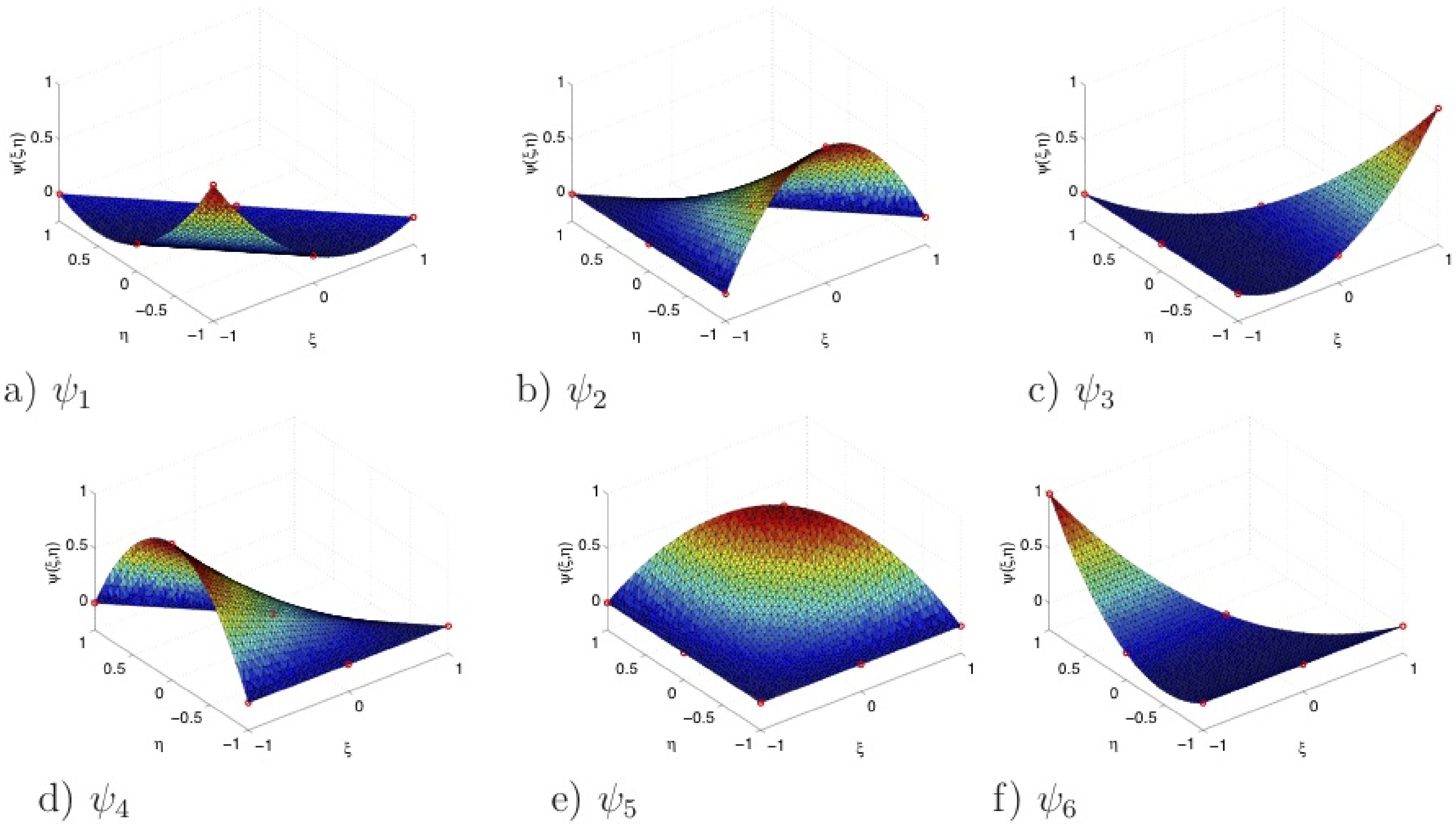
Triangular basis functions



a) ψ_1 b) ψ_2 c) ψ_3

Triangular basis functions of order N=1 at 3 interpolation points.

Triangular basis functions

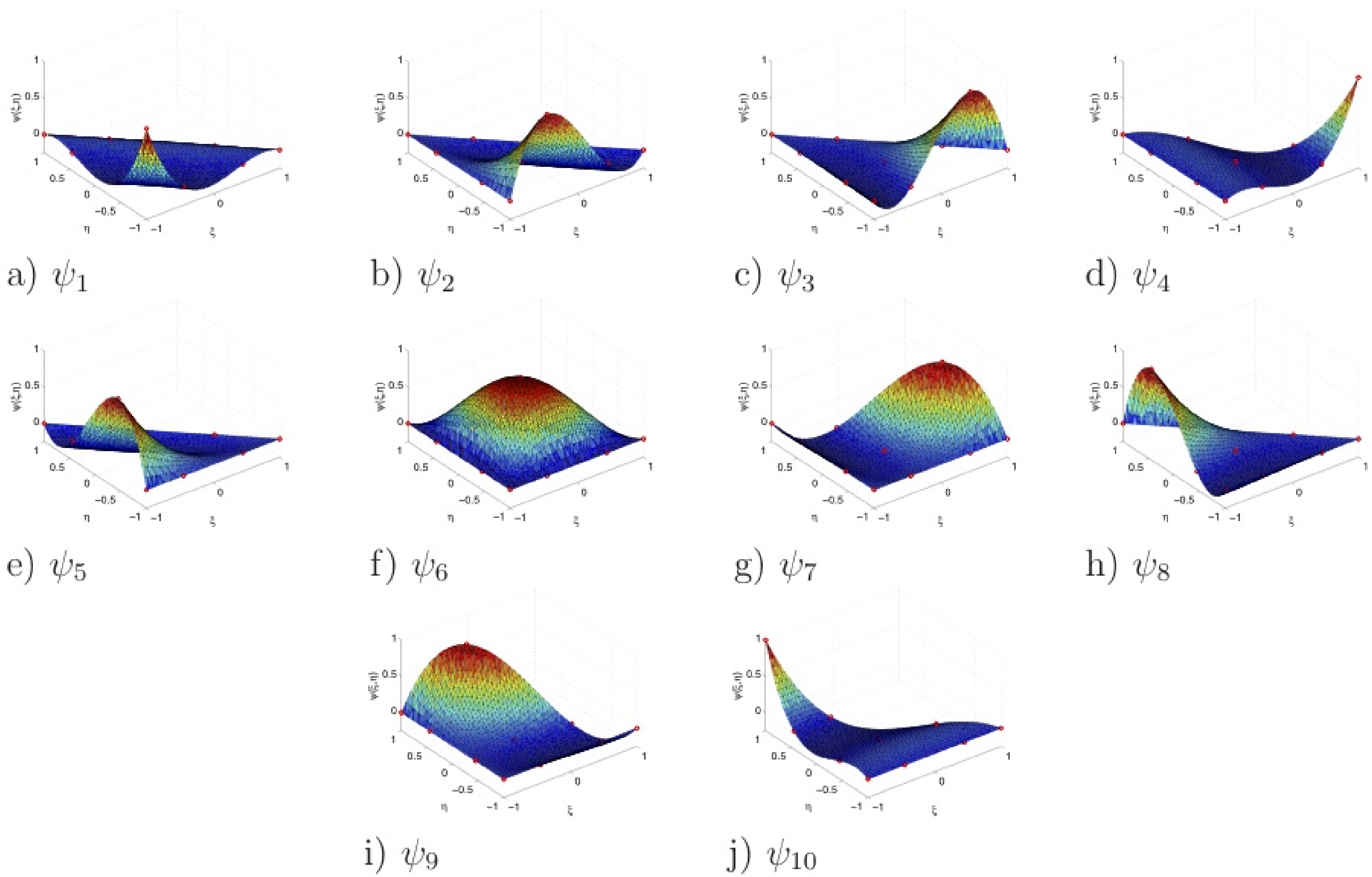


a) ψ_1 b) ψ_2 c) ψ_3

d) ψ_4 e) ψ_5 f) ψ_6

Triangular basis functions of order N=2 at 6 interpolation points.

Triangular basis functions



Triangular basis functions of order N=3 at 10 interpolation points.

Discontinuous Galerkin Method

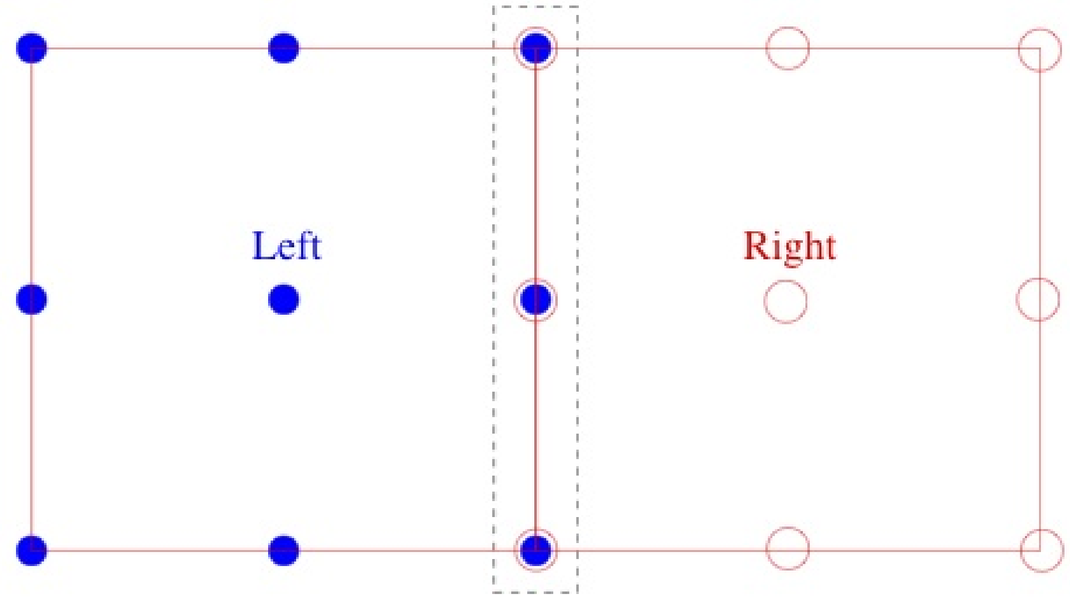
Applying DG to the Shallow water equations to obtain the weak form

$$\int_{\Omega_e} \left(\frac{\partial q_N^{(e)}}{\partial t} - F_N^{(e)} \cdot \nabla - S_N^{(e)} \right) \psi_i (x) \, dx$$
$$= - \sum_{l=1}^{3} \int_{\Gamma_e} \psi_i (x) \, n^{(e,l)} \cdot F_N^{(*,l)} dx$$

Integrating the above equation by parts,

$$\int_{\Omega_e} \psi_i (x) \left(\frac{\partial q_N^{(e)}}{\partial t} + \nabla \cdot F_N^{(e)} - S_N^{(e)} \right) dx$$
$$= \sum_{l=1}^{3} \int_{\Gamma_e} \psi_i (x) \, n^{(e,l)} \cdot \left(F_N^{(e)} - F_N^{(*,l)} \right) dx$$

Discontinuous Galerkin Method



Rusanov Numerical Flux

$$F_N^{(*,l)} = \frac{1}{2}\left[F_N\left(q_N^{(e)}\right) + F_N\left(q_N^{(l)}\right) - |\lambda^{(l)}|\left(q_N^{(l)} - q_N^{(e)}\right) n^{(e,l)}\right]$$

Where,

$$\lambda^{(l)} = max\left(|U^{(e)}| + \sqrt{\phi^{(e)}}, |U^{(l)}| + \sqrt{\phi^{(l)}}\right)$$

with,

$$U^{(e,l)} = u^{(e,l)} \cdot n^{(l)}$$

Matrix form of semi−discrete equations

Using the polynomial approximation $q_N = \sum_{i=1}^{M_N} \psi_i q_i$

$$\int_{\Omega_e} \psi_i \psi_j dx \frac{\partial q_j^{(e)}}{\partial t} + \int_{\Omega_e} \psi_i \nabla \psi_j dx \cdot F_j^{(e)} - \int_{\Omega_e} \psi_i \psi_j dx S_j^{(e)}$$
$$= \sum_{l=1}^{3} \int_{\Gamma_e} \psi_i \psi_j n^{(e,l)} dx \cdot \left(F^{(e)} - F^{(*,l)} \right)_j$$

Defining element matrices as

$$M_{ij}^{(e)} = \int_{\Omega_e} \psi_i \psi_j dx, \; M_{ij}^{(e,l)} = \int_{\Gamma_e} \psi_i \psi_j n^{(e,l)} dx, \; D_{ij}^{(e)} = \int_{\Omega_e} \psi_i \nabla \psi_j dx$$

Matrix form of semi−discrete equations

Eliminating mass matrix on LHS

$$\hat{D}^{(e)} = \left(M^{(e)}\right)^{-1} D^{(e)}, \hat{M}^{(e,l)} = \left(M^{(e)}\right)^{-1} M^{(e,l)}$$

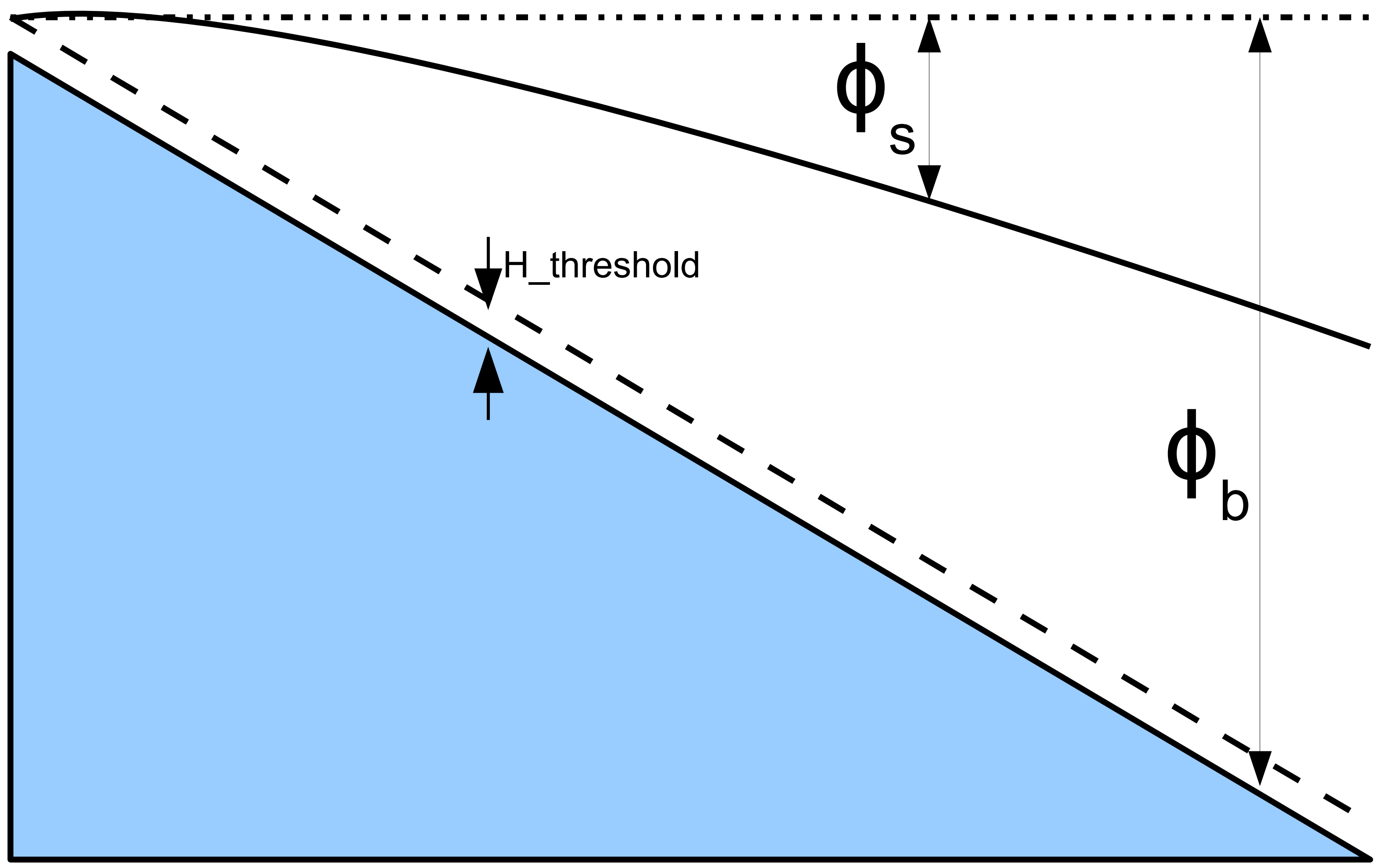
$$\frac{\partial q_i^{(e)}}{\partial t} + \left(\hat{D}_{ij}^{(e)}\right)^T F_j^{(e)} - S_i^{(e)} = \sum_{l=1}^{3} \left(\hat{M}_{ij}^{(e,l)}\right)^T \left(F^{(e)} - F^{(*,l)}\right)_j$$

Inundation Modeling

Coastal ocean modeling

- The shoreline is represented as a moving boundary condition where $\phi = \phi_s + \phi_b = 0$
- Moving front is described as $x = x_b + \int v_b dt$.
- Where x_b is initial position and v the velocity of the front.
- Approaches used to model the wetting and drying of land.
  - Fixed grid methods. – Easier to implement. Additional algorithms required to maintain depth positivity.
  - Moving grid methods. – traditionally perceived as cumbersome. (Lynch and Gray 1978)

Wetting and Drying Algorithm



Wetting and Drying Algorithm – based on Gourgue et al 2009

Conservation of Mass

$$\frac{\partial \phi_s}{\partial t} = -F(U)$$

where $\phi = \phi_s + \phi_b$, and operator $F(U) \equiv F(\phi_s, \bar{u})$

- Step 1 – limit ϕ to a threshold value.

$$\phi_s^* = max\left(\phi_s^n, H_{threshold} - \phi_b\right)$$

- Step 2

$$\frac{\phi_s^{**}}{\Delta t} = -F(\phi_s^*, \bar{u})$$

- Step 3 – ensure free surface does not move to dry areas.

$$\frac{\phi_s^{n+1}}{\Delta t} = -F^*(\phi_s^*, \bar{u})$$

Wetting and Drying Algorithm – based on Gourgue et al 2009

Conservation of Mass in matrix form

$$\frac{\partial \phi_i^{**(e)}}{\partial t} = -\left(\hat{D}_{ij}^{(e)}\right)^T F_j^{(e)} + \sum_{l=1}^{3} \left(\hat{M}_{ij}^{(e,l)}\right)^T \left(F^{(e)} - F^{(*,l)}\right)_j$$

Let,

$$F_j^s\left(\phi_s^*, \bar{u}\right) = -\left(\hat{D}_{ij}^{(e)}\right)^T F_j^{(e)}$$

$$F_j^c\left(\phi_s^*, \bar{u}\right) = \sum_{l=1}^{3} \left(\hat{M}_{ij}^{(e,l)}\right)^T \left(F^{(e)} - F^{(*,l)}\right)_j$$

Wetting and Drying Algorithm – based on Gourgue et al 2009

$$\frac{\phi_s^{n+1}}{\Delta t} = F_j^{c*}\left(\phi_s^*, \bar{u}\right) + F_j^{s*}\left(\phi_s^*, \bar{u}\right)$$

Where,

$$F_j^{c*} = \begin{cases} 0 \ if \ F_c^j < 0 \ \& \ \phi^n < H_{threshold} \\ F_c^j \ otherwise \end{cases}$$

$$F_j^{s*} = \begin{cases} 0 \ if \ there \ is \ a \ node \ i \ \in \ \Omega_e \ with \ F_s^j < 0 \ \& \ \phi < H_{threshold} \\ F_s^j \ otherwise \end{cases}$$

*limited to using linear elements.

Steady state test

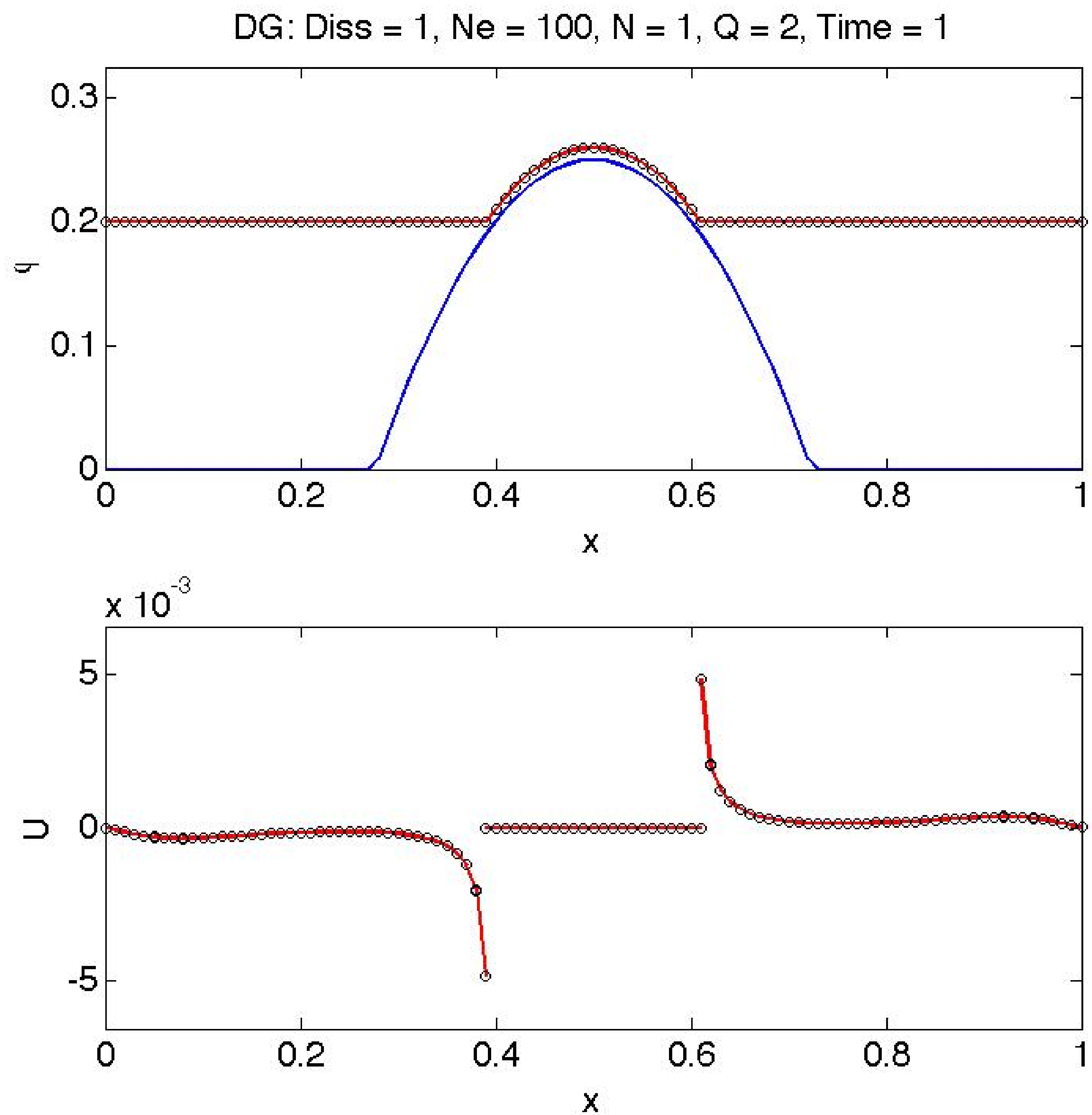
Is the model well balanced ?
Bottom topography is defined as,

$$h_b(x) = max\left(0, 0.25 - 5\left(x - 0.5^2\right)\right), 0 \leq x \;\; eq \; 1$$

Initial condition

$$h_s + h_b = max\,(0.2, b)$$
$$\phi U = 0 \quad \text{over entire domain}$$

Steady state test case



Balzano [1998] Test Case 1

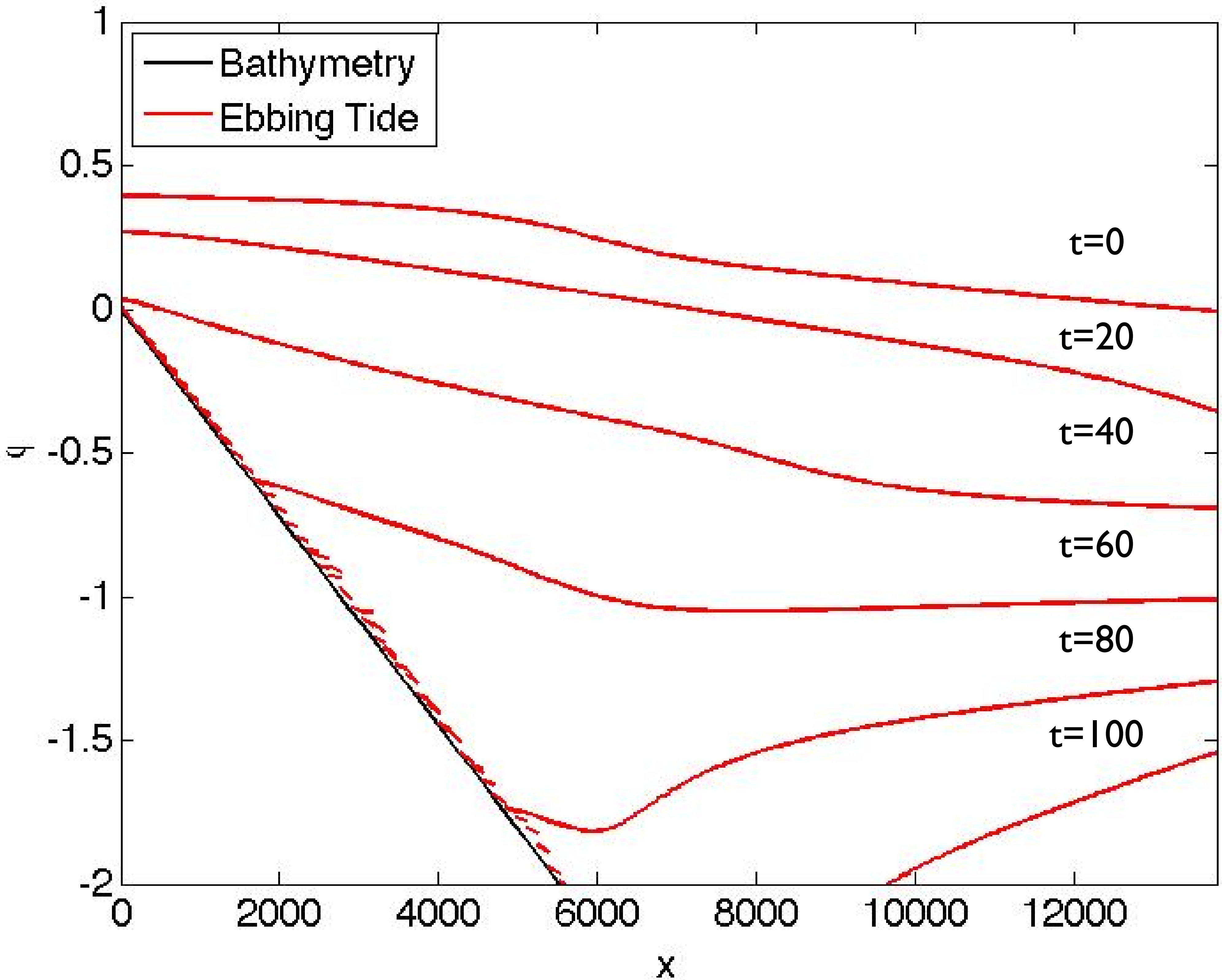
Bottom topography is defined as,

$$h_b(x) = \frac{x}{2760}$$

Domain size is 13,800 meters. Sinusoidal forcing at the open end is given by,

$$\phi_s = g * (2sin\left(\frac{2\pi t}{43200}\right))$$

Balzano [1998] Test Case 1



Balzano [1998] Test Case 2

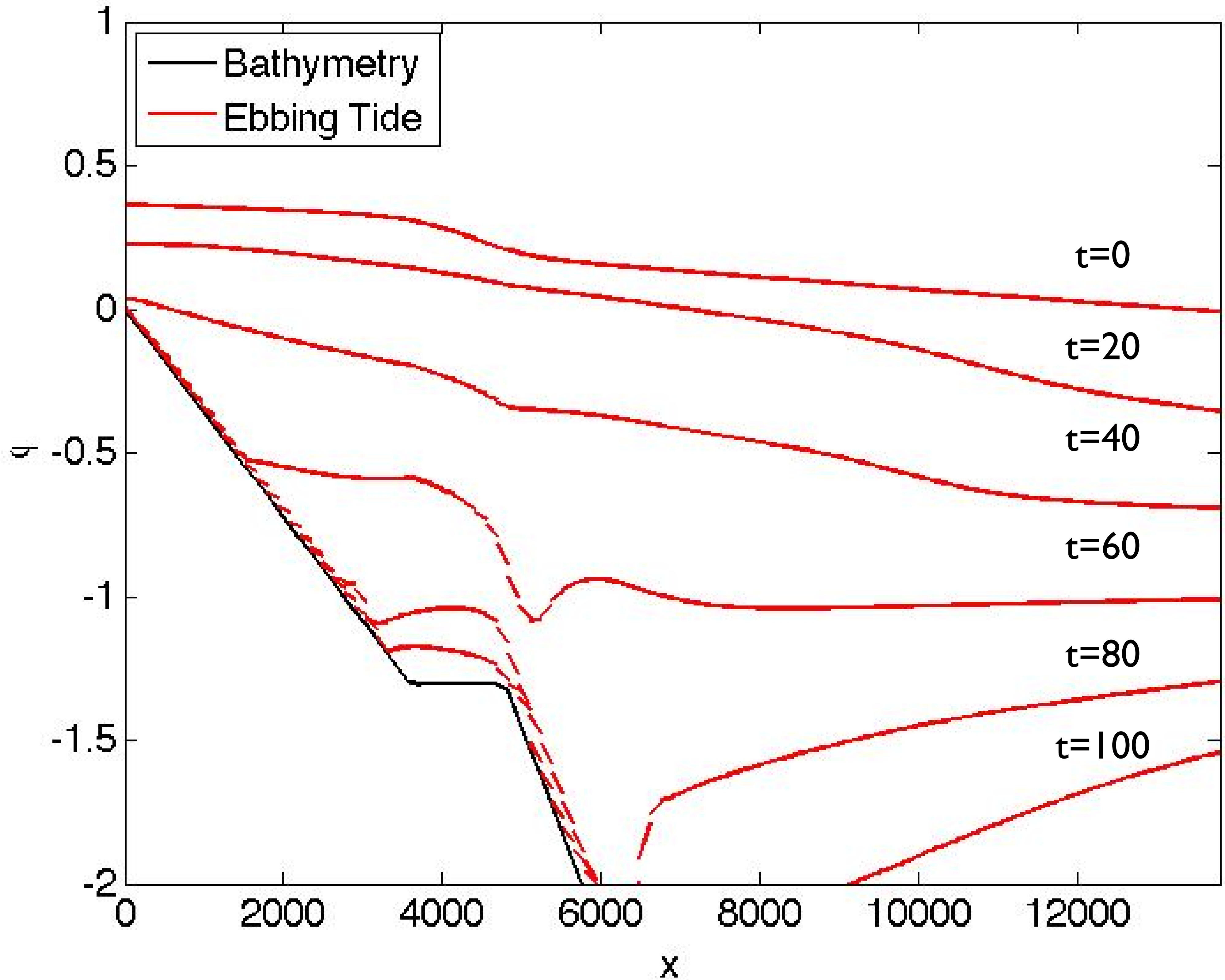
Bottom topography is defined as,

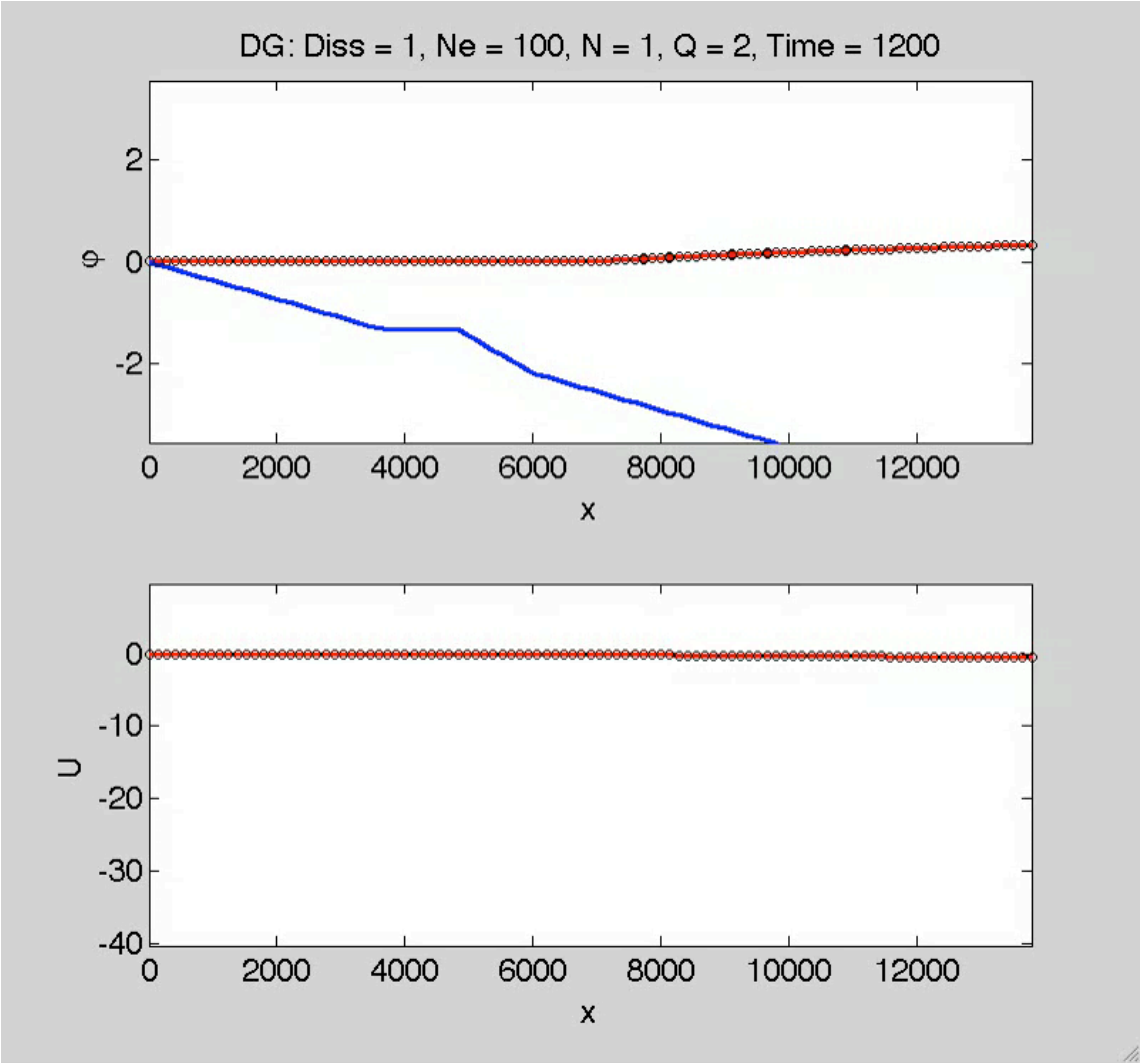
$$h_b(x) = \begin{cases} \frac{x}{2760} & \text{if x} \leq \text{3600m, or if x} \geq \text{6000m} \\ \frac{30}{23} & \text{if 3600m} \leq \text{x} \leq \text{4800m} \\ \frac{x}{920} - \frac{100}{23} & \text{if 4800m} \leq \text{x} \leq \text{6000m} \end{cases}$$

Domain size is 13,800 meters. Sinusoidal forcing at the open end is given by,

$$\phi_s = g * (2sin\left(\frac{2\pi t}{43200}\right))$$

Balzano [1998] Test Case 2





DG: Diss = 1, Ne = 100, N = 1, Q = 2, Time = 1200
φ
2
0
-2
0
2000
4000
6000
8000
10000
12000
x
U
0
-10
-20
-30
-40
0
2000
4000
6000
8000
10000
12000
x

Balzano [1998] Test Case 3

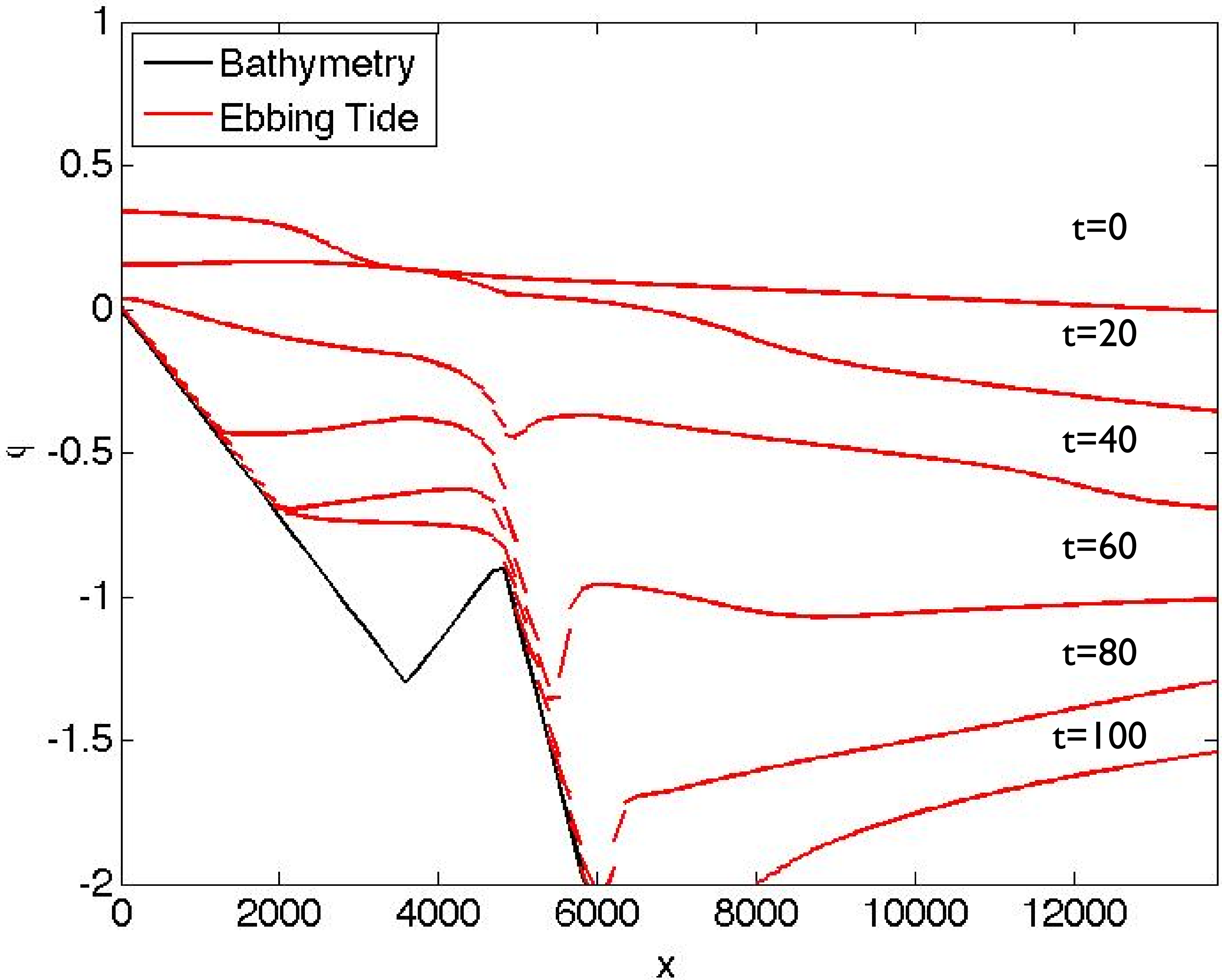
Bottom topography is defined as,

$$h_b(x) = \begin{cases} \frac{x}{2760} & \text{if x} \leq \text{3600m, or if x} \geq \text{6000m} \\ \frac{-x}{2760} + \frac{60}{23} & \text{if 3600m} \leq \text{x} \leq \text{4800m} \\ \frac{x}{920} - \frac{100}{23} & \text{if 4800m} \leq \text{x} \leq \text{6000m} \end{cases}$$

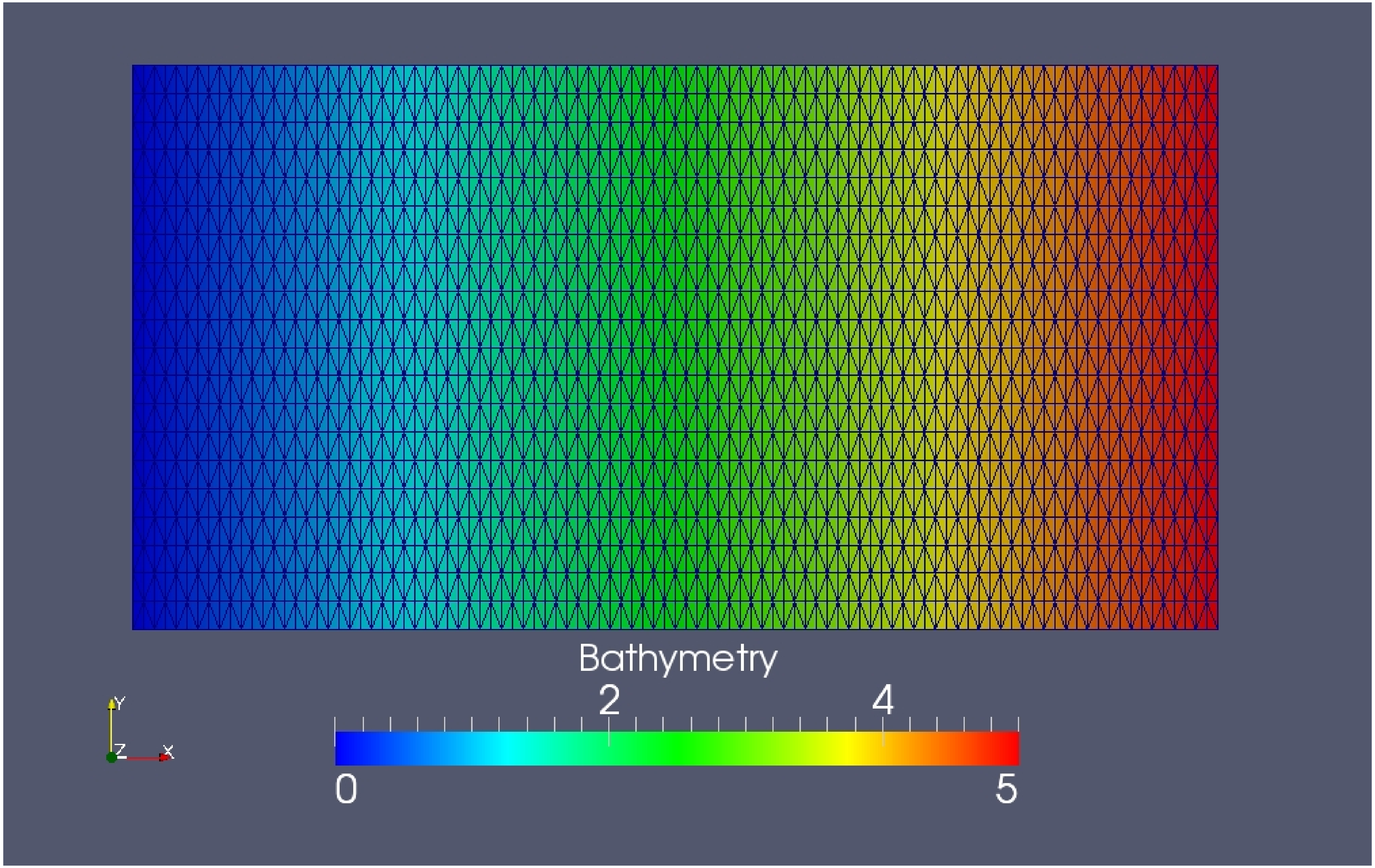
Domain size is 13,800 meters. Sinusoidal forcing at the open end is given by,

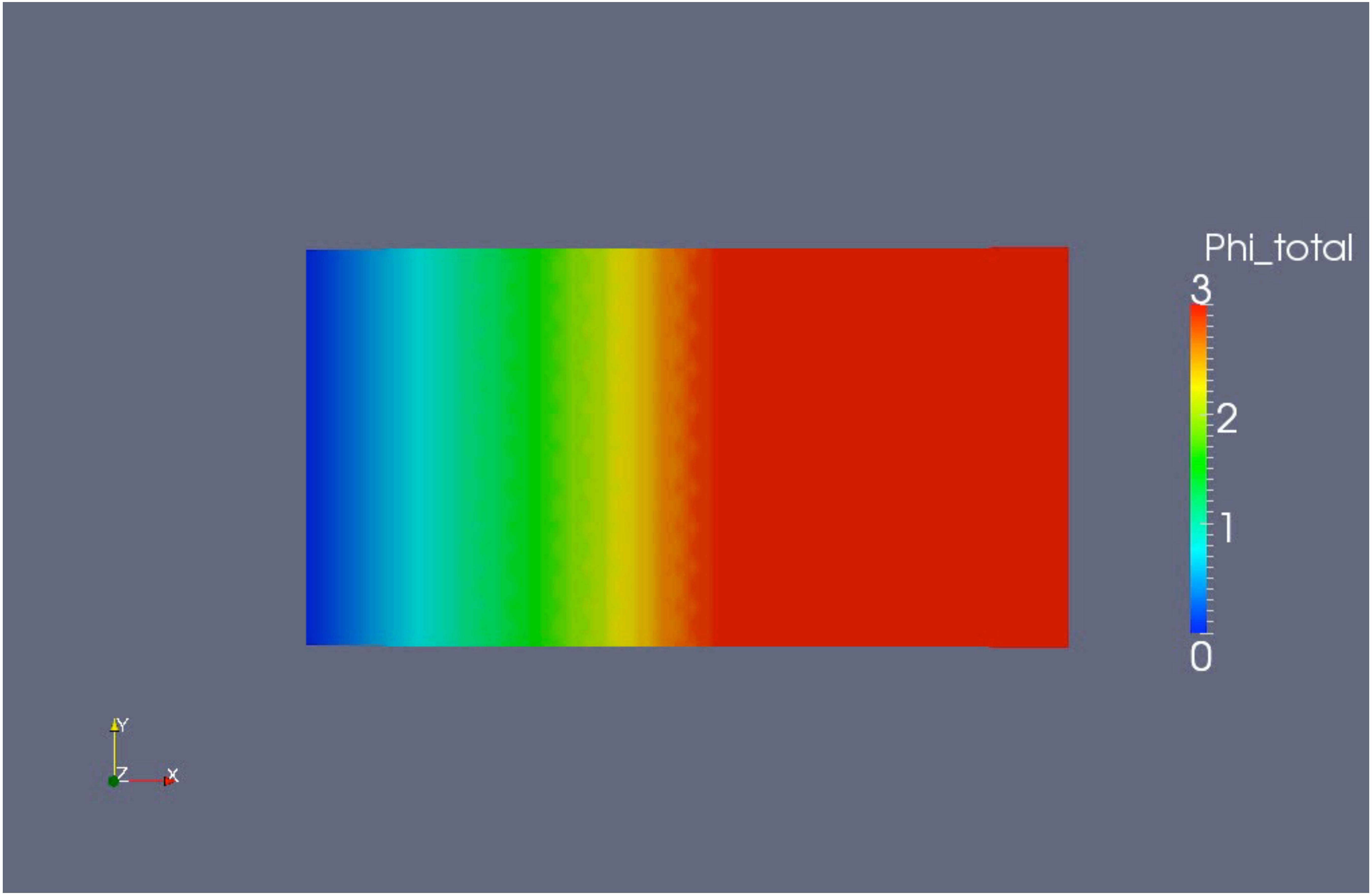
$$\phi_s = g * (2sin\left(\frac{2\pi t}{43200}\right))$$

Balzano [1998] Test Case 3

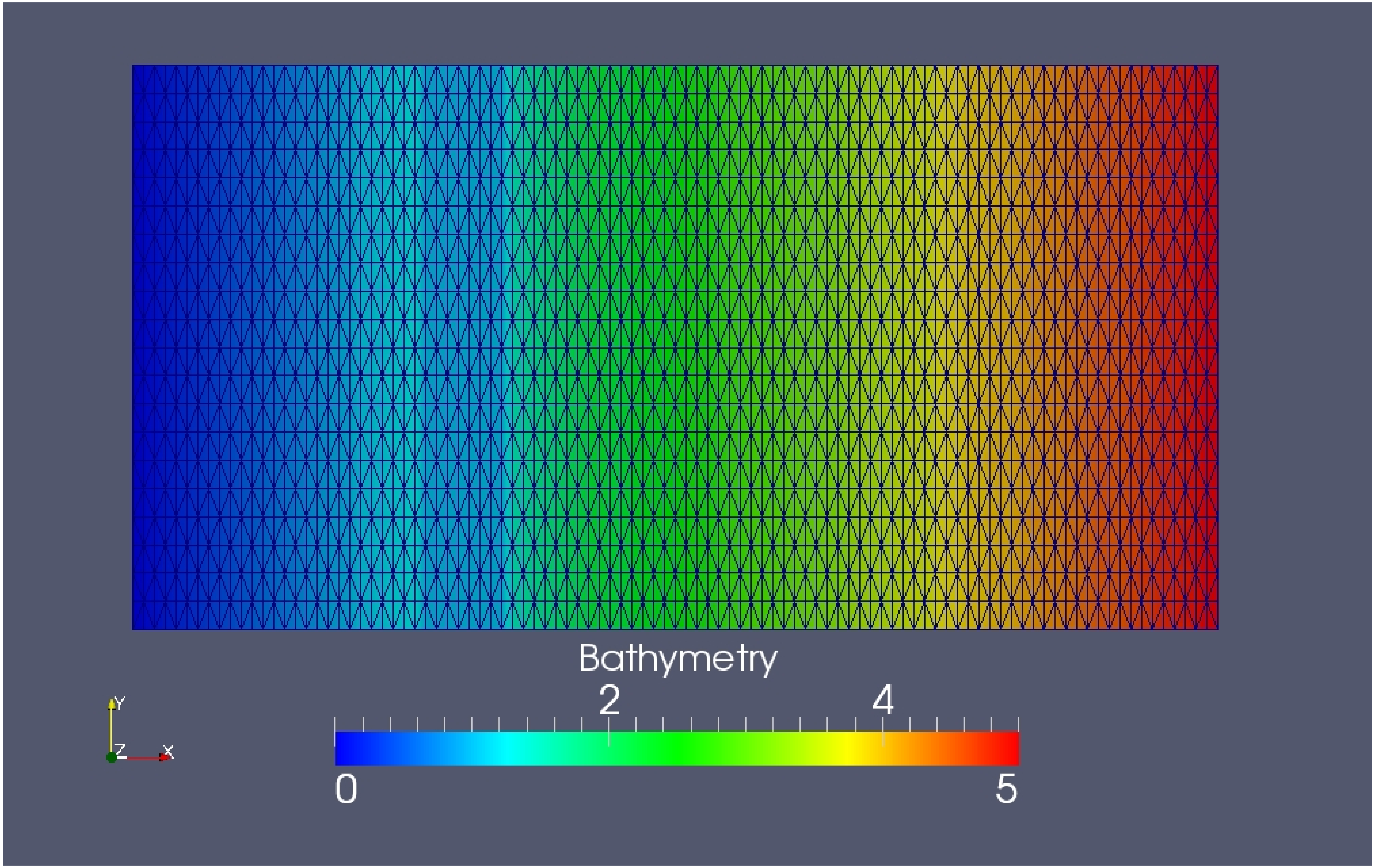


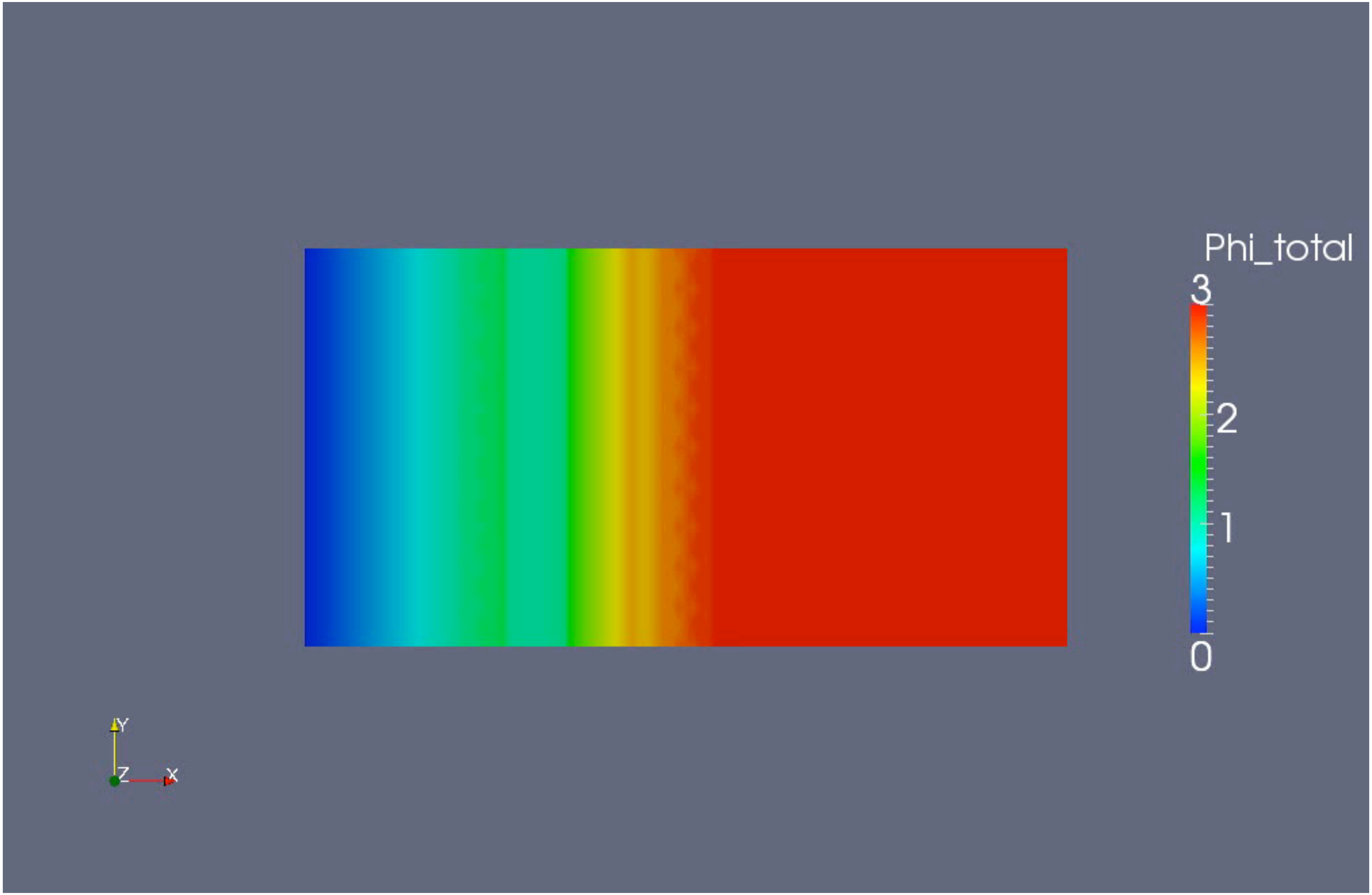
Balzano [1998] 2D Test Case 1




Phi_total
3
2
1
0
Y
Z
X

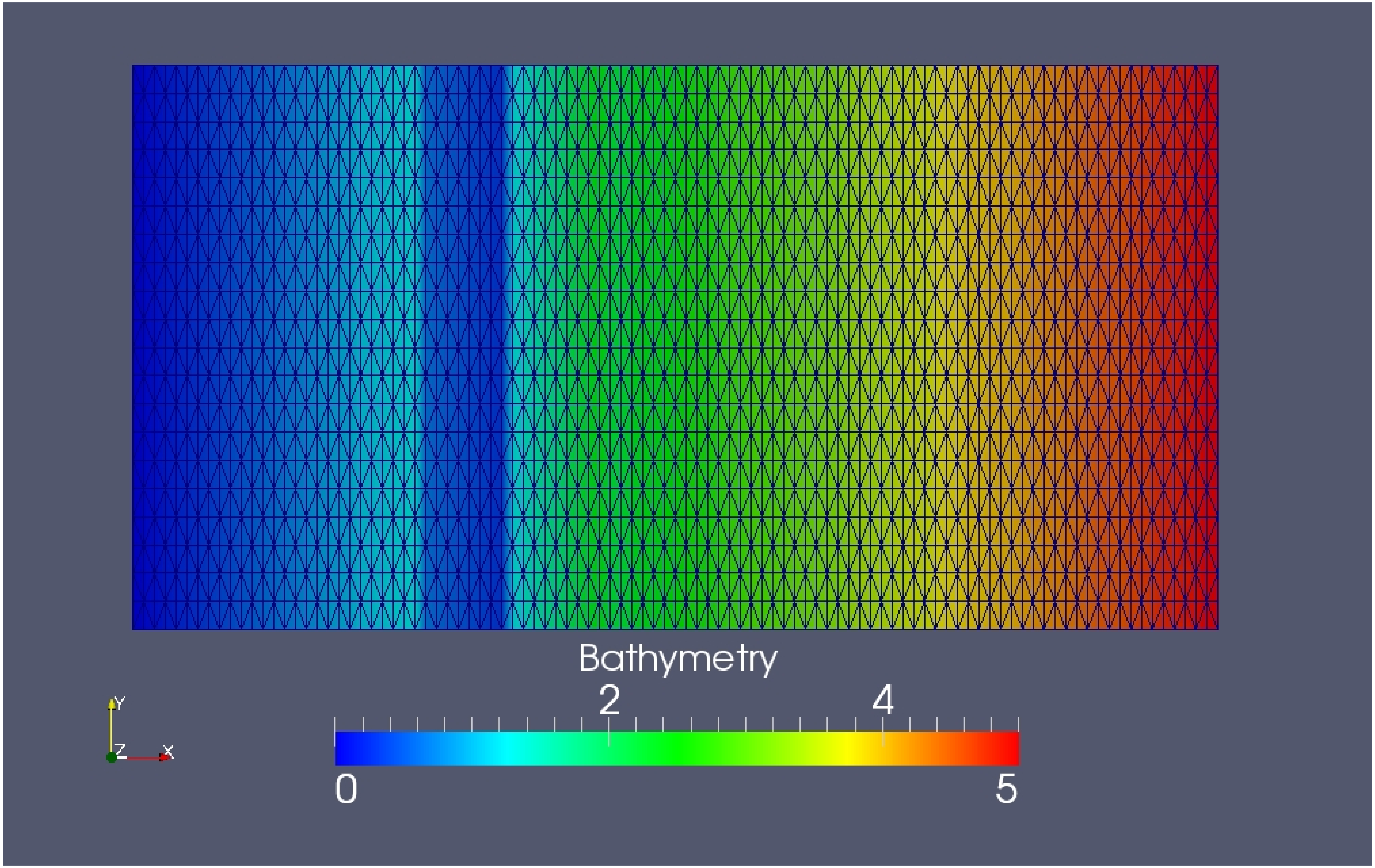
Balzano [1998] 2D Test Case 2

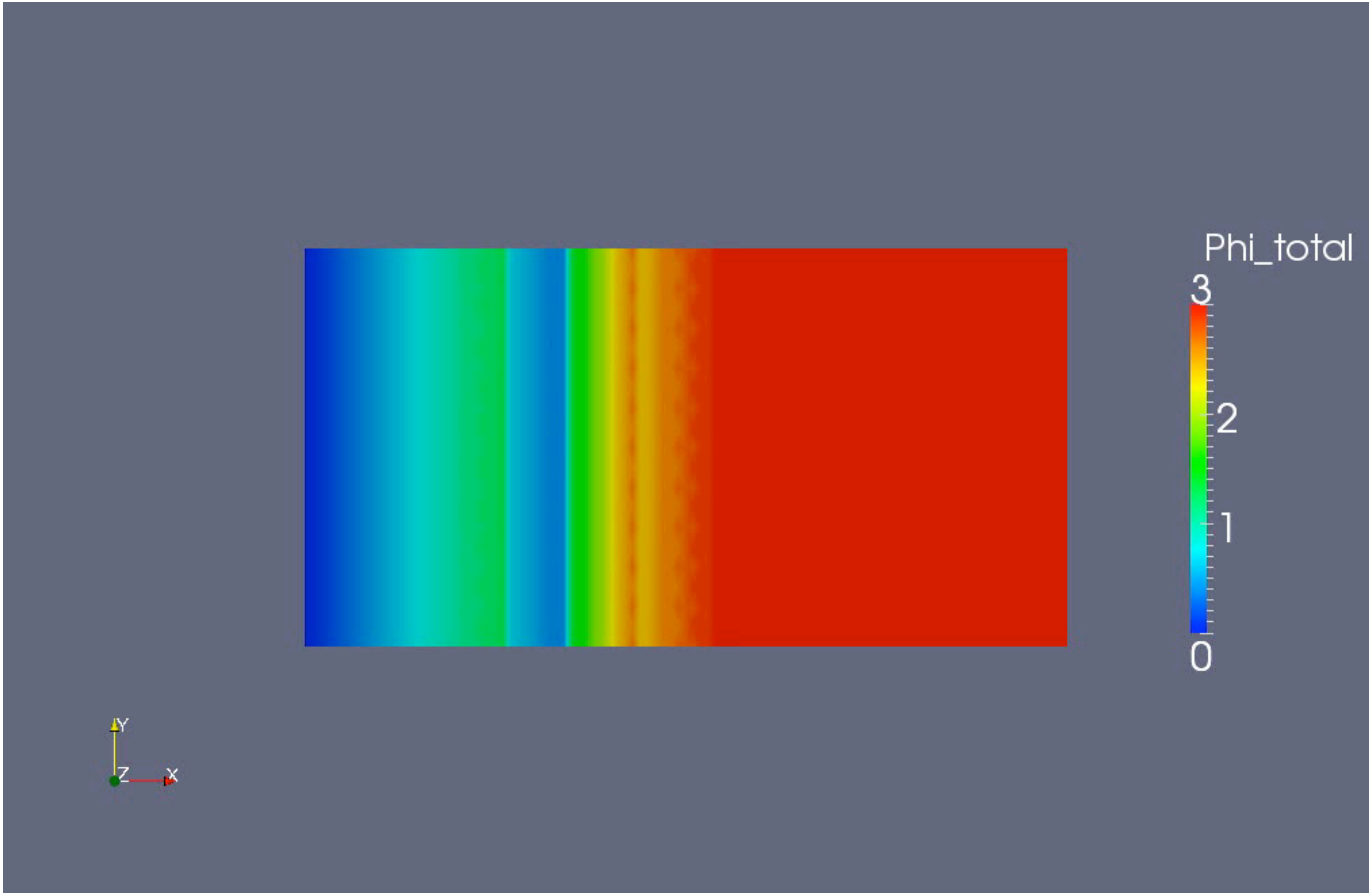




Phi_total
3
2
1
0
Y
Z
X

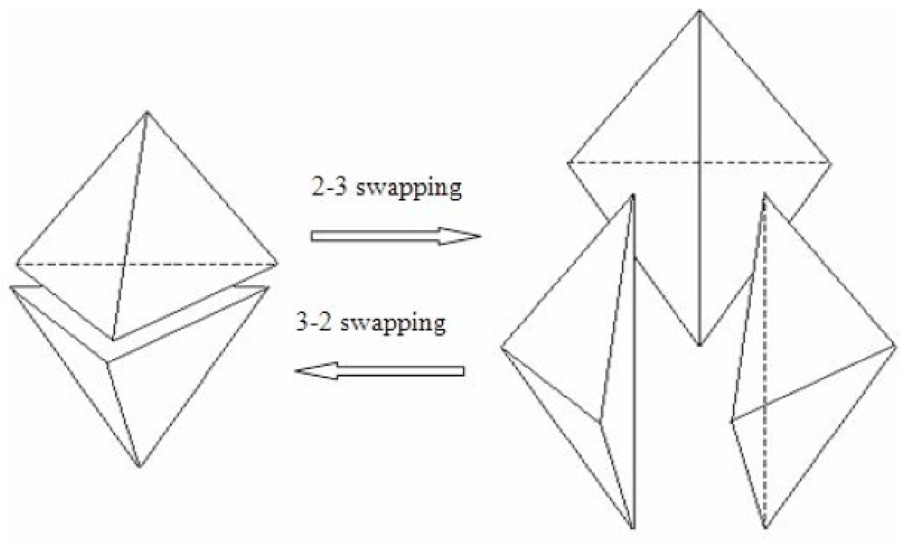
Balzano [1998] Test Case 3



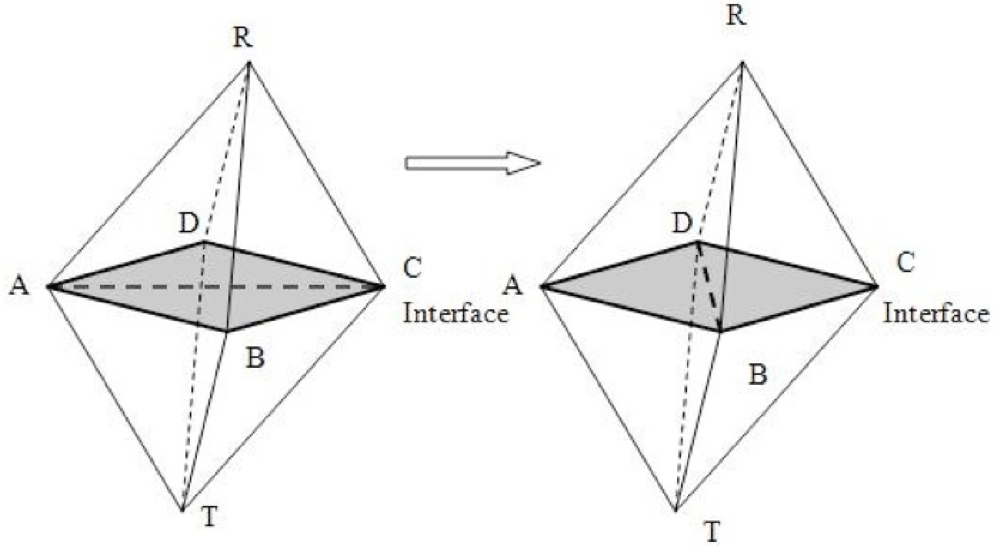

Phi_total
3
2
1
0
Y
Z
X

Adaptive Mesh Refinement

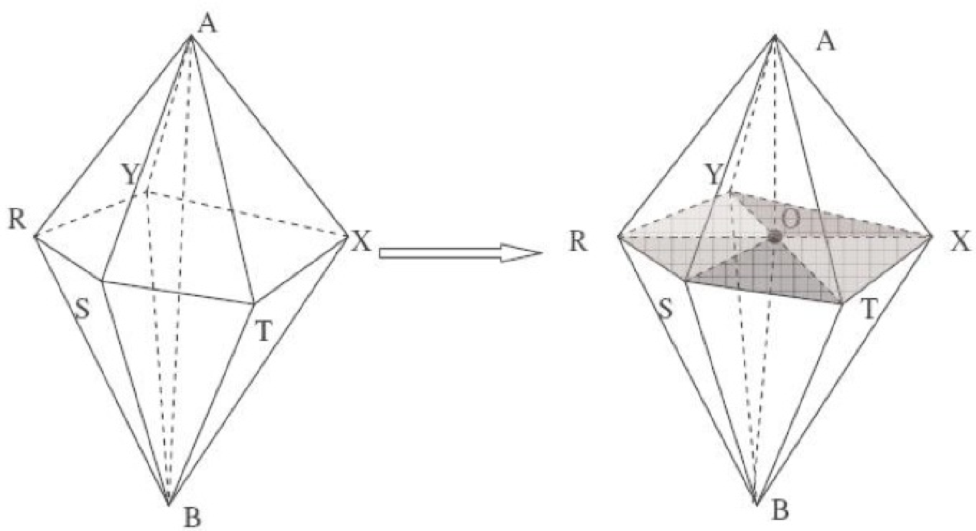
Adaptive Mesh Refinement



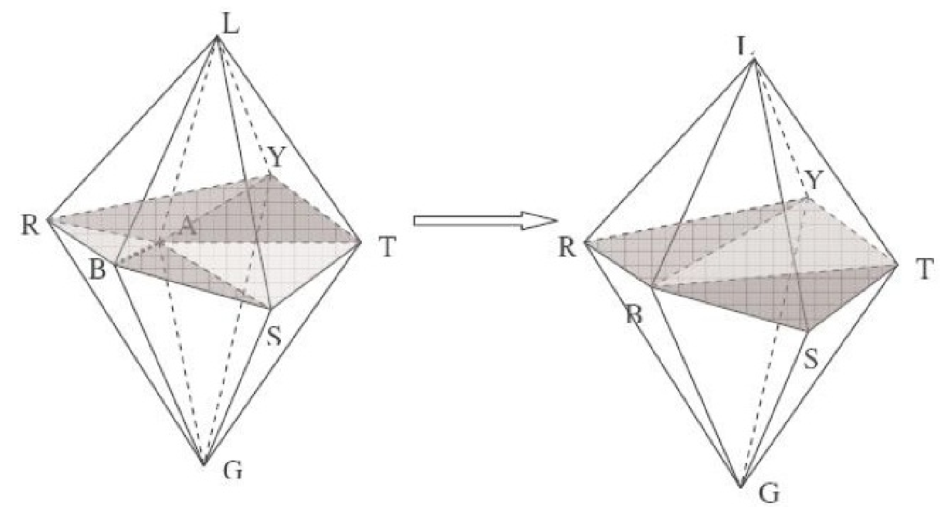
2–3 and 3–2 flipping



4–4 flipping

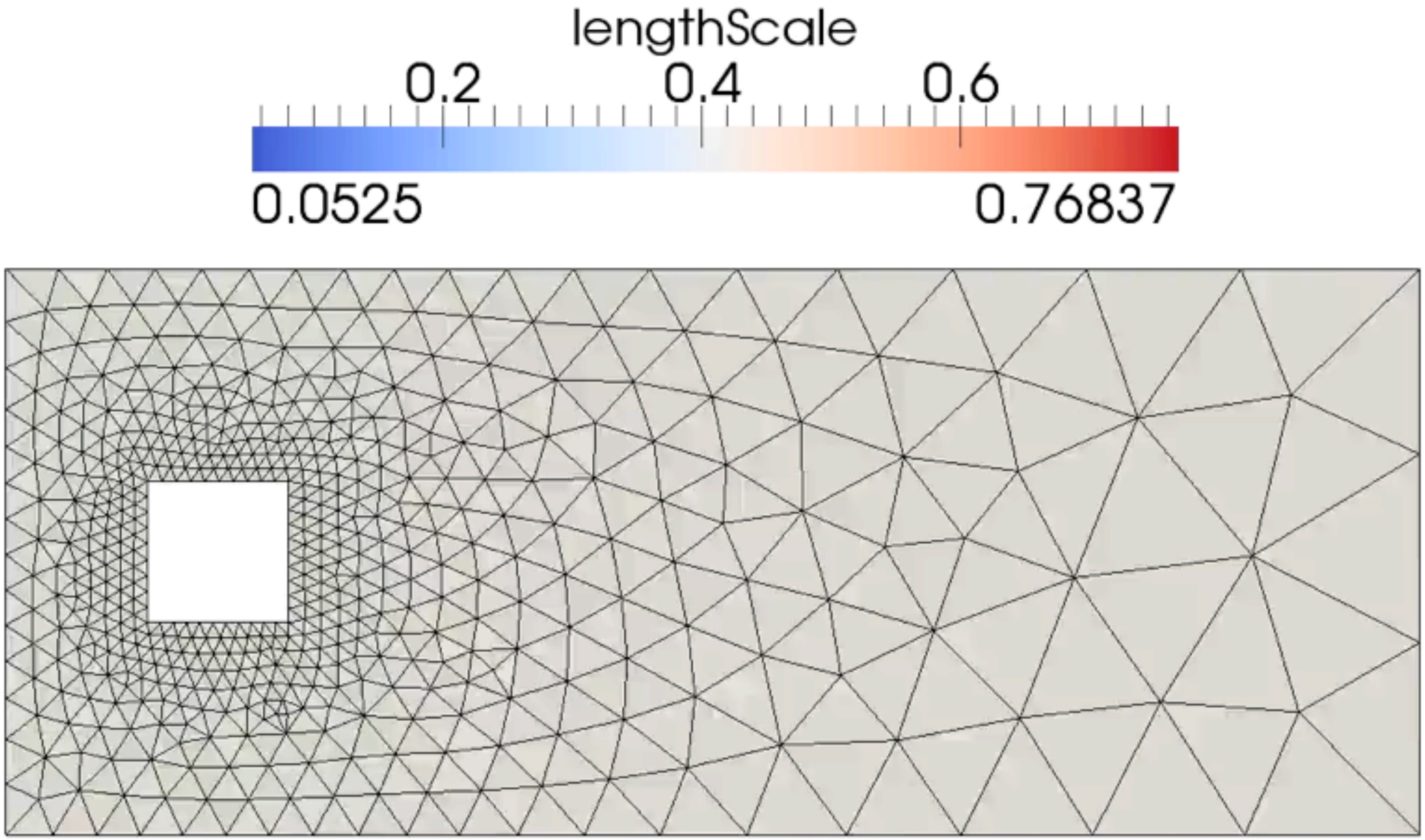


Edge bisection

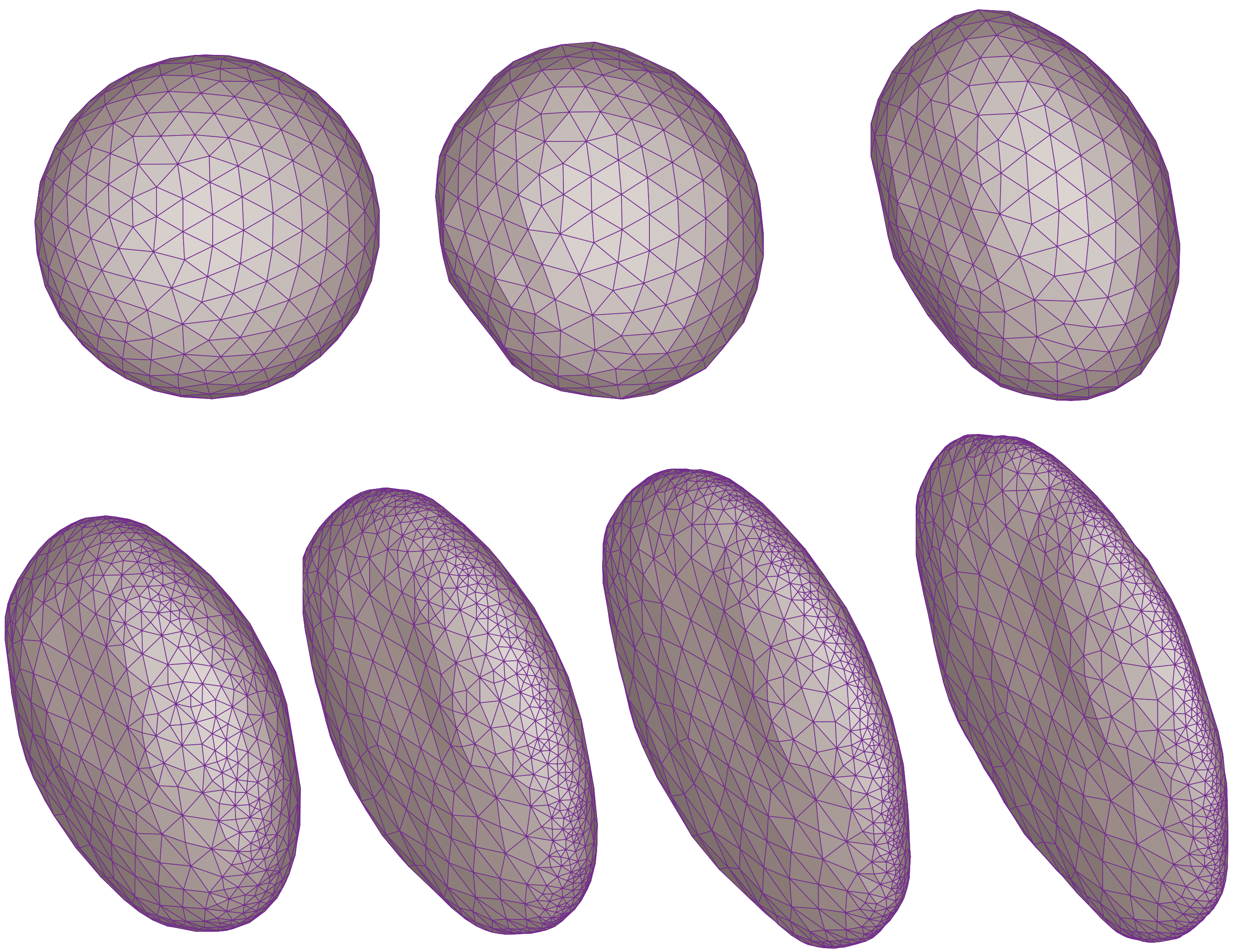


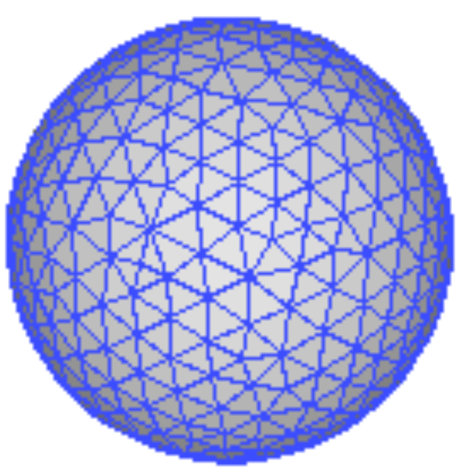
Edge collapse

2D Unstructured AMR – S. Menon (In Progress)



3D Unstructured AMR – Gopalakrishnan, Quan and Schmidt [2006]

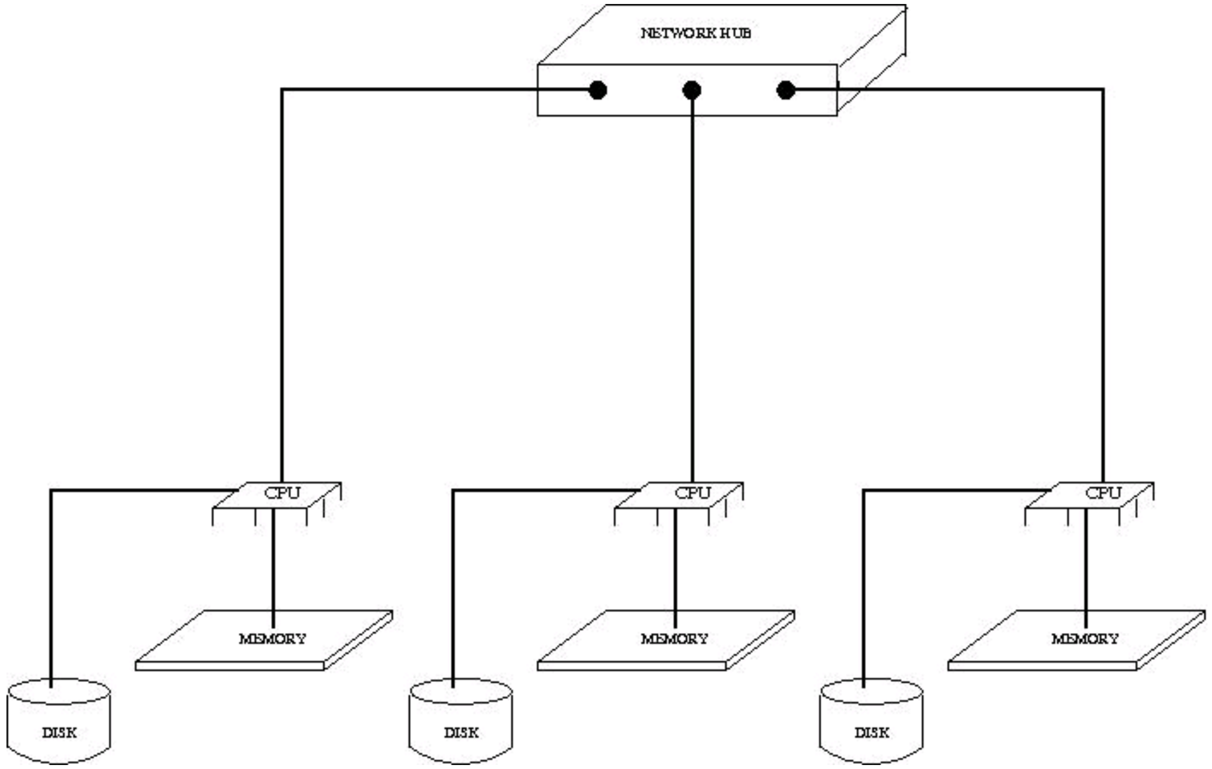




Distributed – Memory Machines

- Each node in the computer has a locally addressable memory space
- The computers are connected together via some high-speed network
  - Infiniband, Myrinet, Giganet, etc..

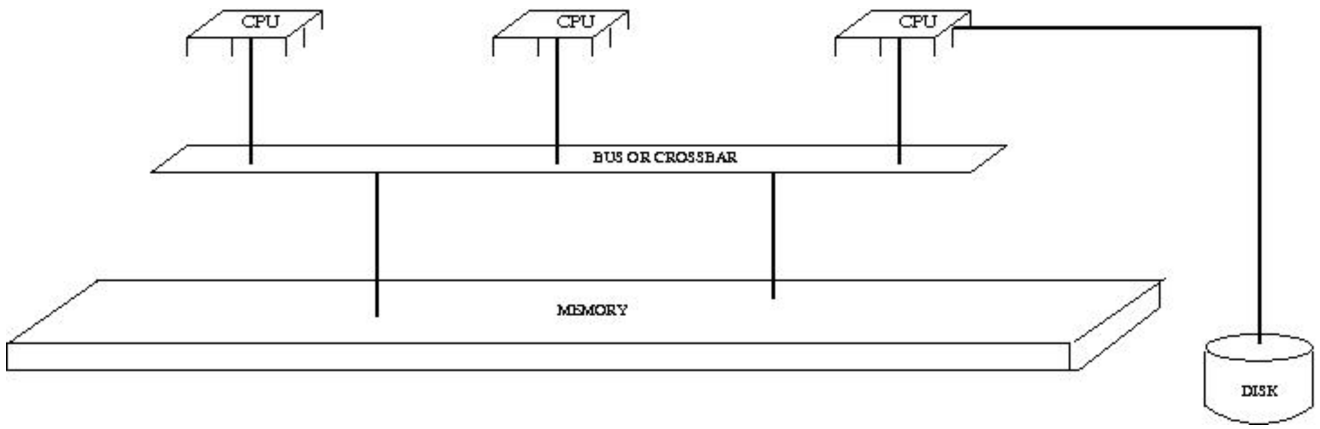
- Pros
  - Really large machines
  - Size limited only by gross physical considerations:
    - Room size
    - Cable lengths (10's of meters)
    - Power/cooling capacity
    - Money!
  - Cheaper to build and run
- Cons
  - Harder to program
    Data Locality



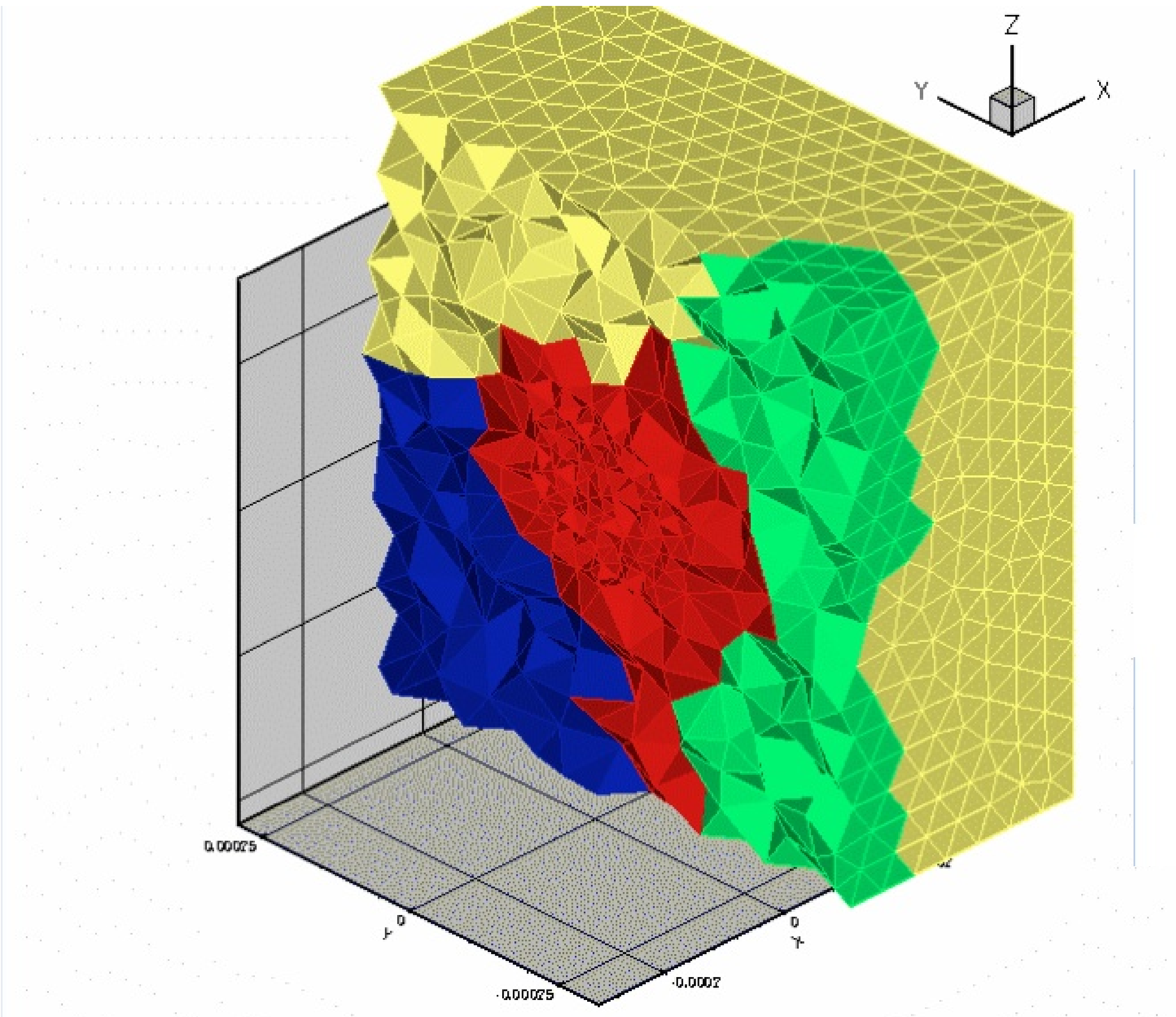
Shared-Memory Processing

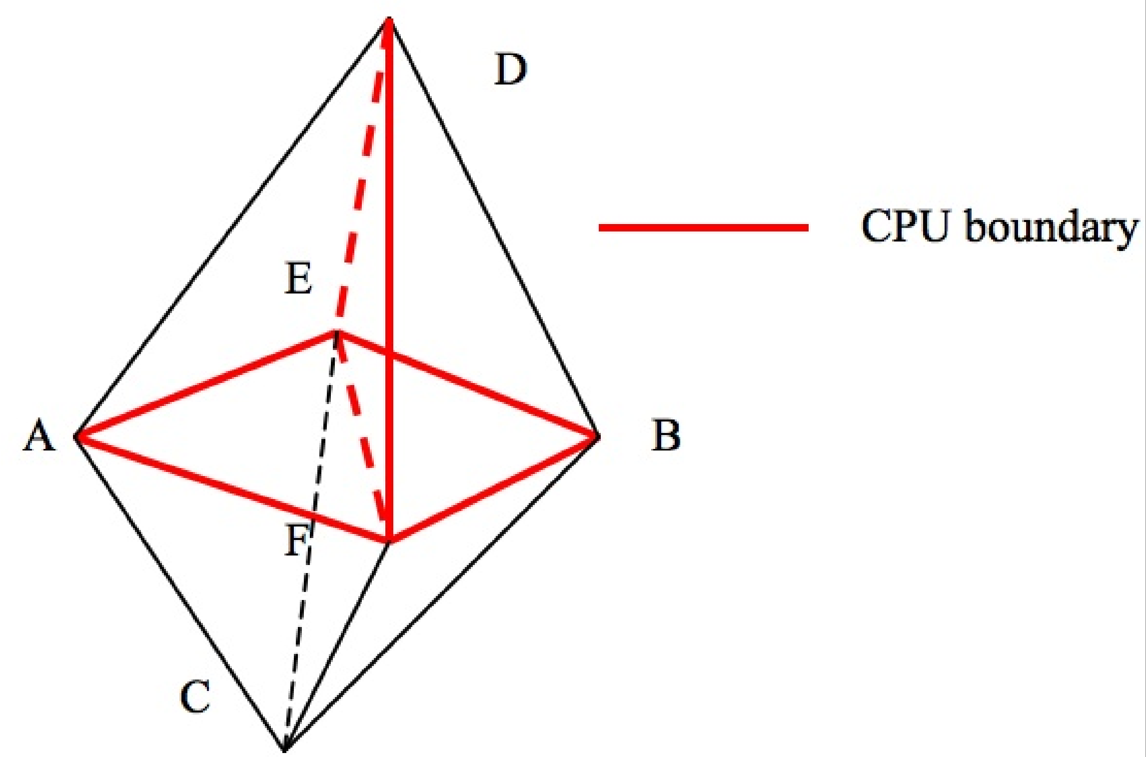
Each processor can access the entire data space

- Pro's
  - Easier to program
  - Amenable to automatic parallelism
  - Can be used to run large memory serial programs
- Con's
  - Expensive
  - Difficult to implement on the hardware level
  - Processor count limited by contention/coherency (currently around 512)
  - Watch out for "NU" part of "NUMA"



Parallel AMR





Conclusions and Future Directions

- DG provides a robust framework to implement SWE.
- Use of triangular DG method enables the modeling of complex coastlines.
- Adaptive mesh refinement helps in providing adequate resolution to resolve interesting features.
- Higher order Wetting and Drying Methods.
- Wind forcing data from mesoscale atmospheric codes - NUMA.
- Moving mesh technique may provide a useful alternative to model inundation as a moving boundary.
- Collaboration with Randy LeVeque and Kyle Mandli. Comparison and verification of test cases.