Well-Balanced Positivity Preserving Central-Upwind Scheme on Triangular Grids for the Saint-Venant System

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Modeling and Computations of Shallow-Water Coastal Flows
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Outline

- **Motivation**
- **Saint-Venant System of Shallow Water Equations**
- **Brief Overview of the Semi-Discrete Central-Upwind Scheme**
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Motivation

- **Saint-Venant System of shallow water equations describes the fluid flow as a conservation law with an additional source term**
- **The general characteristic of shallow water flows is that vertical scales of motion are much smaller than the horizontal scales**
- **The shallow water equations are derived from the incompressible Navier-Stokes**

Motivation

- **This Saint-Venant System is widely used in many scientific and engineering applications related to**
- **Modeling of water flows in rivers, lakes and coastal areas**
- **The Development of robust and accurate numerical methods for Shallow Water Equations is an important and challenging problem**

2-D Saint-Venant system of shallow water equations

$$
\begin{cases}
h_t + (hu)_x + (hv)_y = 0, \\
(hu)_t + \left(hu^2 + \frac{1}{2}gh^2\right)_x + (huv)_y = -ghB_x, \\
(hv)_t + (huv)_x + \left(hv^2 + \frac{1}{2}gh^2\right)_y = -ghB_y,
\end{cases} \quad (1)
$$

- **the function** $B(x, y)$ **represents the bottom elevation**
- h **is the fluid depth above the bottom**
- $(u, v)^T$ **is the velocity vector**
- g **is the gravitational constant**

One of the difficulties encountered:

- **that system (1) admits nonsmooth solutions: shocks, rarefaction waves,**
- **the bottom topography function** B **can be discontinuous.**

2-D Saint-Venant system of shallow water equations

A good numerical method for Saint-Venant System should have at least two major properties, which are crucial for its stability:

(i) **The method should be well-balanced, that is, it should exactly preserve the stationary steady-state solutions** $h + B \equiv \text{const}, \ u \equiv v \equiv 0$ **(lake at rest states).**
This property diminishes the appearance of unphysical waves of magnitude proportional to the grid size (the so-called "numerical storm"), which are normally present when computing quasi steady-states;

(ii) **The method should be positivity preserving, that is, the water depth** h **should be nonnegative at all times.**
This property ensures a robust performance of the method on dry $(h = 0)$ **and almost dry** $(h \sim 0)$ **states.**

Semi-discrete central-upwind scheme

Central-Upwind schemes were developed for multidimensional hyperbolic systems of conservation laws in $2000 - 2007$ *by Kurganov, Lin, Noelle, Petrova, Tadmor, ...*

- **Central-Upwind schemes are Godunov-type finite-volume projection-evolution methods:**
- **At each time level a solution is globally approximated by a piecewise polynomial function,**
- **Which is then evolved to the new time level using the integral form of the conservation law system.**

Key ideas of the scheme development for Saint-Venant system

- **Change of conservative variables from** $(h, hu, hv)^T$ **to** $(w := h + B, hu, hv)^T$
- **Replacement of the bottom topography function** B **with its continuous piecewise linear (or bilinear in the 2-D case) approximation**
- **Special positivity preserving correction of the piecewise linear reconstruction for the water surface** w
- **Development of a special finite-volume-type quadrature for the discretization of the cell averages of the geometric source term.**

Description of the scheme

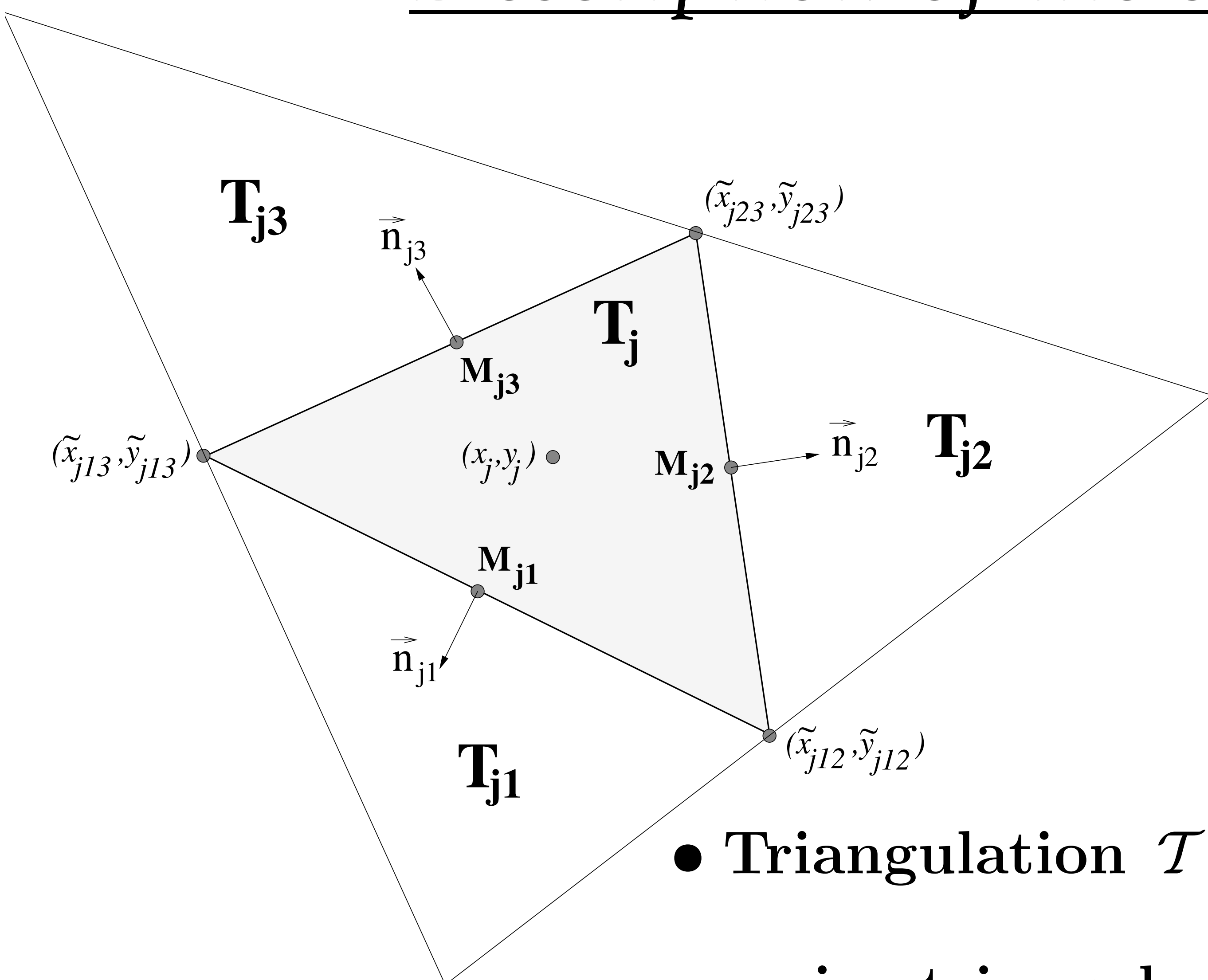
- **We describe now, our new second-order semi-discrete central-upwind scheme for solving the Saint-Venant system of shallow water equations on triangular grids**
- **We first denote the water surface by $w := h + B$ and rewrite the original Saint-Venant system in terms of the vector $\mathbf{U} := (w, hu, hv)^T$:**

$$\mathbf{U}_t + \mathbf{F}(\mathbf{U}, B)_x + \mathbf{G}(\mathbf{U}, B)_y = \mathbf{S}(\mathbf{U}, B)$$

where the fluxes and the source terms are:

$$\begin{aligned}
\mathbf{F}(\mathbf{U}, B) &= \left(hu, \frac{(hu)^2}{w-B} + \frac{1}{2}g(w-B)^2, \frac{(hu)(hv)}{w-B}\right)^T \\
\mathbf{G}(\mathbf{U}, B) &= \left(hv, \frac{(hu)(hv)}{w-B}, \frac{(hv)^2}{w-B} + \frac{1}{2}g(w-B)^2\right)^T \\
\mathbf{S}(\mathbf{U}, B) &= \Big(0, -g(w-B)B_x, -g(w-B)B_y\Big)^T.
\end{aligned}$$

Description of the scheme: notations



- **Triangulation** $\mathcal{T} := \bigcup_j T_j$ **of the computational domain: triangular cells** T_j **of size** $|T_j|$
- $\vec{n}_{jk} := (\cos(\theta_{jk}), \sin(\theta_{jk}))$ **are the outer unit normals to the corresponding sides of** T_j **of length** ℓ_{jk}, $k = 1, 2, 3$,
- (x_j, y_j) **are the coordinates of the center of mass for** T_j **and** $M_{jk} = (x_{jk}, y_{jk})$ **is the midpoint of the** k**-th side of the triangle** T_j, $k = 1, 2, 3$
- T_{j1}, T_{j2} **and** T_{j3} **are the neighboring triangles that share a common side with** T_j

Description of the central-upwind scheme on triangular grids

Denote $\overline{\mathbf{U}}_j(t) \approx \frac{1}{|T_j|}\int\limits_{T_j} \mathbf{U}(x,y,t)\,dxdy.$

Second order central-upwind scheme on triangular grid for the Saint-Venant System:

$$\frac{d\overline{\mathbf{U}}_j}{dt} =$$

$$-\frac{1}{|T_j|}\sum_{k=1}^{3}\frac{\ell_{jk}\cos(\theta_{jk})}{a_{jk}^{\text{in}}+a_{jk}^{\text{out}}}\left[a_{jk}^{\text{in}}\mathbf{F}(\mathbf{U}_{jk}(M_{jk}),B(M_{jk}))+a_{jk}^{\text{out}}\mathbf{F}(\mathbf{U}_j(M_{jk}),B(M_{jk}))\right]$$

$$-\frac{1}{|T_j|}\sum_{k=1}^{3}\frac{\ell_{jk}\sin(\theta_{jk})}{a_{jk}^{\text{in}}+a_{jk}^{\text{out}}}\left[a_{jk}^{\text{in}}\mathbf{G}(\mathbf{U}_{jk}(M_{jk}),B(M_{jk}))+a_{jk}^{\text{out}}\mathbf{G}(\mathbf{U}_j(M_{jk}),B(M_{jk}))\right]$$

$$+\frac{1}{|T_j|}\sum_{k=1}^{3}\ell_{jk}\frac{a_{jk}^{\text{in}}a_{jk}^{\text{out}}}{a_{jk}^{\text{in}}+a_{jk}^{\text{out}}}\Big[\mathbf{U}_{jk}(M_{jk})-\mathbf{U}_j(M_{jk})\Big]+\overline{\mathbf{S}}_j,$$

Description of the central-upwind scheme on triangular grids

- **$\mathbf{U}_j(M_{jk})$ and $\mathbf{U}_{jk}(M_{jk})$ are the corresponding values at M_{jk} of the piecewise linear reconstruction**

  $$\widetilde{\mathbf{U}}(x,y) := \overline{\mathbf{U}}_j + (\mathbf{U}_x)_j(x-x_j) + (\mathbf{U}_y)_j(y-y_j), \quad (x,y) \in T_j$$

  of $\mathbf{U}$ at time t

- **The quantity $\overline{\mathbf{S}}_j$ in the scheme is an appropriate discretization of the cell averages of the source term**

- **The directional local speeds $a_{jk}^{\rm in}$ and $a_{jk}^{\rm out}$ are defined by**

  $$a_{jk}^{\rm in}(M_{jk}) = -\min\{\lambda_1[V_{jk}(\mathbf{U}_j(M_{jk}))], \lambda_1[V_{jk}(\mathbf{U}_{jk}(M_{jk})], 0\},$$

  $$a_{jk}^{\rm out}(M_{jk}) = \max\{\lambda_3[V_{jk}(\mathbf{U}_j(M_{jk}))], \lambda_3[V_{jk}(\mathbf{U}_{jk}(M_{jk})], 0\},$$

  where $\lambda_1[V_{jk}] \le \lambda_2[V_{jk}] \le \lambda_3[V_{jk}]$ are the eigenvalues of the matrix $V_{jk} = \cos(\theta_{jk})\frac{\partial \mathbf{F}}{\partial \mathbf{U}} + \sin(\theta_{jk})\frac{\partial \mathbf{G}}{\partial \mathbf{U}}$.

- **A fully discrete scheme is obtained by using a stable ODE solver of an appropriate order**

Calculation of the numerical derivatives of the ith component of U

- **Construct three linear interpolations** $L_j^{12}(x, y)$**,** $L_j^{23}(x, y)$ **and** $L_j^{13}(x, y)$**: conservative on** T_j **and two of the neighboring triangles** (T_{j1}, T_{j2})**,** (T_{j2}, T_{j3}) **and** (T_{j1}, T_{j3})

- **Select the linear piece with the smallest magnitude of the gradient, say,** $L_j^{km}(x, y)$**, and set**

$$((\mathbf{U}_x^{(i)})_j, (\mathbf{U}_y^{(i)})_j)^T = \nabla L_j^{km}$$

- **Minimize the oscillations by checking the appearance of local extrema at the points** M_{jk}**,** $1, 2, 3$

Piecewise linear approximation of the bottom

- **Replace the bottom topography function B with its continuous piecewise linear approximation $\widetilde{B}$, which over each cell T_j is given by the formula:**

$$\begin{vmatrix} x-\tilde{x}_{j_{12}} & y-\tilde{y}_{j_{12}} & \widetilde{B}(x,y)-\mathcal{B}_{j_{12}} \\ \tilde{x}_{j_{23}}-\tilde{x}_{j_{12}} & \tilde{y}_{j_{23}}-\tilde{y}_{j_{12}} & \mathcal{B}_{j_{23}}-\mathcal{B}_{j_{12}} \\ \tilde{x}_{j_{13}}-\tilde{x}_{j_{12}} & \tilde{y}_{j_{13}}-\tilde{y}_{j_{12}} & \mathcal{B}_{j_{13}}-\mathcal{B}_{j_{12}} \end{vmatrix} = 0, \qquad (x,y)\in T_j.$$

- $\mathcal{B}_{j_\kappa}$ **are the values of $\widetilde{B}$ at the vertices $(\tilde{x}_{j_\kappa},\tilde{y}_{j_\kappa})$, $\kappa = 12, 23, 13$, of the cell T_j**

- $\mathcal{B}_{j_\kappa} := \frac{1}{2}(\max_{\xi^2+\eta^2=1}\lim_{h,\ell\to 0} B(\tilde{x}_{j_\kappa}+h\xi,\tilde{y}_{j_\kappa}+\ell\eta) + \min_{\xi^2+\eta^2=1}\lim_{h,\ell\to 0} B(\tilde{x}_{j_\kappa}+h\xi,\tilde{y}_{j_\kappa}+\ell\eta)),$

- **If the function B is continuous at $(\tilde{x}_{j_\kappa},\tilde{y}_{j_\kappa})$: $\mathcal{B}_{j_\kappa} = B(\tilde{x}_{j_\kappa},\tilde{y}_{j_\kappa})$**

Positivity preserving reconstruction for w

The idea of the algorithm that guarantees positivity of the reconstructed values of the water depth $h_j(M_{jk}) := w_j(M_{jk}) - B_{jk},\ k = 1, 2, 3$, for all j:

- **The reconstruction $\widetilde{w}$ should be corrected only in those triangles, where $\widetilde{w}(\tilde{x}_{j_\kappa}, \tilde{y}_{j_\kappa}) < \mathcal{B}_{j_\kappa}$ for some κ, $\kappa = 12, 23, 13$**
- **Since $\overline{w}_j \ge B_j$, it is impossible to have $\widetilde{w}(\tilde{x}_{j_\kappa}, \tilde{y}_{j_\kappa}) < \mathcal{B}_{j_\kappa}$ for all three values of κ: at all three vertices of the triangle T_j**
- **Two cases in which a correction is needed are possible:**
  either there are two indices κ_1 and κ_2, for which $\widetilde{w}(\tilde{x}_{j_{\kappa_1}}, \tilde{y}_{j_{\kappa_1}}) < \mathcal{B}_{j_{\kappa_1}}$ and $\widetilde{w}(\tilde{x}_{j_{\kappa_2}}, \tilde{y}_{j_{\kappa_2}}) < \mathcal{B}_{j_{\kappa_2}}$,
  or there is only one index κ_1, for which $\widetilde{w}(\tilde{x}_{j_{\kappa_1}}, \tilde{y}_{j_{\kappa_1}}) < \mathcal{B}_{j_{\kappa_1}}$

Well-balanced discretization of the source term

- **The well-balanced property of the scheme is guaranteed if the discretized cell average of the source term, $\overline{\mathbf{S}}_j$, exactly balances the numerical fluxes**
- **The desired quadrature for the source term that will preserve stationary steady states ($\mathbf{U}_{jk}(M_{jk}) \equiv \mathbf{U}_j(M_{jk}) \equiv (C, 0, 0)^T, \ \forall j, k$) is given by:**

$$\overline{\mathbf{S}}_j^{(2)} = \frac{g}{2|T_j|} \sum_{k=1}^{3} \ell_{jk}(w_j(M_{jk}) - B_{jk})^2 \cos(\theta_{jk}) - g(w_x)_j(\overline{w}_j - B_j)$$

$$\overline{\mathbf{S}}_j^{(3)} = \frac{g}{2|T_j|} \sum_{k=1}^{3} \ell_{jk}(w_j(M_{jk}) - B_{jk})^2 \sin(\theta_{jk}) - g(w_y)_j(\overline{w}_j - B_j)$$

Main theorem: positivity property of the new scheme

Theorem 1 ***Consider the Saint-Venant system in the new variables*** $\mathbf{U} := (w, hu, hv)^T$ ***and the central-upwind semi-discrete scheme (with well-balanced quadrature for the source*** S***, positivity preserving reconstruction for*** w***)***

- ***Assume that the system of ODEs for the fully discrete scheme is solved by the forward Euler method and that for all*** j***,*** $\overline{w}_j^n - B_j \geq 0$ ***at time*** $t = t^n$
- ***Then, for all*** j***,*** $\overline{w}_j^{n+1} - B_j \geq 0$ ***at time*** $t = t^{n+1} = t^n + dt$***, provided that*** $dt \leq \frac{1}{6a} \min_{j,k} \{r_{jk}\}$***, where*** $a := \max_{j,k}\{a_{jk}^{\text{out}}, a_{jk}^{\text{in}}\}$ ***and*** r_{jk}***,*** $k = 1, 2, 3$***, are the altitudes of triangle*** T_j

Remark. *Theorem 1 is still valid if one uses a higher-order SSP ODE solver (either the Runge-Kutta or the multistep one), because such solvers can be written as a convex combination of several forward Euler steps.*

Accuracy test

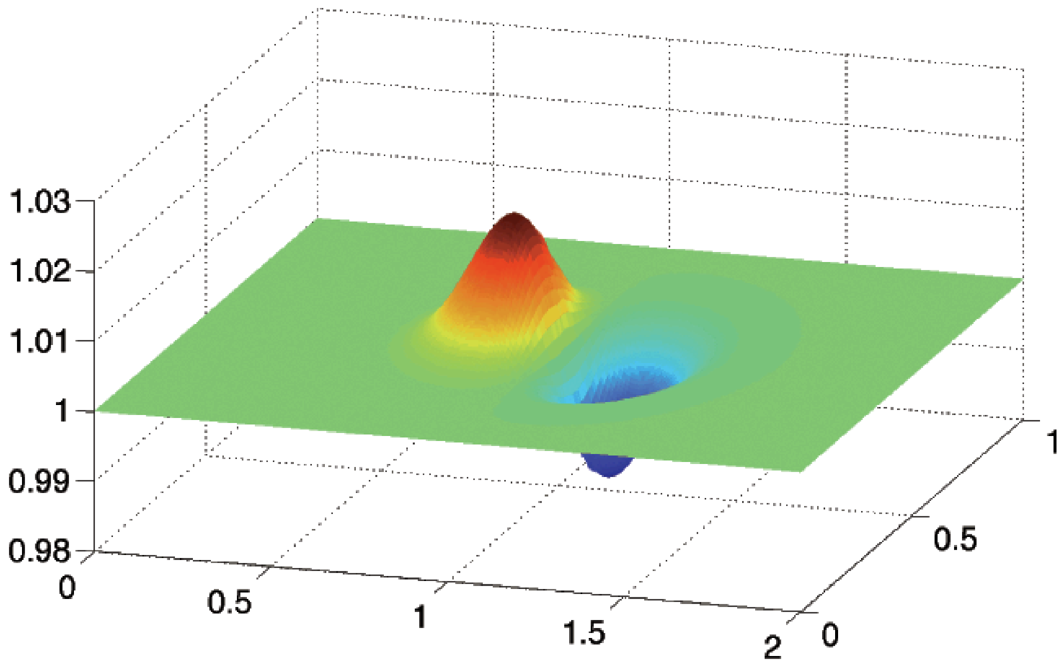
The scheme is applied to the Saint-Venant system subject to the following initial data and the bottom topography:

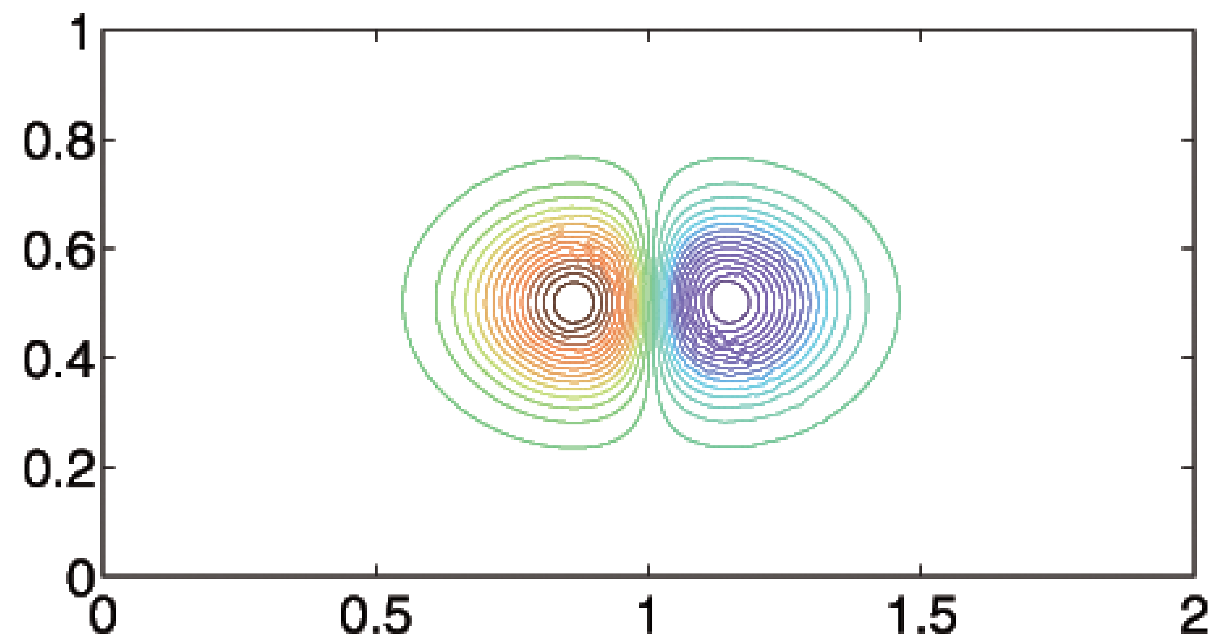
$$w(x, y, 0) = 1, \quad u(x, y, 0) = 0.3,$$
$$B(x, y) = 0.5 \exp(-25(x-1)^2 - 50(y-0.5)^2).$$

- **For a reference solution, we solve this problem with our method on a** $2 \times 400 \times 400$ **triangular grid**
- **By** $t = 0.07$ **the solution converges to the steady state**

Accuracy test

- w component of the reference solution of the IVP on a $2 \times 400 \times 400$ grid: the 3-D view (left) and the contour plot (right).





- L^1- and L^∞-errors and numerical orders of accuracy.

Number of cells	L^1-error	Order	L^∞-error	Order
$2 \times 50 \times 50$	**6.59e-04**	–	**8.02e-03**	–
$2 \times 100 \times 100$	**2.87e-04**	**1.20**	**3.59e-03**	**1.16**
$2 \times 200 \times 200$	**1.00e-04**	**1.52**	**1.21e-03**	**1.57**

Small perturbation of a stationary steady-state solution

- **Solve the initial value problem (IVP) proposed by R.Leveque.**
- **The computational domain is $[0,2] \times [0,1]$ and the bottom consists of an elliptical shaped hump:**

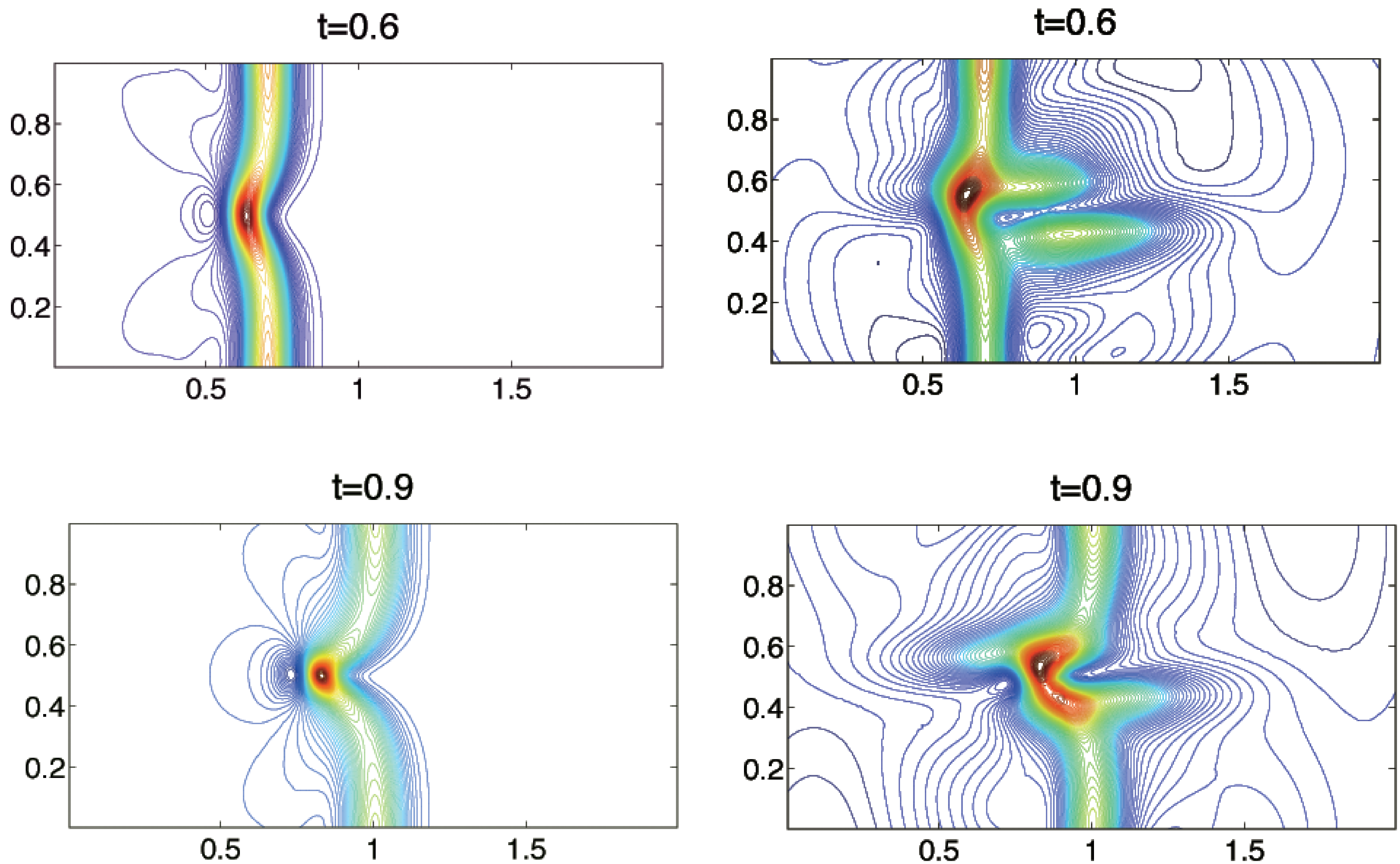
$$B(x,y) = 0.8\exp(-5(x-0.9)^2 - 50(y-0.5)^2).$$

- **Initially, the water is at rest and its surface is flat everywhere except for $0.05 < x < 0.15$:**

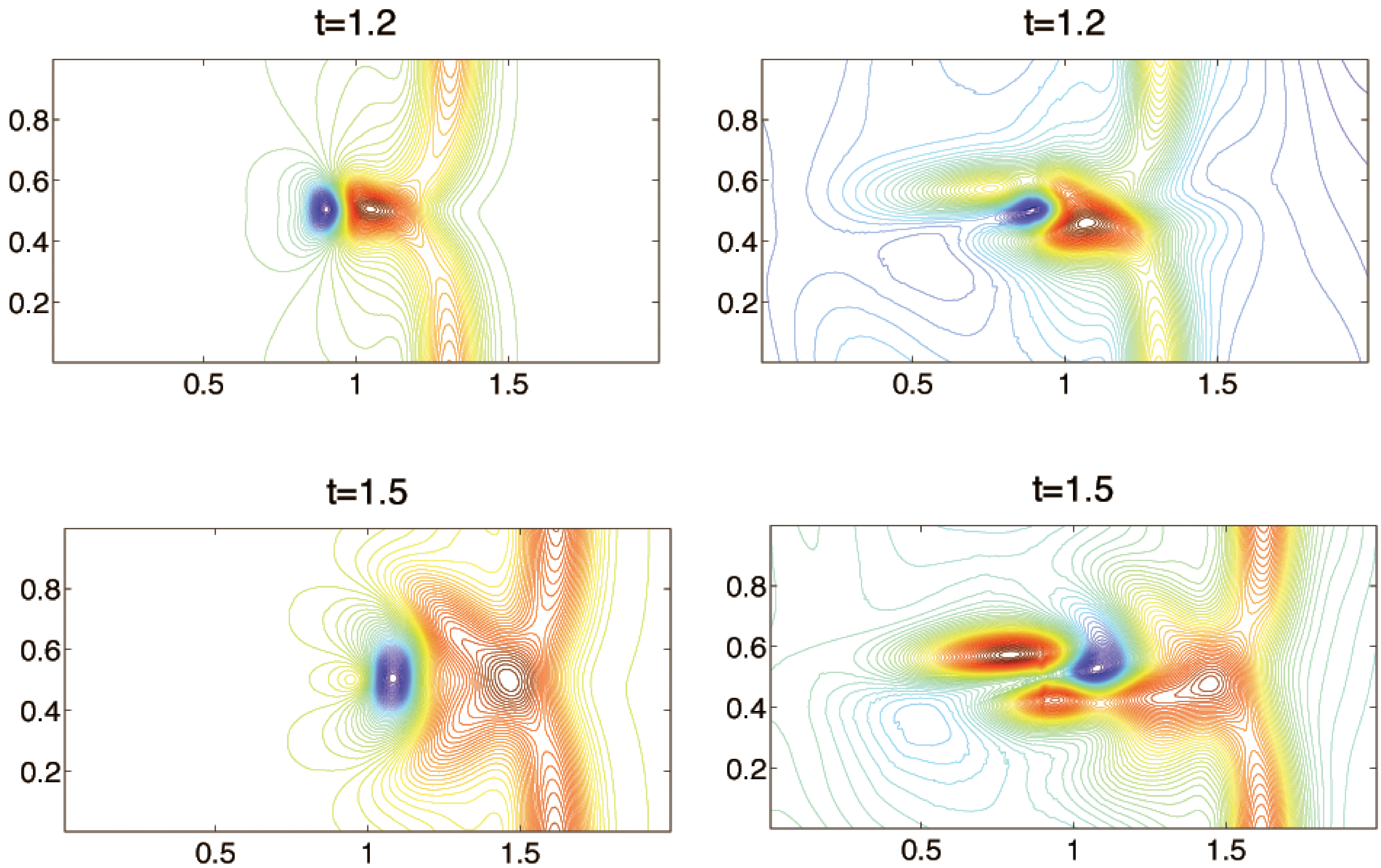
$$w(x,y,0) = \begin{cases} 1+\varepsilon, & 0.05 < x < 0.15, \\ 1, & \textbf{otherwise}, \end{cases} \qquad u(x,y,0) \equiv v(x,y,0) \equiv 0,$$

where the perturbation height is $\varepsilon = 10^{-4}$

Perturbation of a stationary steady-state: well-balanced scheme (left) and non well-balanced (right)



Perturbation of a stationary steady-state: well-balanced scheme (left) and non well-balanced (right)



Saint-Venant System with friction and discontinuous bottom

- **More realistic shallow water models include additional friction and/or viscosity terms**
- **Presence of friction and viscosity terms guarantees uniqueness of the steady state solution**
- **We consider the simplest model in which only friction terms, $-\kappa(h)u$ and $-\kappa(h)v$, are added to the rhs of the second and third equations of the Saint-Venant System**

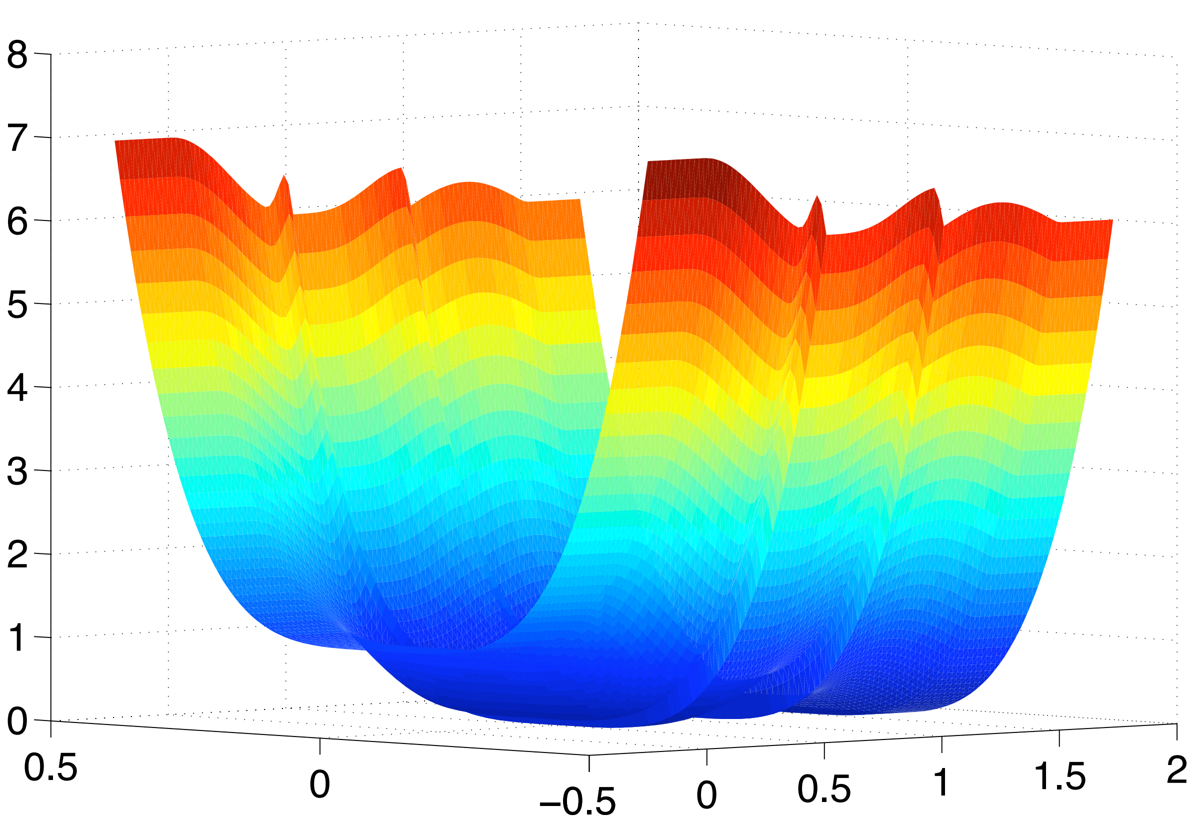
$$
\begin{cases}
h_t + (hu)_x + (hv)_y = 0, \\
(hu)_t + \left(hu^2 + \dfrac{1}{2}gh^2\right)_x + (huv)_y = -ghB_x - \kappa(h)u, \\
(hv)_t + (huv)_x + \left(hv^2 + \dfrac{1}{2}gh^2\right)_y = -ghB_y - \kappa(h)v.
\end{cases}
$$

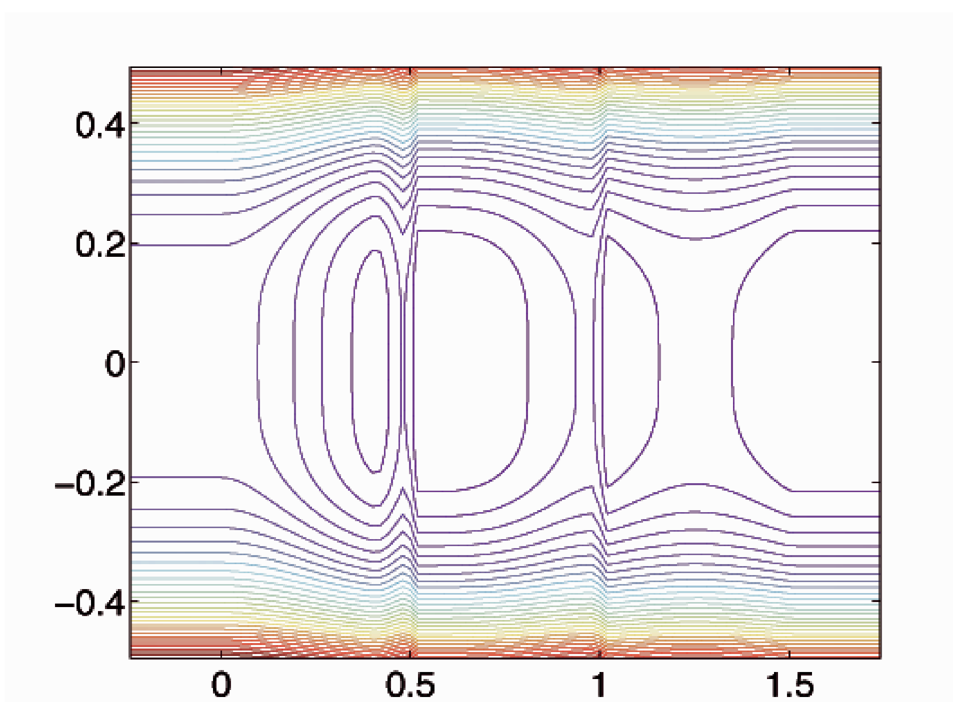
Saint-Venant System with friction and discontinuous bottom

- **We numerically solve the shallow water model with friction term on the domain** $[-0.25, 1.75] \times [-0.5, 0.5]$
- **We assume that the friction coefficient is**

$$\kappa(h) = 0.001(1 + 10h)^{-1}$$

- **The bottom topography function has a discontinuity along the vertical line** $x = 1$ **and it mimics a mountain river valley**





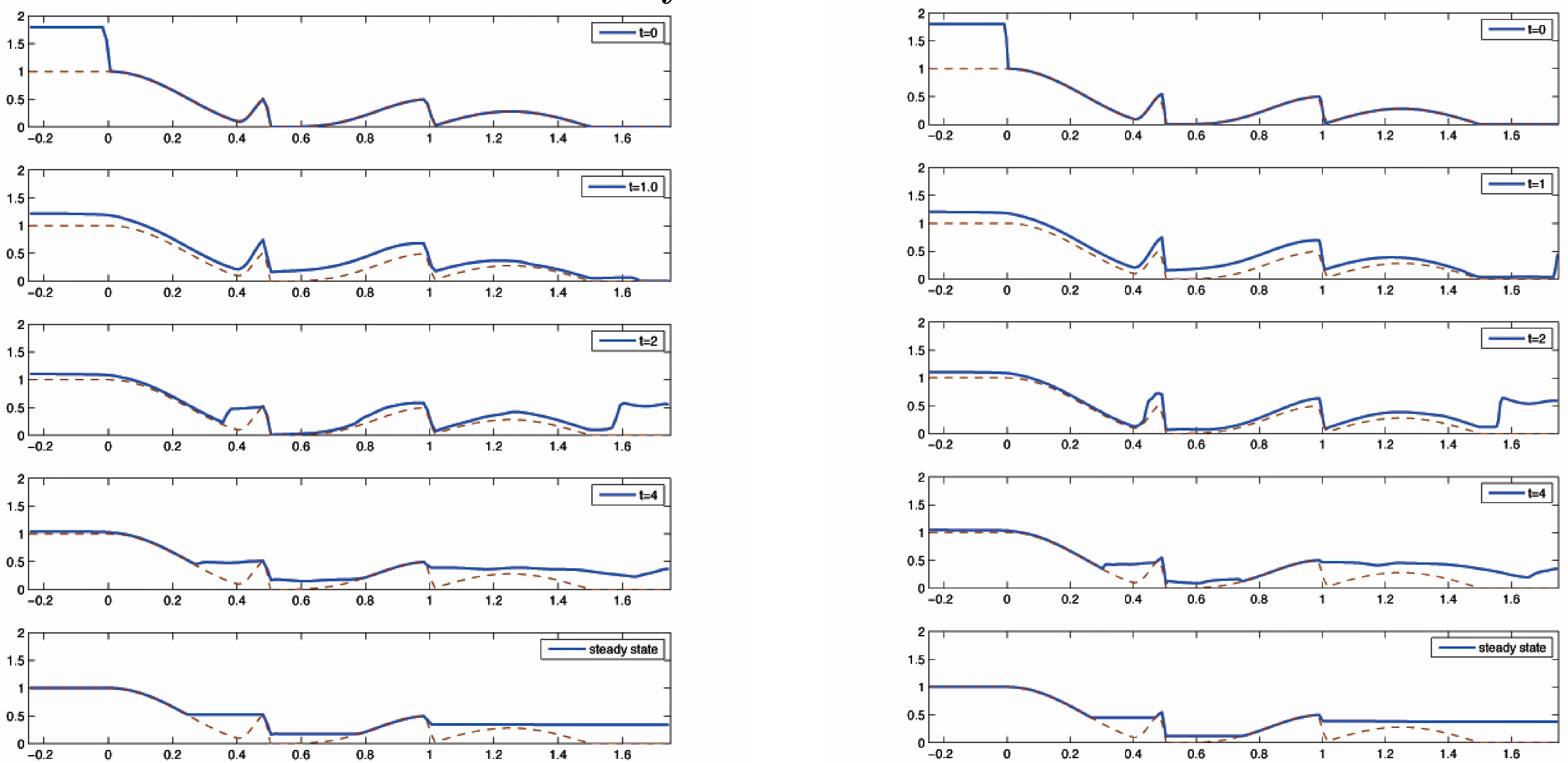
Saint-Venant System with friction and discontinuous bottom: description of the initial and boundary conditions

- **We implement reflecting (solid wall) boundary conditions at all boundaries**

- **Our initial data correspond to the situation when the second of the three dams, initially located at the vertical lines**
  $x = -0.25$ (*the left boundary of the computational domain*), $x = 0$, **and** $x = 1.75$ (*the right boundary of the computational domain*),
  breaks down at time $t = 0$**, and the water propagates into the initially dry area** $x > 0$**, and a "lake at rest" steady state is achieved after a certain period of time**

- **We plot 1-D slices of the numerical solution along the $y = 0$ line**

- **Plots clearly show the dynamics of the fluid flow as it moves from the region $x < 0$ into the initially dry area $x > 0$ and gradually settles down into a "lake at rest" steady state**



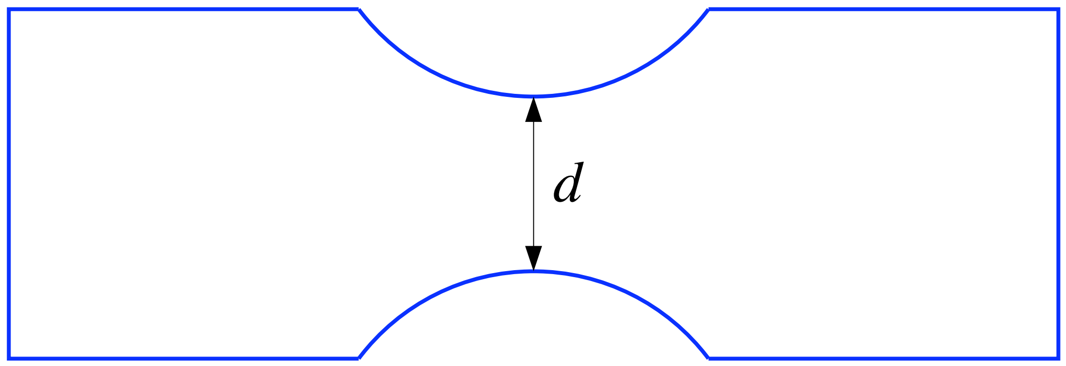
- **This state includes dry areas and therefore its computation requires a** *method that is both well-balanced and positivity preserving on the entire computational domain*

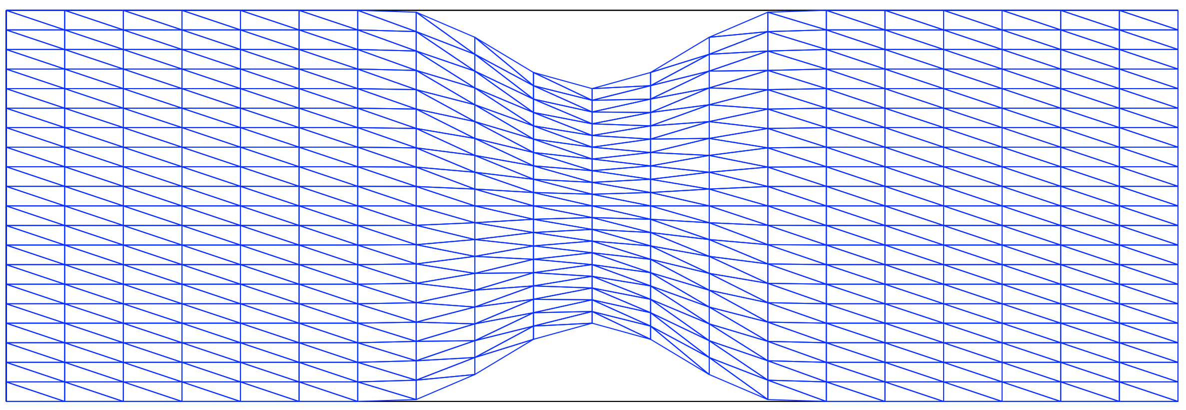
Flow in converging-diverging channel

- **The exact geometry of each channel is determined by its breadth, which is equal to $2y_{\rm b}(x)$, where**

$$y_{\rm b}(x) = \begin{cases} 0.5 - 0.5(1-d)\cos^2(\pi(x-1.5)), & |x-1.5| \le 0.5, \\ 0.5, & \textbf{otherwise}, \end{cases}$$

- **$d = 0.6$ is the minimum channel breadth**





Flow in converging-diverging channel

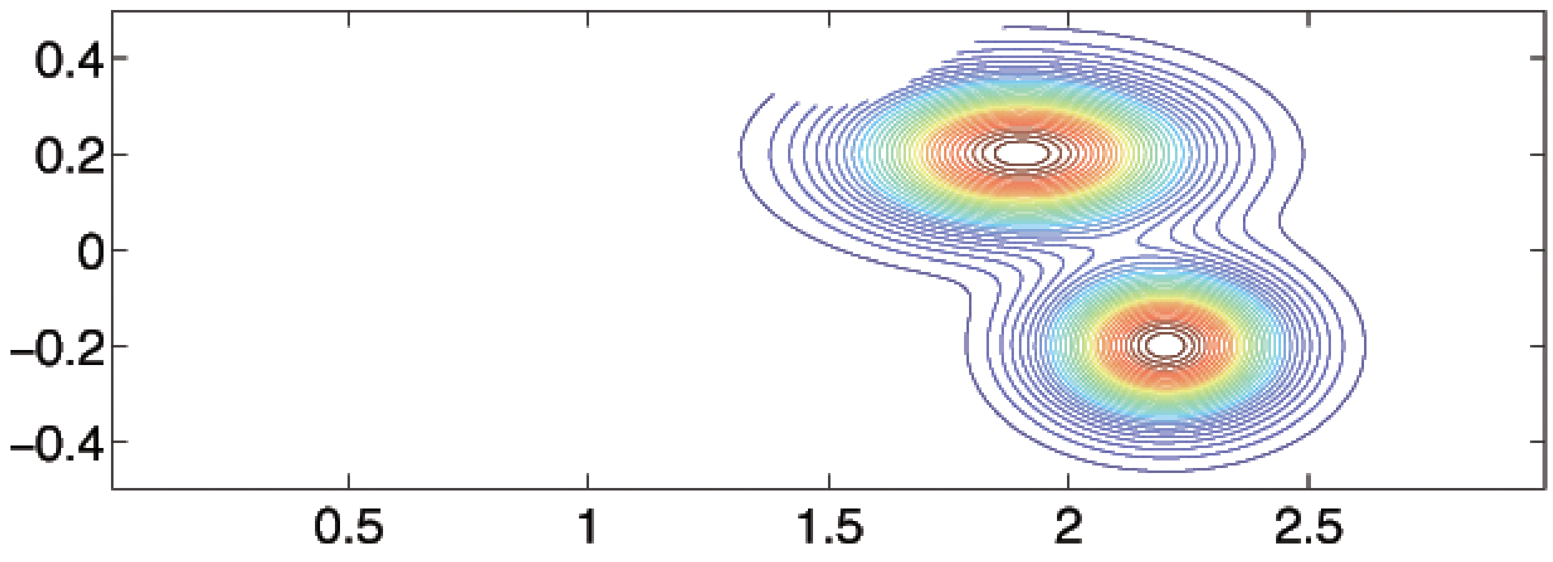
- **The initial conditions:**

$$w(x,y,0) = \max\Big\{1, B(x,y)\Big\}, \quad u(x,y,0) = 2, \quad v(x,y,0) = 0.$$

- **The upper and lower y-boundaries are reflecting (solid wall), the left x-boundary is an inflow boundary with $u = 2$ and the right x-boundary is a zero-order outflow boundary**

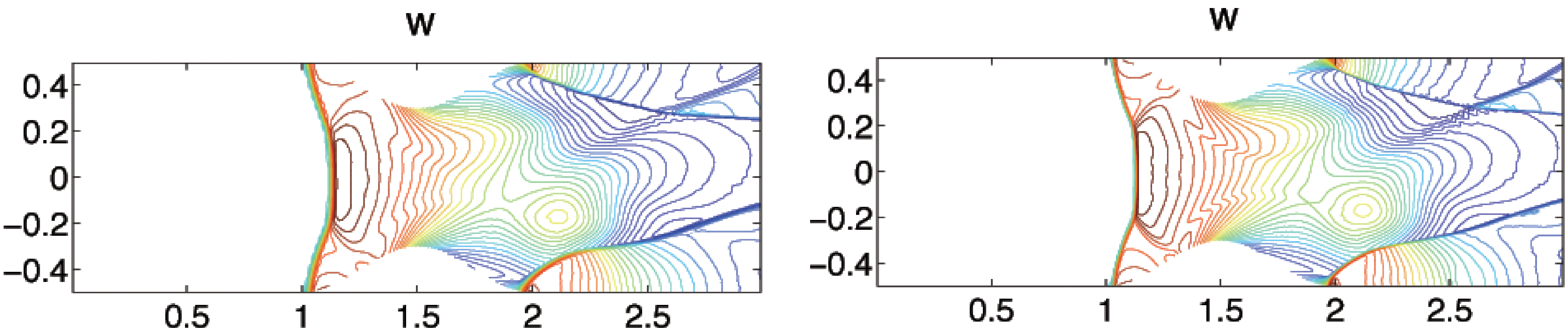
- **The bottom topography is given by**

$$B(x,y) = \left(e^{-10(x-1.9)^2-50(y-0.2)^2} + e^{-20(x-2.2)^2-50(y+0.2)^2}\right),$$



Flow in converging-diverging channel: w

Steady-state solution (w) for $(d, B_{\max}) = (0.6, 1)$ on $2 \times 200 \times 200$ (left) and $2 \times 400 \times 400$ (right) grids.



Conclusions/Difficulties

- **We developed a simple central-upwind scheme for the Saint-Venant system on triangular grids**
- **We proved that the scheme both preserves stationary steady states (lake at rest) and guarantees the positivity of the computed fluid depth**
- **It can be applied to models with discontinuous bottom topography and irregular channel widths**
- **Method is sensitive to the accuracy of the boundary representation**
- **S. Bryson, Y. Epshteyn, A. Kurganov and G. Petrova, Well-Balanced Positivity Preserving Central-Upwind Scheme on Triangular Grids for the Saint-Venant System, to appear, ESAIM: M2AN 2010.**