Central-Upwind Schemes for Shallow Water Models

Alexander Kurganov

Tulane University
Mathematics Department

www.math.tulane.edu/∼kurganov

joint work with

Andreas Bollermann, RWTH Aachen University, Germany

Doron Levy, University of Maryland

Sebastian Noelle, RWTH Aachen University, Germany

Guergana Petrova, Texas A&M University

Saint-Venant System of Shallow Water Equations

$$\begin{cases} h_t + q_x = 0 \\ q_t + \left(hu^2 + \frac{g}{2}h^2\right)_x = -ghB_x \end{cases}$$

This is a system of hyperbolic balance laws

$$\mathbf{U}_t + \mathbf{F}(\mathbf{U}, B)_x = \mathbf{S}(\mathbf{U}, B), \quad \mathbf{U} := (h, q)$$

h: depth

u: velocity

$q := hu$: discharge

B: bottom topography

g: gravitational constant

Finite-Volume Methods

1-D System: $\mathbf{U}_t + \mathbf{F}(\mathbf{U})_x = \mathbf{S}$

$\bar{\mathbf{U}}_j(t) \approx \dfrac{1}{\Delta x} \int\limits_{C_j} \mathbf{U}(x,t)\,dx$: cell averages over $C_j := (x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}})$

This solution is approximated by a piecewise linear (conservative, second-order accurate, non-oscillatory) reconstruction:

$\widetilde{\mathbf{U}}(x) = \bar{\mathbf{U}}_j + (\mathbf{U}_x)_j(x - x_j)$ for $x \in C_j$

Central-Upwind Schemes

Godunov-type central schemes with a built-in upwind nature

[Kurganov, Tadmor; 2000]

[Kurganov, Petrova; 2000]

[Kurganov, Petrova; 2001]

[Kurganov, Noelle, Petrova; 2001]

[Kurganov, Lin; 2007]

The discontinuities appearing at the reconstruction step at the interface points $\{x_{j+\frac{1}{2}}\}$ propagate at finite speeds estimated by:

$$a^+_{j+\frac{1}{2}} := \max\left\{\lambda_N\left(\frac{\partial \mathbf{F}}{\partial \mathbf{U}}(\mathbf{U}^-_{j+\frac{1}{2}})\right), \lambda_N\left(\frac{\partial \mathbf{F}}{\partial \mathbf{U}}(\mathbf{U}^+_{j+\frac{1}{2}})\right), 0\right\}$$

$$a^-_{j+\frac{1}{2}} := \min\left\{\lambda_1\left(\frac{\partial \mathbf{F}}{\partial \mathbf{U}}(\mathbf{U}^-_{j+\frac{1}{2}})\right), \lambda_1\left(\frac{\partial \mathbf{F}}{\partial \mathbf{U}}(\mathbf{U}^+_{j+\frac{1}{2}})\right), 0\right\}$$

$\lambda_1 < \lambda_2 < \ldots < \lambda_N$: N eigenvalues of the Jacobian $\frac{\partial \mathbf{F}}{\partial \mathbf{U}}$

$$\mathbf{U}^-_{j+\frac{1}{2}} := \lim_{x \to x_{j+\frac{1}{2}}-} \widetilde{\mathbf{U}}(x) = \bar{\mathbf{U}}_j + \frac{\Delta x}{2}(\mathbf{U}_x)_j$$

$$\mathbf{U}^+_{j+\frac{1}{2}} := \lim_{x \to x_{j+\frac{1}{2}}+} \widetilde{\mathbf{U}}(x) = \bar{\mathbf{U}}_{j+1} - \frac{\Delta x}{2}(\mathbf{U}_x)_{j+1}$$

1-D Semi-Discrete Central-Upwind Scheme

$$\frac{d}{dt}\bar{\mathbf{U}}_j(t) = -\frac{\mathbf{H}_{j+\frac{1}{2}}(t) - \mathbf{H}_{j-\frac{1}{2}}(t)}{\Delta x}$$

The central-upwind numerical flux is:

$$\mathbf{H}_{j+\frac{1}{2}} = \frac{a^+_{j+\frac{1}{2}}\mathbf{F}(\mathbf{U}^-_{j+\frac{1}{2}}) - a^-_{j+\frac{1}{2}}\mathbf{F}(\mathbf{U}^+_{j+\frac{1}{2}})}{a^+_{j+\frac{1}{2}} - a^-_{j+\frac{1}{2}}} + a^+_{j+\frac{1}{2}} a^-_{j+\frac{1}{2}} \left[\frac{\mathbf{U}^+_{j+\frac{1}{2}} - \mathbf{U}^-_{j+\frac{1}{2}}}{a^+_{j+\frac{1}{2}} - a^-_{j+\frac{1}{2}}} - \boxed{\mathbf{d}_{j+\frac{1}{2}}}\right]$$

The built-in "anti-diffusion" term is:

$$\mathbf{d}_{j+\frac{1}{2}} = \text{minmod}\left(\frac{\mathbf{U}^+_{j+\frac{1}{2}} - \mathbf{U}^{\text{int}}_{j+\frac{1}{2}}}{a^+_{j+\frac{1}{2}} - a^-_{j+\frac{1}{2}}}, \frac{\mathbf{U}^{\text{int}}_{j+\frac{1}{2}} - \mathbf{U}^-_{j+\frac{1}{2}}}{a^+_{j+\frac{1}{2}} - a^-_{j+\frac{1}{2}}}\right)$$

The intermediate values $\mathbf{U}^{\text{int}}_{j+\frac{1}{2}}$ are:

$$\mathbf{U}^{\text{int}}_{j+\frac{1}{2}} = \frac{a^+_{j+\frac{1}{2}}\mathbf{U}^+_{j+\frac{1}{2}} - a^-_{j+\frac{1}{2}}\mathbf{U}^-_{j+\frac{1}{2}} - \left\{\mathbf{F}(\mathbf{U}^+_{j+\frac{1}{2}}) - \mathbf{F}(\mathbf{U}^-_{j+\frac{1}{2}})\right\}}{a^+_{j+\frac{1}{2}} - a^-_{j+\frac{1}{2}}}$$

Remarks

1. $\mathbf{d}_{j+\frac{1}{2}} \equiv 0$ corresponds to the central-upwind scheme from [Kurganov, Noelle, Petrova; 2001]

2. For the system of balance laws,

$$\mathbf{U}_t + \mathbf{F}(\mathbf{U})_x = \mathbf{S},$$

the central-upwind scheme is:

$$\frac{d}{dt}\bar{\mathbf{U}}_j(t) = -\frac{\mathbf{H}_{j+\frac{1}{2}}(t) - \mathbf{H}_{j-\frac{1}{2}}(t)}{\Delta x} + \boxed{\boxed{\bar{\mathbf{S}}_j(t)}},$$

where

$$\bar{\mathbf{S}}_j(t) \approx \frac{1}{\Delta x} \int\limits_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} \mathbf{S}(x,t)\,dx$$

2-D Semi-Discrete Central-Upwind Scheme

Rectangular Grid

- [Kurganov, Petrova; 2001]
- [Kurganov, Noelle, Petrova; 2001]
- [Kurganov, Lin; 2007]

Triangular Grid

- [Kurganov, Petrova; 2004]

Shallow Water Equations — Numerical Challenges

$$\begin{cases} h_t + q_x = 0 \\ q_t + \left(hu^2 + \frac{g}{2}h^2\right)_x = -ghB_x \end{cases}$$

- Steady-state solutions:

$$q = \text{Const}, \quad \frac{u^2}{2} + g(h+B) = \text{Const}$$

- Stationary steady-state solutions (lake at rest):

$$u = 0, \quad h + B = \text{Const}$$

- Dry ($h = 0$) or near dry ($h \sim 0$) states

Shallow Water Equations — Naïve Source Approximation

$$\begin{cases} h_t + q_x = 0 \\ q_t + \left(hu^2 + \frac{g}{2}h^2\right)_x = -ghB_x \end{cases}$$

$$\frac{d}{dt}\bar{\mathbf{U}}_j(t) = -\frac{\mathbf{H}_{j+\frac{1}{2}}(t) - \mathbf{H}_{j-\frac{1}{2}}(t)}{\Delta x} + \boxed{\boxed{\bar{\mathbf{S}}_j(t)}}, \qquad \bar{\mathbf{U}}_j(t) := (\bar{h}_j(t), \bar{q}_j(t))^T$$

where we use the midpoint quadrature:

$$\bar{\mathbf{S}}_j \approx \frac{1}{\Delta x} \int\limits_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} \mathbf{S}(\mathbf{U}(x,t), B(x))\, dx \approx \mathbf{S}(\mathbf{U}_j(t), B(x_j)),$$

that is, we take

$$\bar{\mathbf{S}}_j = (0, -g\bar{h}_j B_x(x_j))^T$$

Example — Small Perturbation of a Steady State

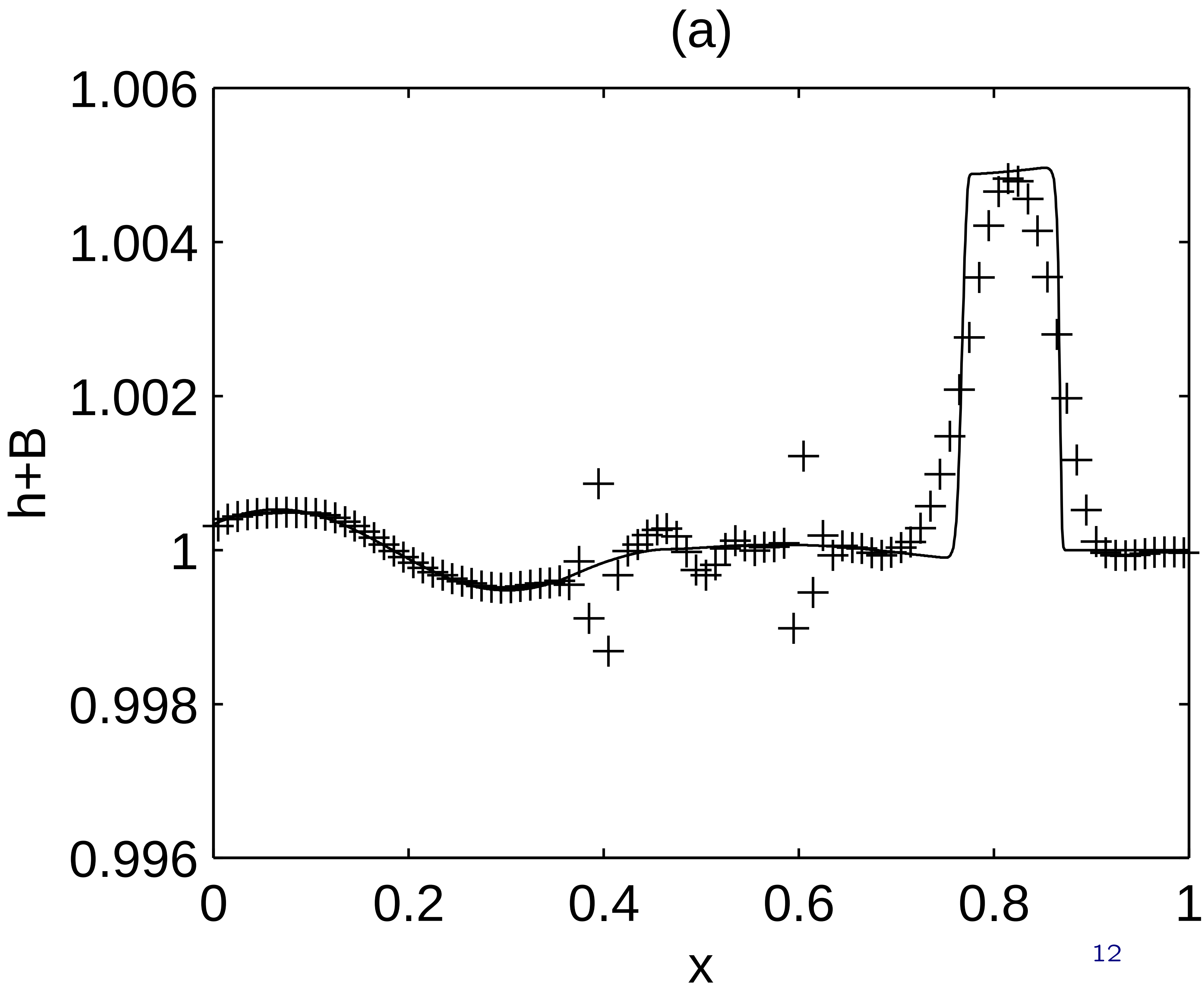
The bottom topography contains a "hump":

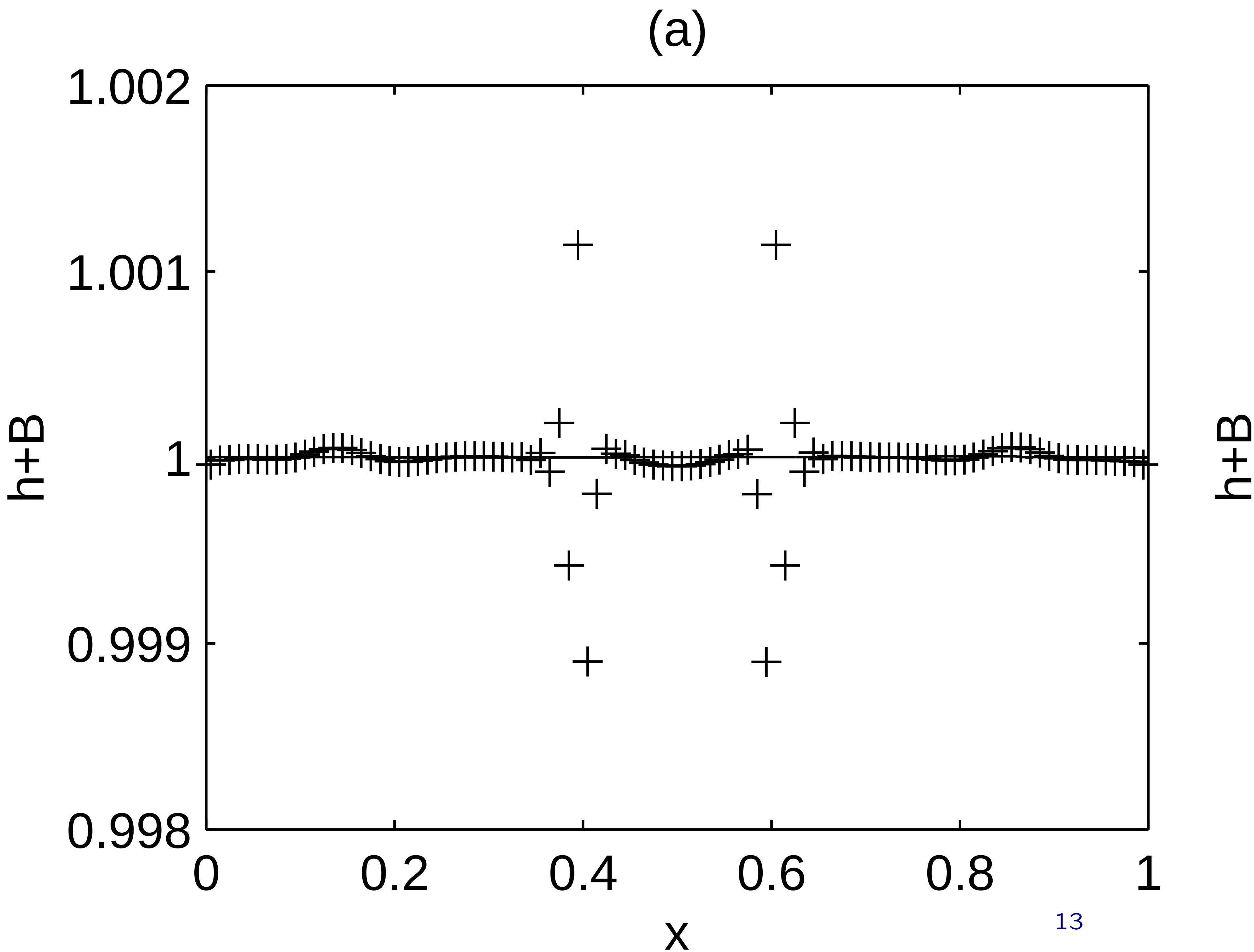
$$B(x) = \begin{cases} 0.25(\cos(\pi(x-0.5)/0.1)+1), & 0.4 < x < 0.6 \\ 0, & \text{otherwise} \end{cases}$$

The initial data are:

$$h(x,0)+B(x) = \begin{cases} 1+\varepsilon, & 0.1 < x < 0.2, \\ 1, & \text{otherwise,} \end{cases} \qquad u(x,0)=0$$

$\varepsilon = 10^{-2}$ and $\varepsilon = 10^{-5}$


(a)
1.006
1.004
1.002
1
0.998
0.996
h+B
0
0.2
0.4
0.6
0.8
1
x



(a)
1.002
1.001
1
0.999
0.998
0
0.2
0.4
0.6
0.8
1
h+B
x

Well-Balanced Central-Upwind Scheme

[Kurganov, Levy; 2002]

$w = h + B$: water surface

$$\begin{cases} h_t + q_x = 0 \\ q_t + \left(hu^2 + \frac{g}{2}h^2\right)_x = -ghB_x \end{cases}$$

$$\Updownarrow \qquad (h,q) \to (w,q)$$

$$\begin{cases} w_t + q_x = 0 \\ q_t + \left(\frac{q^2}{w-B} + \frac{g}{2}(w-B)^2\right)_x = -g(w-B)B_x \end{cases}$$

Stationary steady-states: $\boxed{u = 0, \ w = \text{Const}}$

At the stationary steady state: $q = 0,\ w = \text{Const}$

$\Longrightarrow$ the flux is $\mathbf{F} = (q, \frac{q^2}{w-B} + \frac{g}{2}(w-B)^2)^T = (0, \frac{g}{2}(w-B)^2)^T$

$\Longrightarrow$ the second component of the numerical flux is

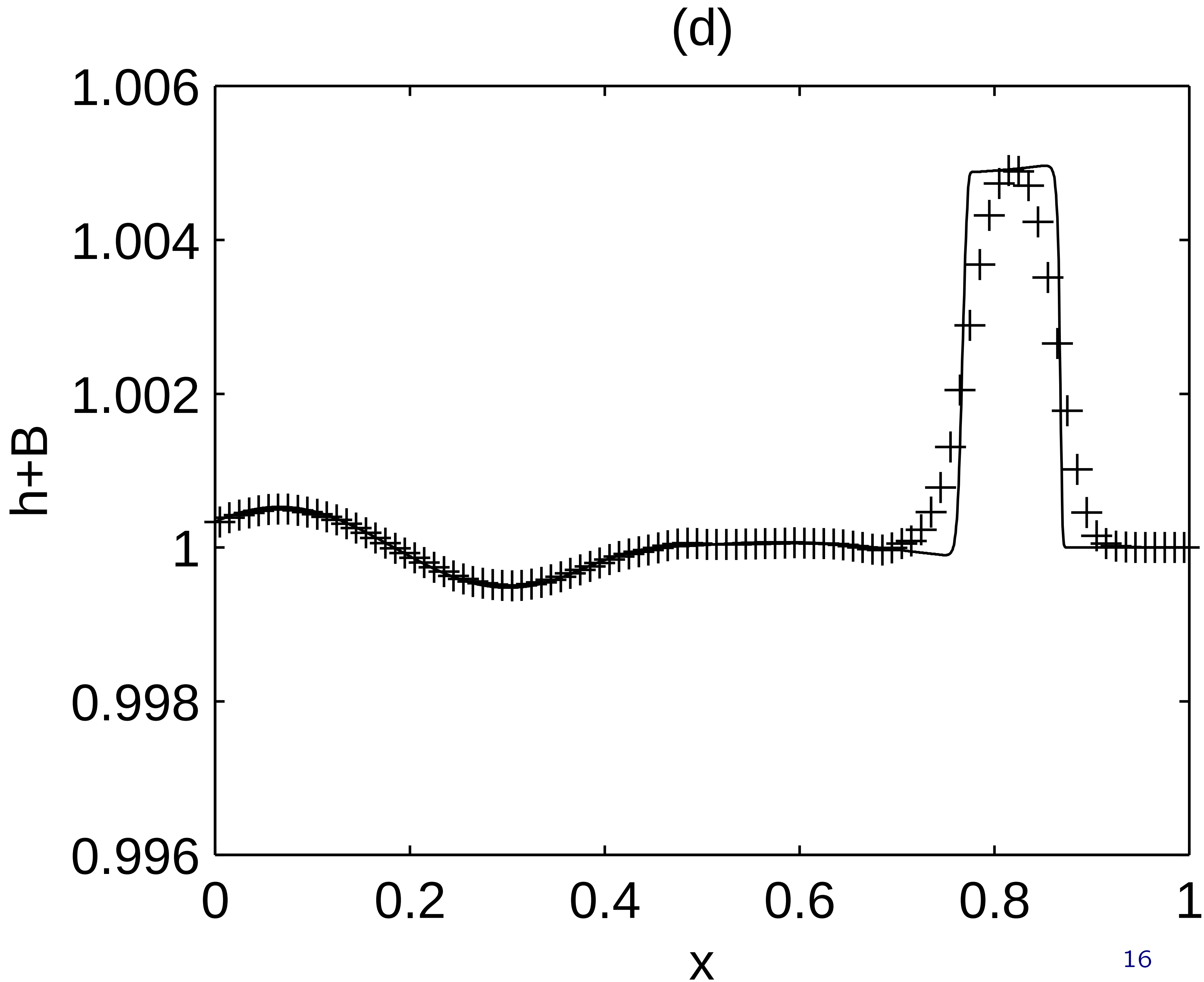
$$\mathbf{H}^{(2)}_{j+\frac{1}{2}} = \frac{g}{2}\left(w - B(x_{j+\frac{1}{2}})\right)^2, \quad \mathbf{H}^{(2)}_{j-\frac{1}{2}} = \frac{g}{2}\left(w - B(x_{j-\frac{1}{2}})\right)^2$$

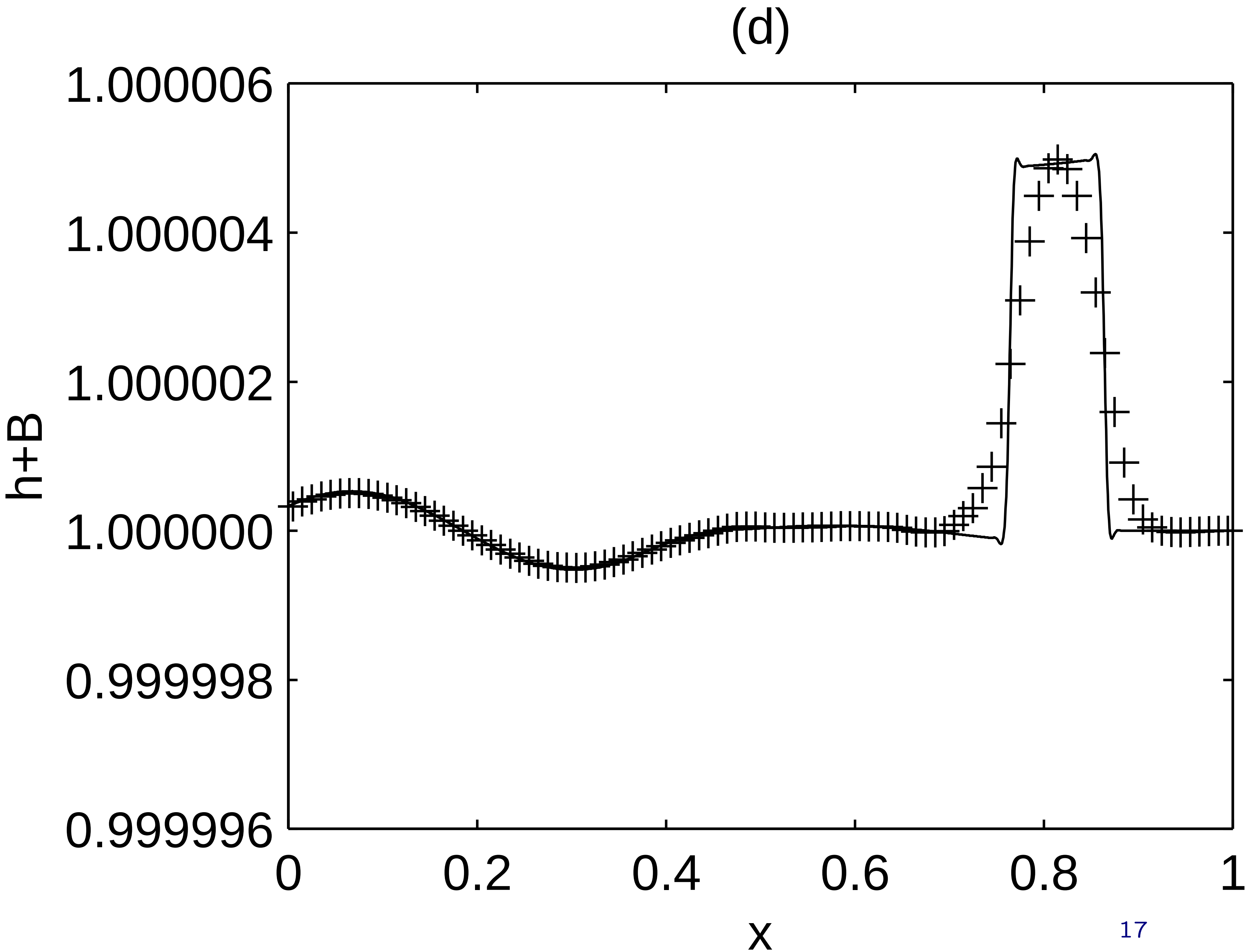
$$\Longrightarrow \frac{d}{dt}\bar{q}_j(t) = -\frac{\mathbf{H}^{(2)}_{j+\frac{1}{2}}(t) - \mathbf{H}^{(2)}_{j-\frac{1}{2}}(t)}{\Delta x} + \bar{\mathbf{S}}^{(2)}_j(t)$$

$$= g \cdot \frac{B(x_{j+\frac{1}{2}}) - B(x_{j-\frac{1}{2}})}{\Delta x} \cdot \frac{(w - B(x_{j+\frac{1}{2}})) + (w - B(x_{j-\frac{1}{2}}))}{2} + \bar{\mathbf{S}}^{(2)}_j(t)$$

$\Longrightarrow$ The well-balanced quadrature is

$$\bar{\mathbf{S}}^{(2)}_j(t) = -g \cdot \frac{B(x_{j+\frac{1}{2}}) - B(x_{j-\frac{1}{2}})}{\Delta x} \cdot \left(\bar{w}_j - \frac{B(x_{j+\frac{1}{2}}) + B(x_{j-\frac{1}{2}})}{2}\right)$$

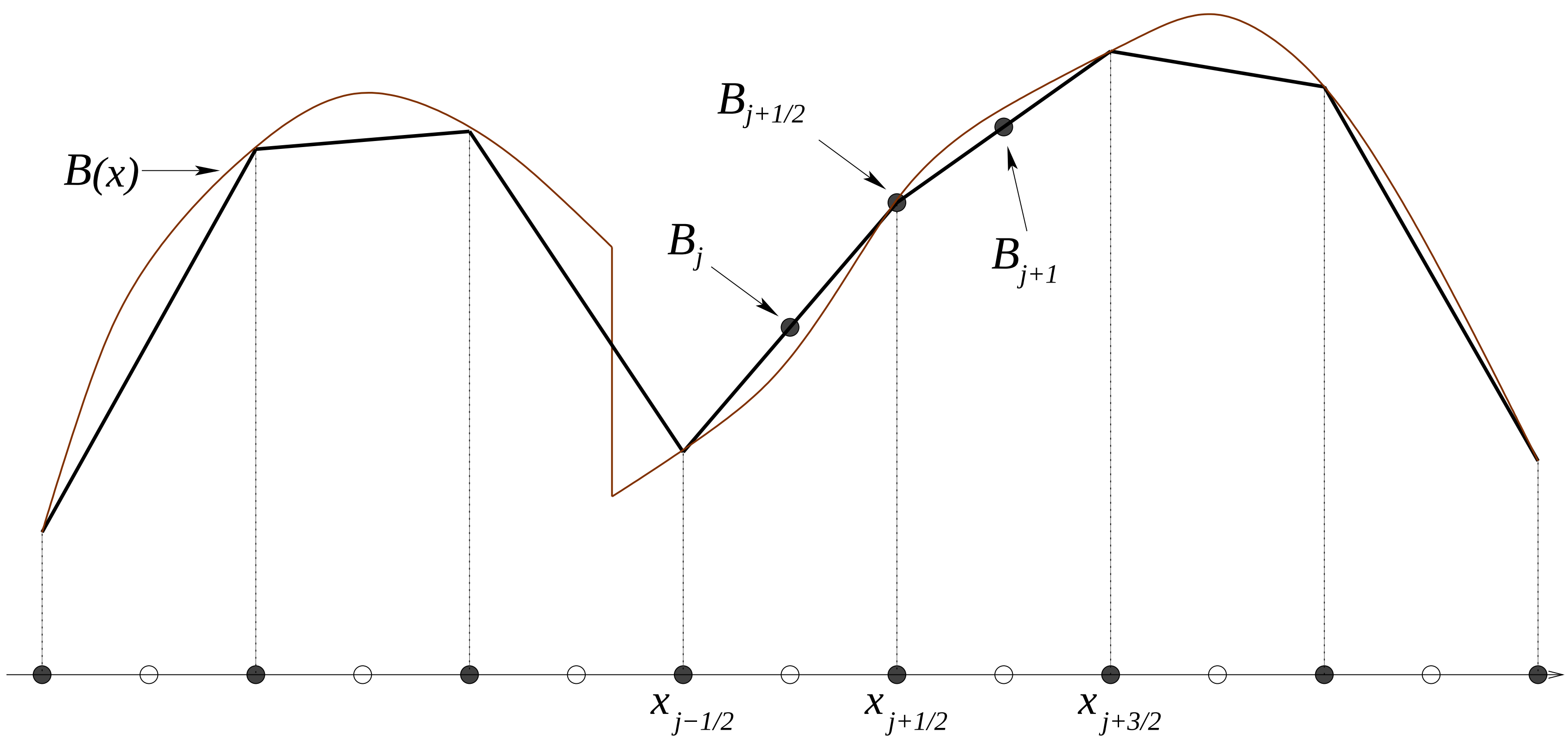

(d)
1.006
1.004
1.002
1
0.998
0.996
h+B
0
0.2
0.4
0.6
0.8
1
x


(d)
1.000006
1.000004
1.000002
1.000000
0.999998
0.999996
h+B
0
0.2
0.4
0.6
0.8
1
x

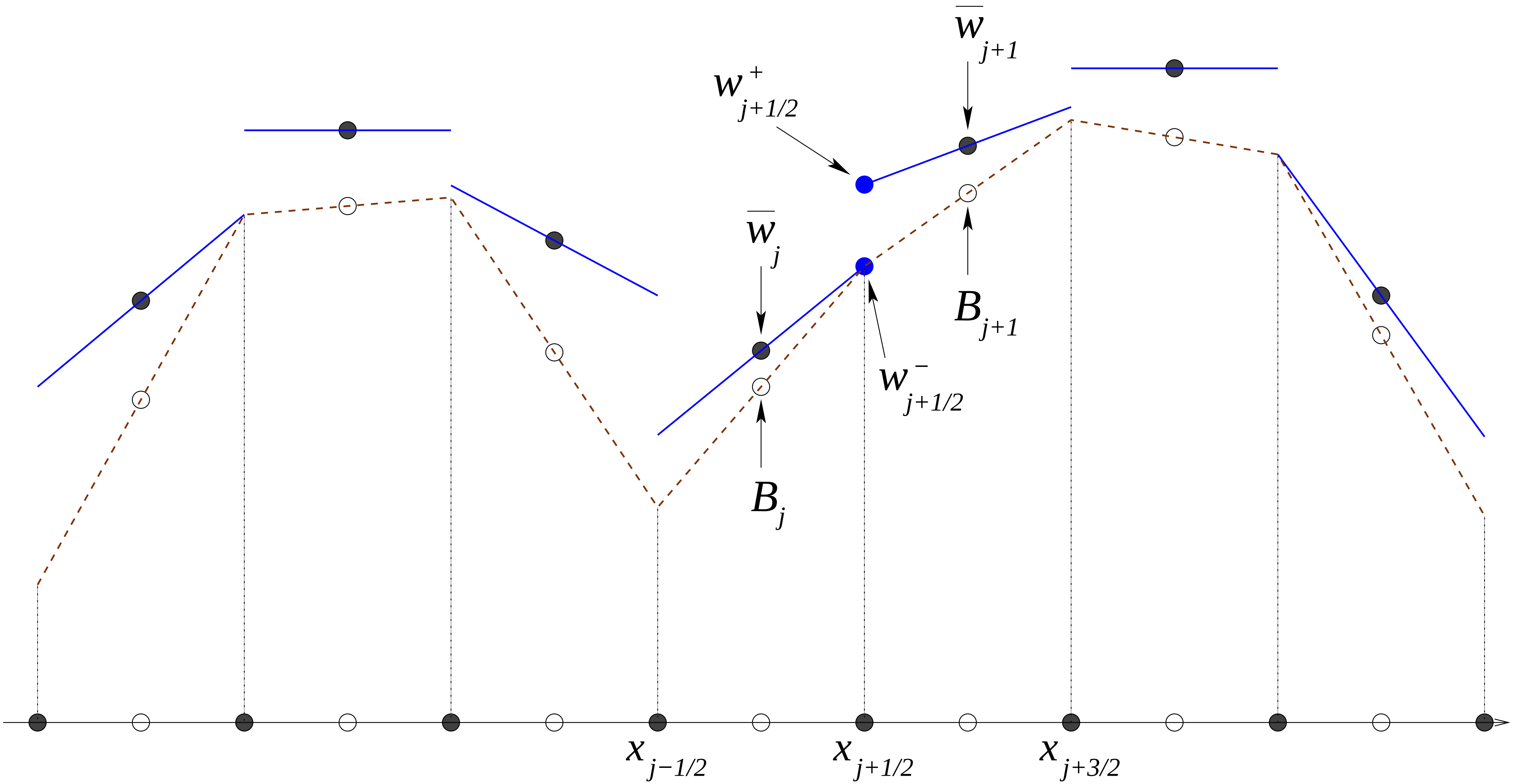
Well-Balanced Positivity Preserving Central-Upwind Scheme

[Kurganov, Petrova; 2007]

Step 1: Piecewise linear reconstruction of the bottom



Step 2: Positivity preserving reconstruction of w



Remarks

1. $u \neq \dfrac{hu}{h}$, but $u = \dfrac{\sqrt{2}\, h\,(hu)}{\sqrt{h^4 + \max(h^4, \varepsilon)}}$

2. $hu \neq q$, but $hu = h \cdot u$

3. We have proved that if an SSP ODE solver is used, then

$$\overline{h}_j^{n+1} = \alpha^-_{j-\frac{1}{2}} h^-_{j-\frac{1}{2}} + \alpha^+_{j-\frac{1}{2}} h^+_{j-\frac{1}{2}} + \alpha^-_{j+\frac{1}{2}} h^-_{j+\frac{1}{2}} + \alpha^+_{j+\frac{1}{2}} h^+_{j+\frac{1}{2}}$$

where the coefficients $\alpha^{\pm}_{j\pm\frac{1}{2}} > 0$ provided a "1/2" ("1/4" in 2-D) CFL condition is satisfied.

This guarantees positivity of h

Example — Small Perturbation of a Steady State

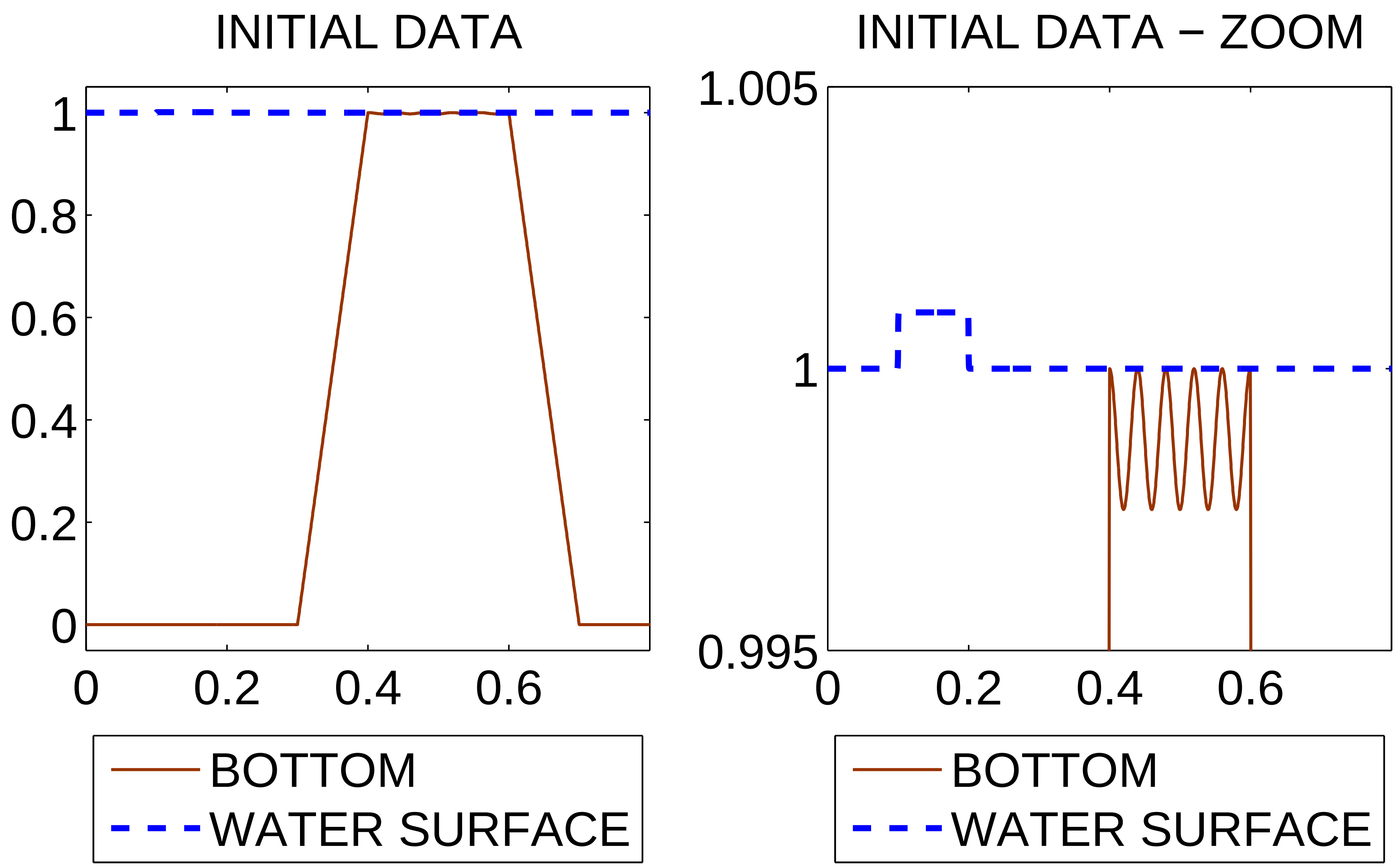
The bottom topography is:

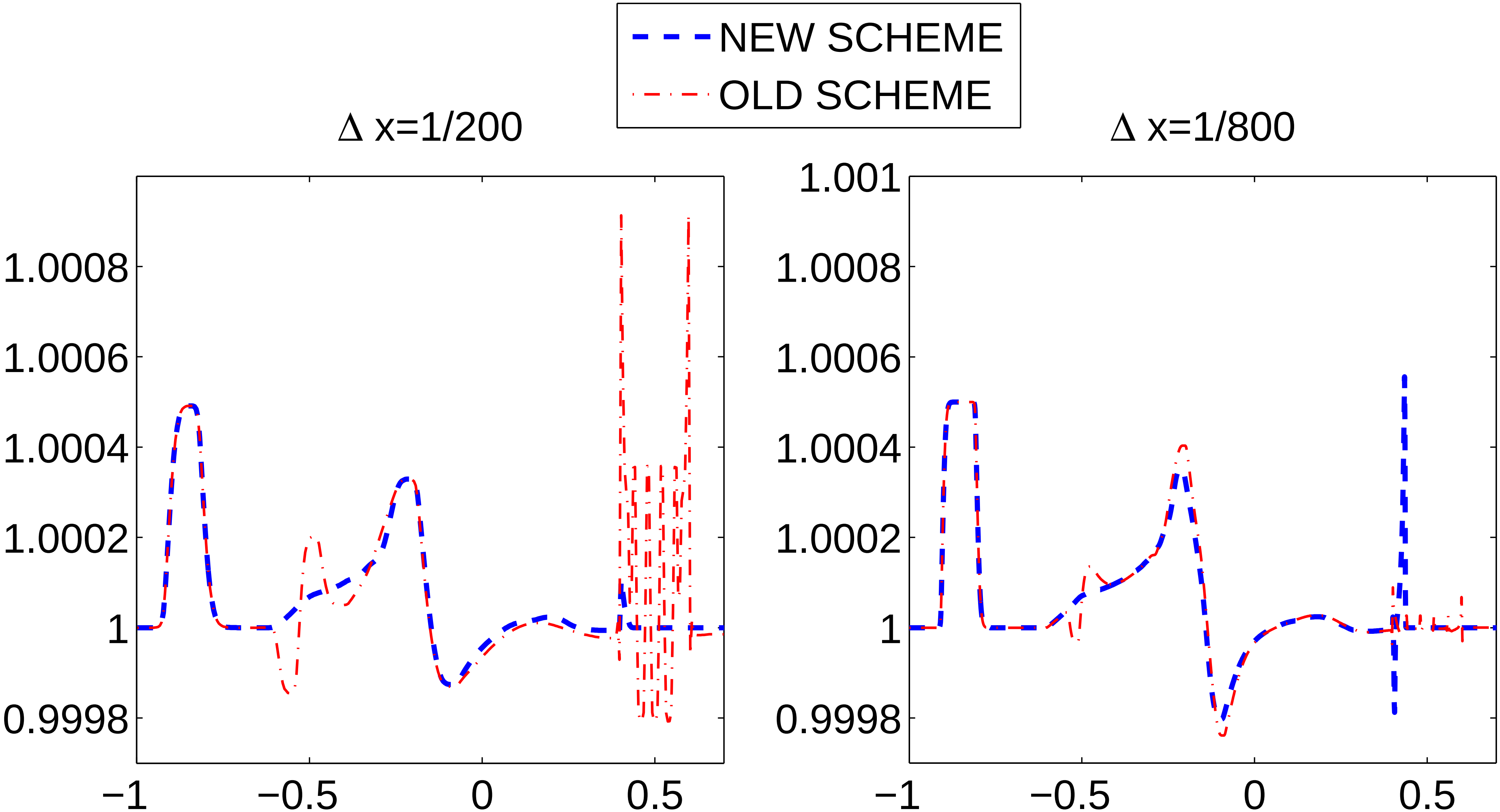
$$B(x) = \begin{cases} 10(x-0.3), & 0.3 \le x \le 0.4 \\ 1 - 0.0025\sin^2(25\pi(x-0.4)), & 0.4 \le x \le 0.6 \\ -10(x-0.7), & 0.6 \le x \le 0.7 \\ 0, & \text{otherwise} \end{cases}$$

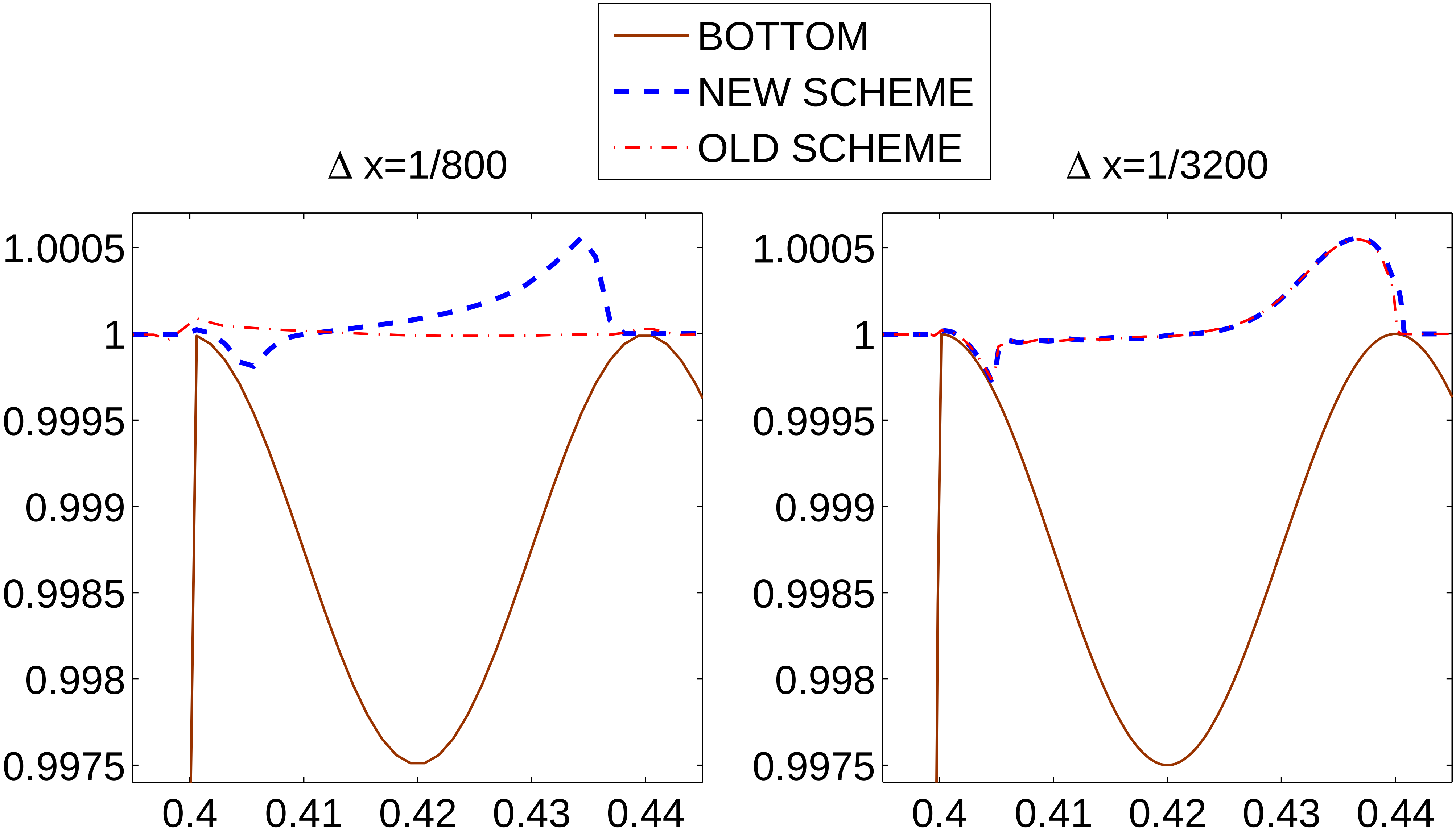
The initial data are:

$$(w(x,0), u(x,0)) = \begin{cases} (1+\varepsilon, 0), & 0.1 < x < 0.2 \\ (1,0), & \text{otherwise} \end{cases}$$

$\varepsilon = 10^{-3}$


INITIAL DATA
1
0.8
0.6
0.4
0.2
0
0
0.2
0.4
0.6
BOTTOM
WATER SURFACE
INITIAL DATA − ZOOM
1.005
1
0.995
0
0.2
0.4
0.6
BOTTOM
WATER SURFACE

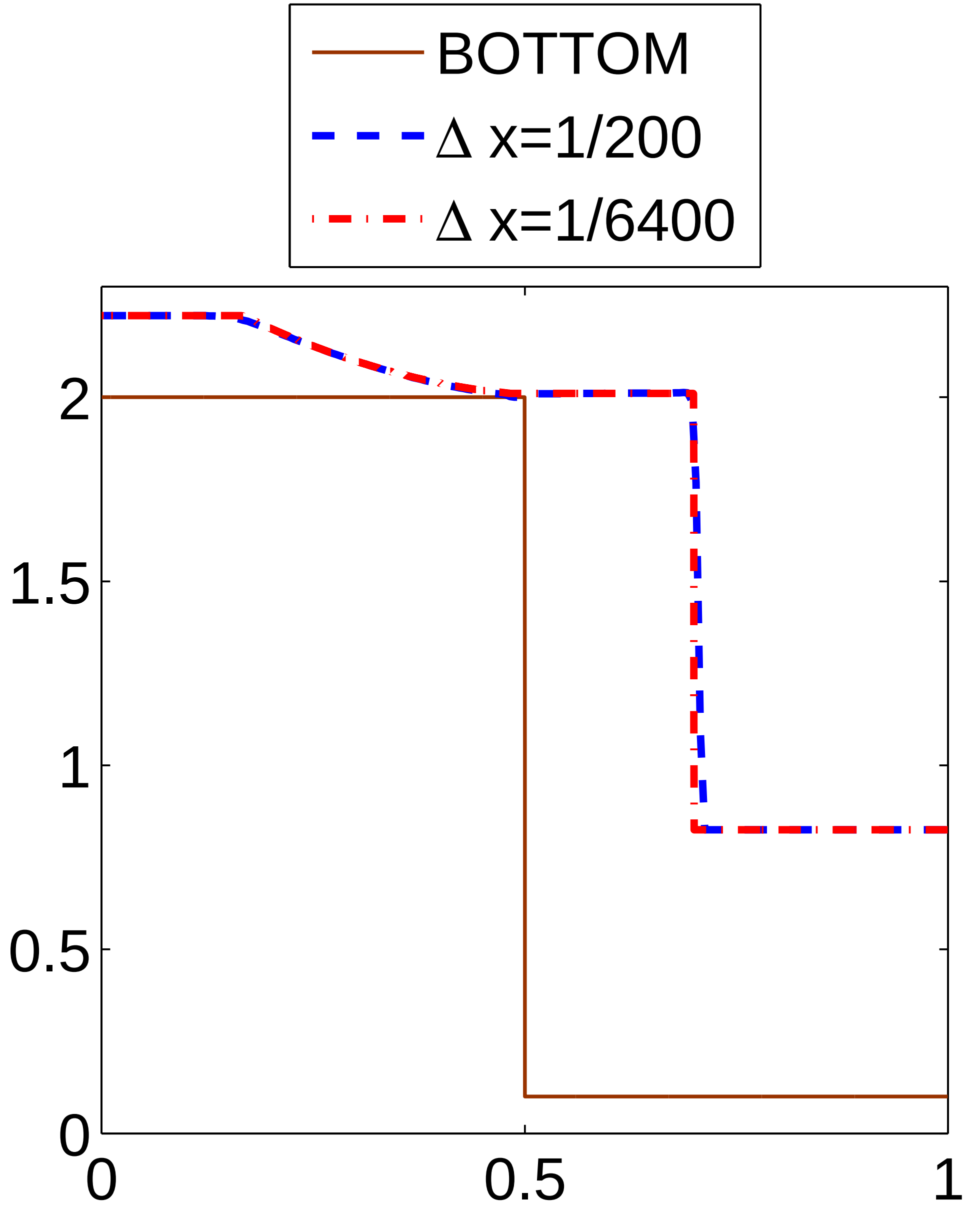

NEW SCHEME
OLD SCHEME
Δ x=1/200
Δ x=1/800
1.001
1.0008
1.0006
1.0004
1.0002
1
0.9998
−1
−0.5
0
0.5

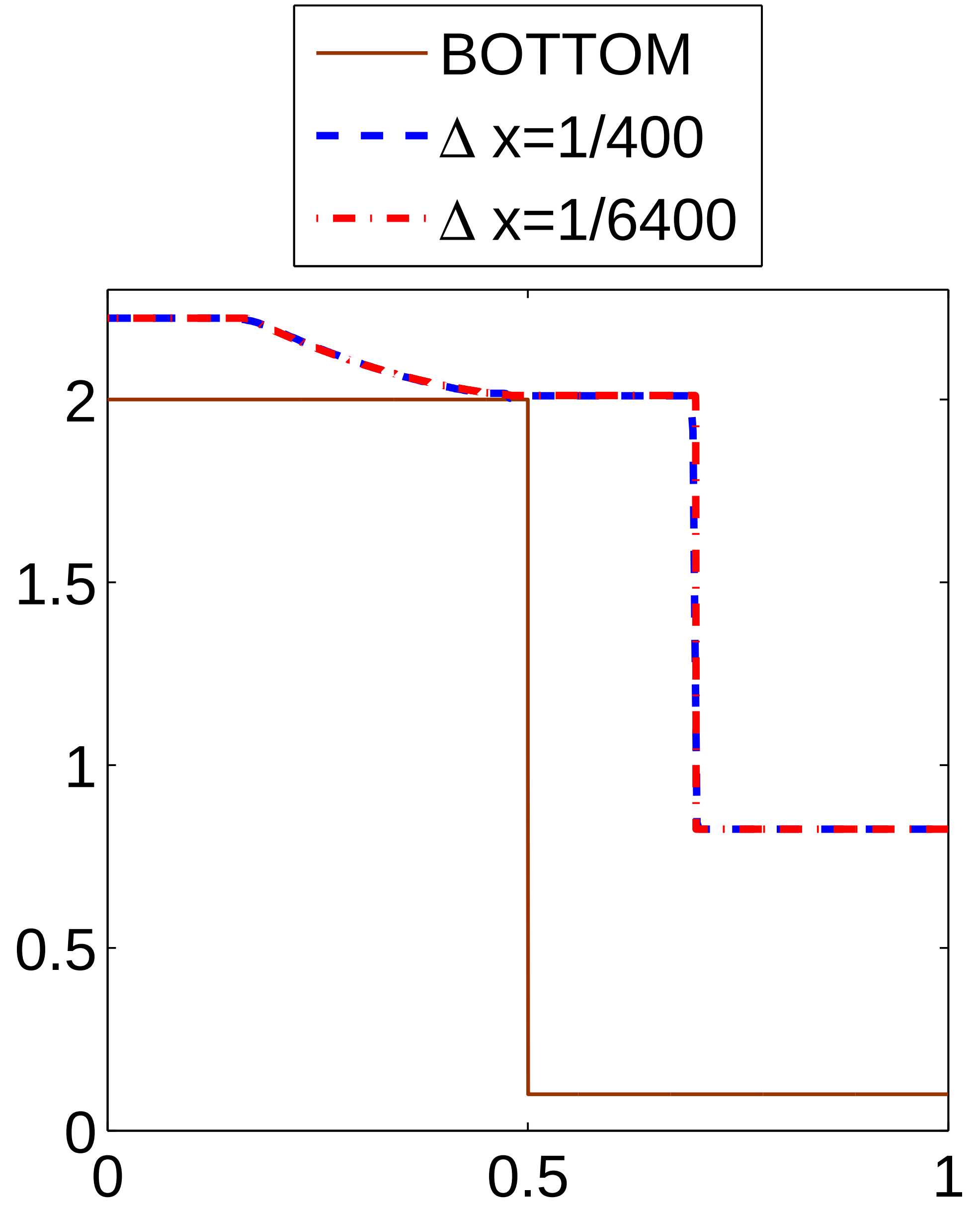

BOTTOM
NEW SCHEME
OLD SCHEME
Δ x=1/800
Δ x=1/3200
1.0005
1
0.9995
0.999
0.9985
0.998
0.9975
0.4
0.41
0.42
0.43
0.44

Example —Dry State and Discontinuous Bottom

$$B(x) = \begin{cases} 2, & x \leq 0.5 \\ 0.1, & x > 0.5 \end{cases}$$

$$(w(x,0), u(x,0)) = \begin{cases} (2.222, -1), & x \leq 0.5 \\ (0.8246, -1.6359), & x > 0.5 \end{cases}$$


BOTTOM
Δ x=1/200
Δ x=1/6400
0
0.5
1
1.5
2
0
0.5
1

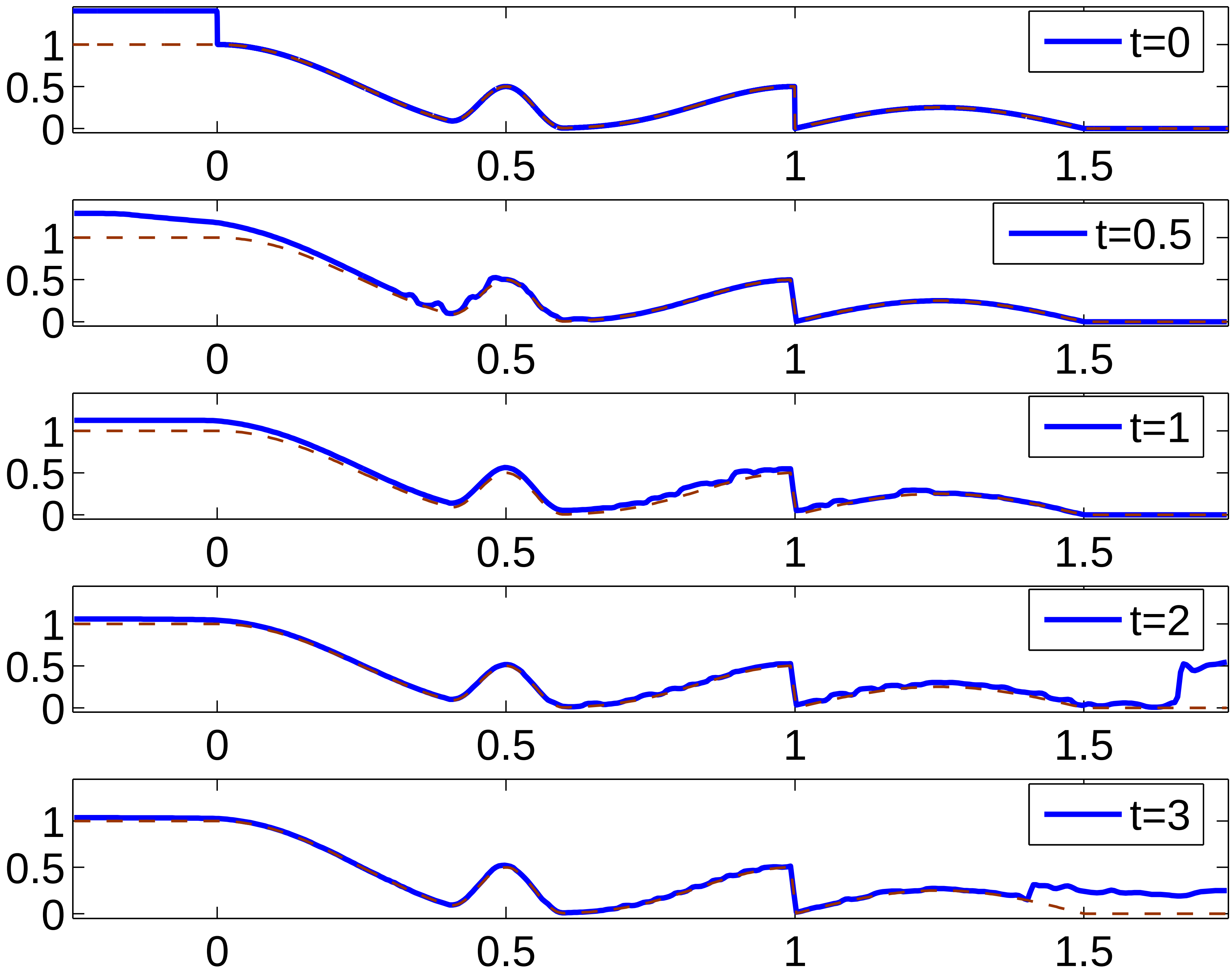

BOTTOM
Δ x=1/400
Δ x=1/6400
0
0.5
1
1.5
2
0
0.5
1

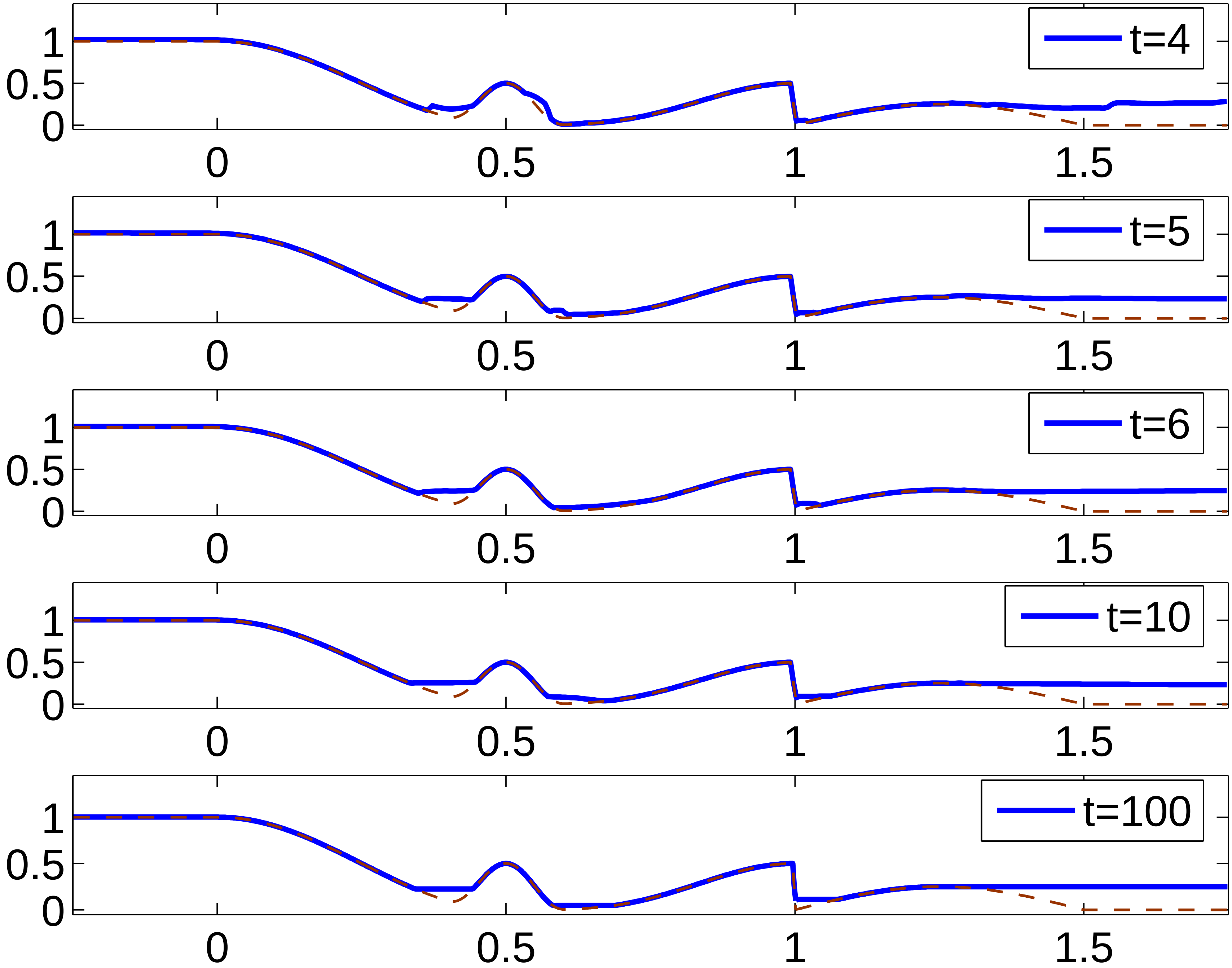
Example — ShW with Friction and Discontinuous Bottom

$$\begin{cases} h_t + q_x = 0 \\ q_t + \left(hu^2 + \frac{g}{2}h^2\right)_x = -ghB_x - \boxed{\boxed{\kappa(h)u}}, \end{cases} \qquad \kappa(h) = \frac{0.001}{1+10h}$$

$$B(x) = \begin{cases} 1, & x < 0 \\ \cos^2(\pi x), & 0 \le x \le 0.4 \\ \cos^2(\pi x) + 0.25(\cos(10\pi(x-0.5)) + 1), & 0.4 \le x \le 0.5 \\ 0.5\cos^4(\pi x) + 0.25(\cos(10\pi(x-0.5)) + 1), & 0.5 \le x \le 0.6 \\ 0.5\cos^4(\pi x), & 0.5 \le x < 1 \\ 0.25\sin(2\pi(x-1)), & 1 < x \le 1.5 \\ 0, & x > 1.5. \end{cases}$$

$$(w(x,0), u(x,0)) = \begin{cases} (1.4, 0), & x < 0 \\ (B(x), 0), & x > 0 \end{cases} \qquad \text{(Dam break)}$$


t=0
t=0.5
t=1
t=2
t=3
1
0.5
0
0
0.5
1
1.5



t=4
t=5
t=6
t=10
t=100
1
0.5
0
0
0.5
1
1.5

Water (Tsunami) Waves Generated by Submarine Landslides

[Kurganov, Petrova; 2008]

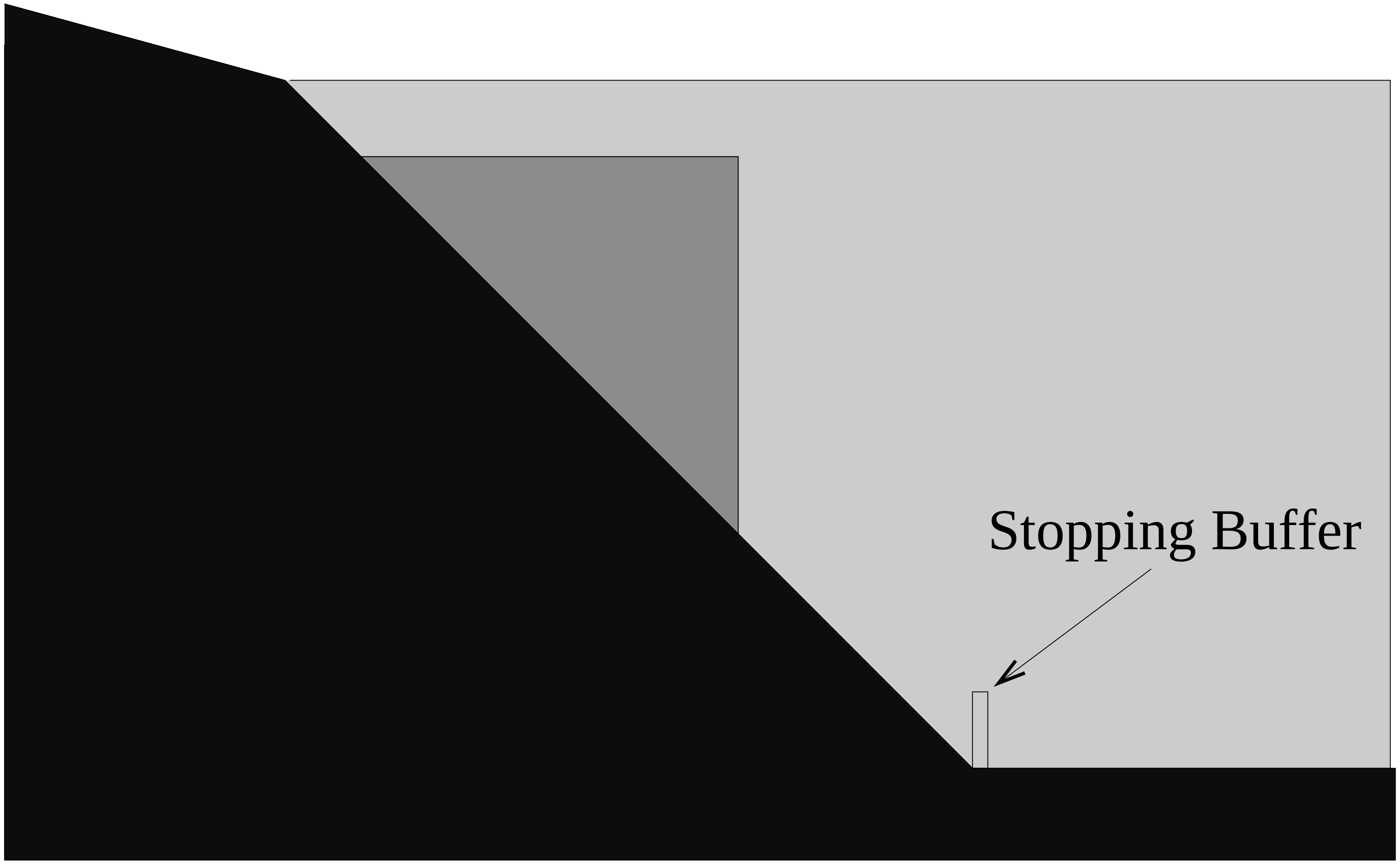
Landslides are natural phenomena that occur under certain conditions such as erosion, earthquakes, storms, heavy rainfalls, or water level fluctuations.

Landslides are capable of generating several types of long waves, including very powerful and destructive tsunami waves.

Simplest approach: The landslide is modeled as a rigid bump translating down the side of the bottom while the water motion is modeled by the shallow water equations.

The initial condition is:

$$h(x,0) = \max\{1 - B(x,0), 0\}, \quad u(x,0) \equiv 0$$



We used the shallow water equations with friction:

$$\begin{cases} h_t + (hu)_x = 0 \\ (hu)_t + \left(hu^2 + \dfrac{gh^2}{2}\right)_x = -ghB_x - \kappa(h)u \end{cases}$$

The friction coefficient used in numerical experiments is:

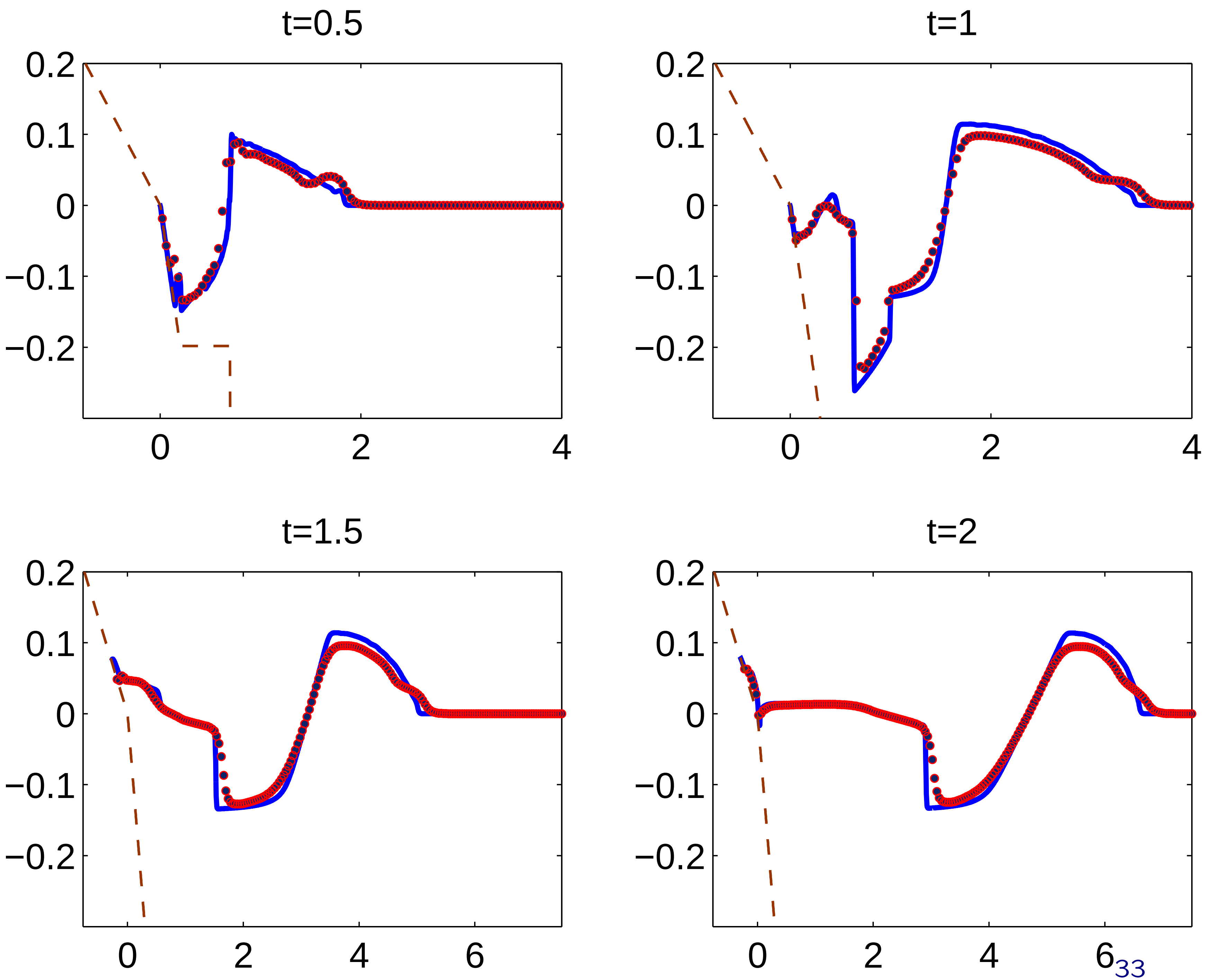
$$\boxed{\kappa(h) = \frac{0.1}{1 + 10h}}$$

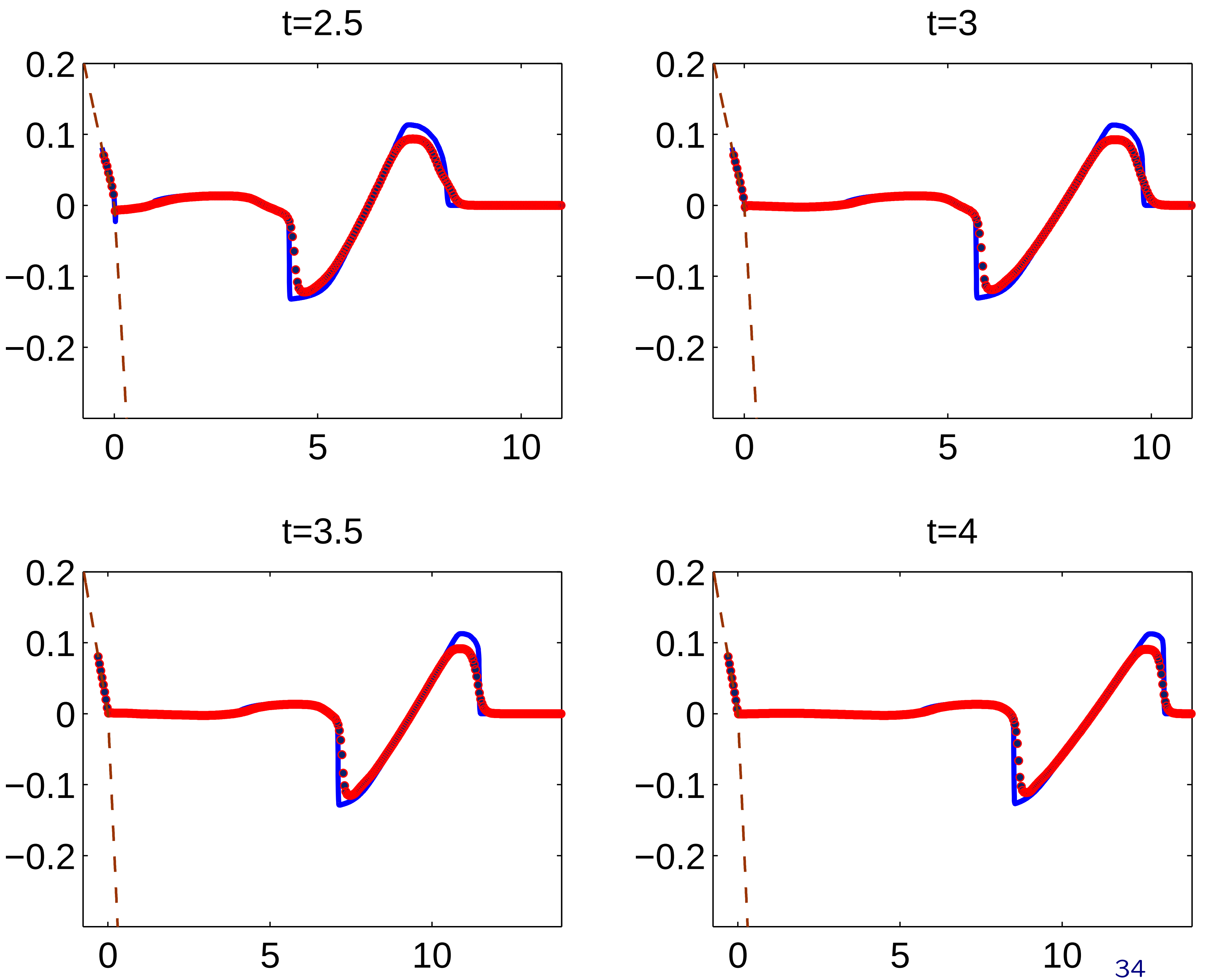
The motion of the triangular box is described by the displacement of its center of mass from its initial position:

$$S(t) = \frac{\sqrt{2\pi(\gamma+1)}}{4} \ln\left(\cosh\left(\frac{t\sqrt{2g(\gamma-1)}}{(\gamma+1)\sqrt{\pi}}\right)\right)$$

$\gamma = 1.925$: ratio of the bulk and water densities

$g = 9.812$: gravity constant

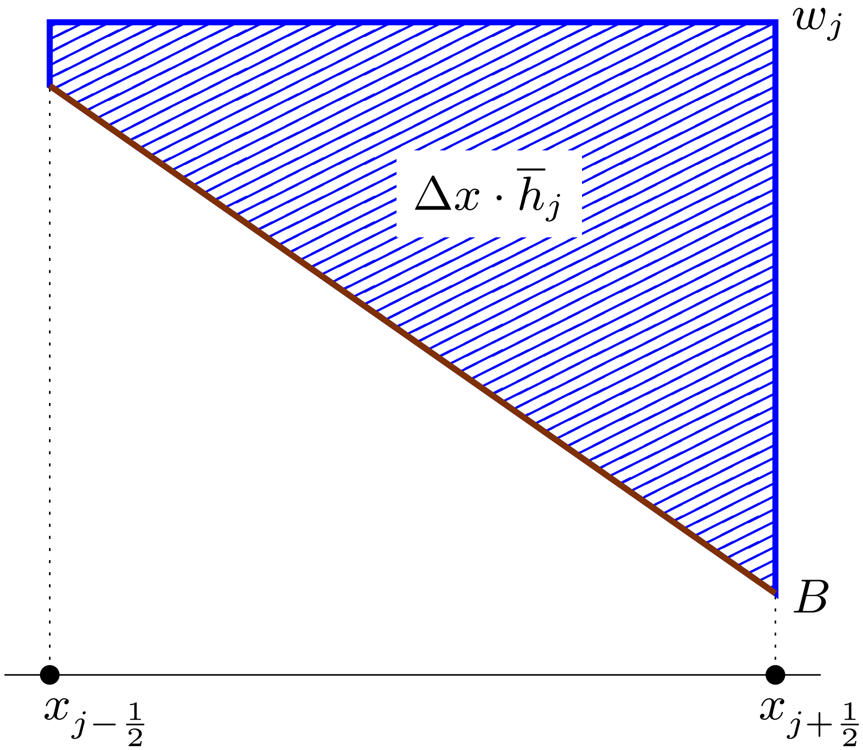

t=0.5
t=1
t=1.5
t=2
0.2
0.1
0
−0.1
−0.2
0
2
4
6



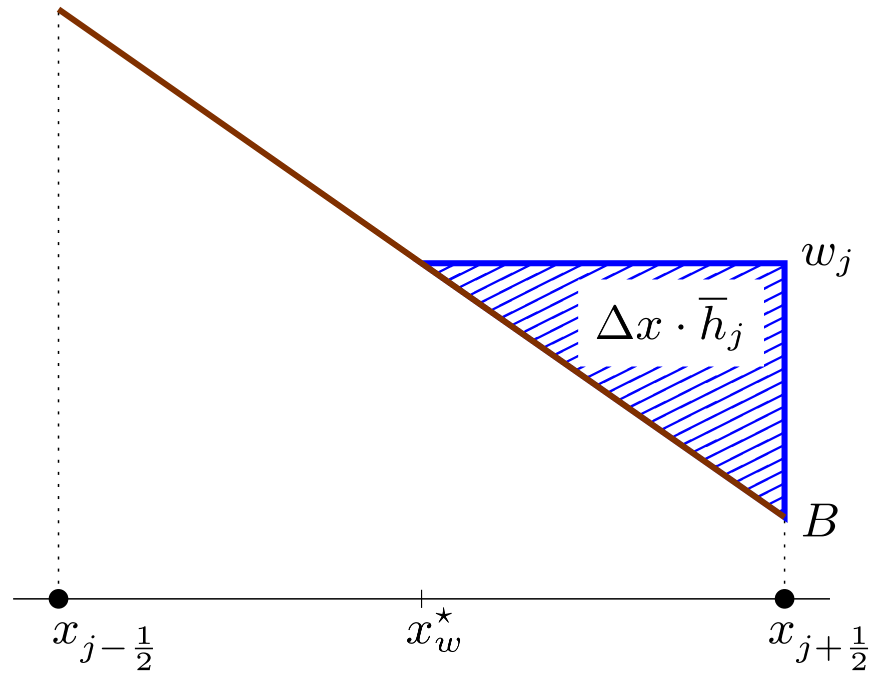
t=2.5
t=3
t=3.5
t=4
0.2
0.1
0
−0.1
−0.2
0
5
10

A Well-Balanced Reconstruction for Wetting/Drying Fronts

[Bollermann, Kurganov, Noelle; submitted]

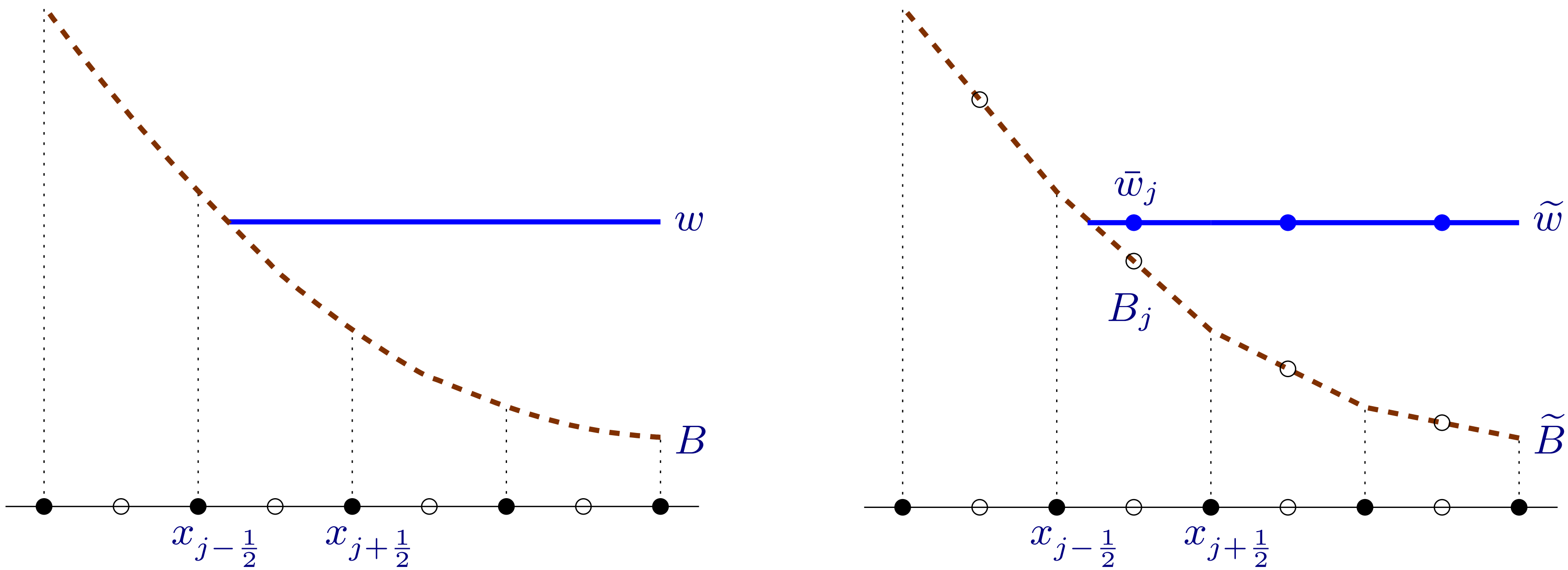


Fully flooded cell



Partially flooded cell

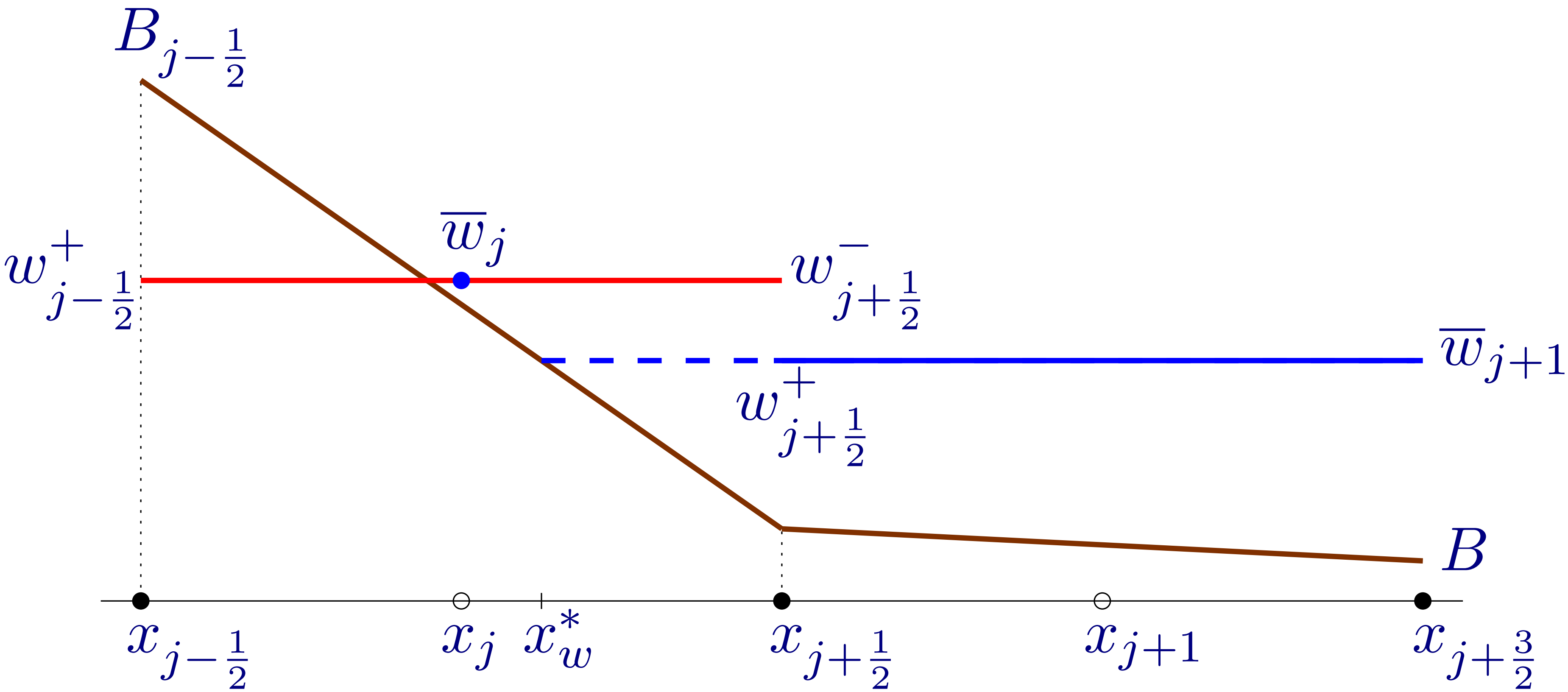
Problem: Assume that the lake is at rest. Then in any partially flooded cell, the cell average of w is larger than the water surface level at the rest of the lake.



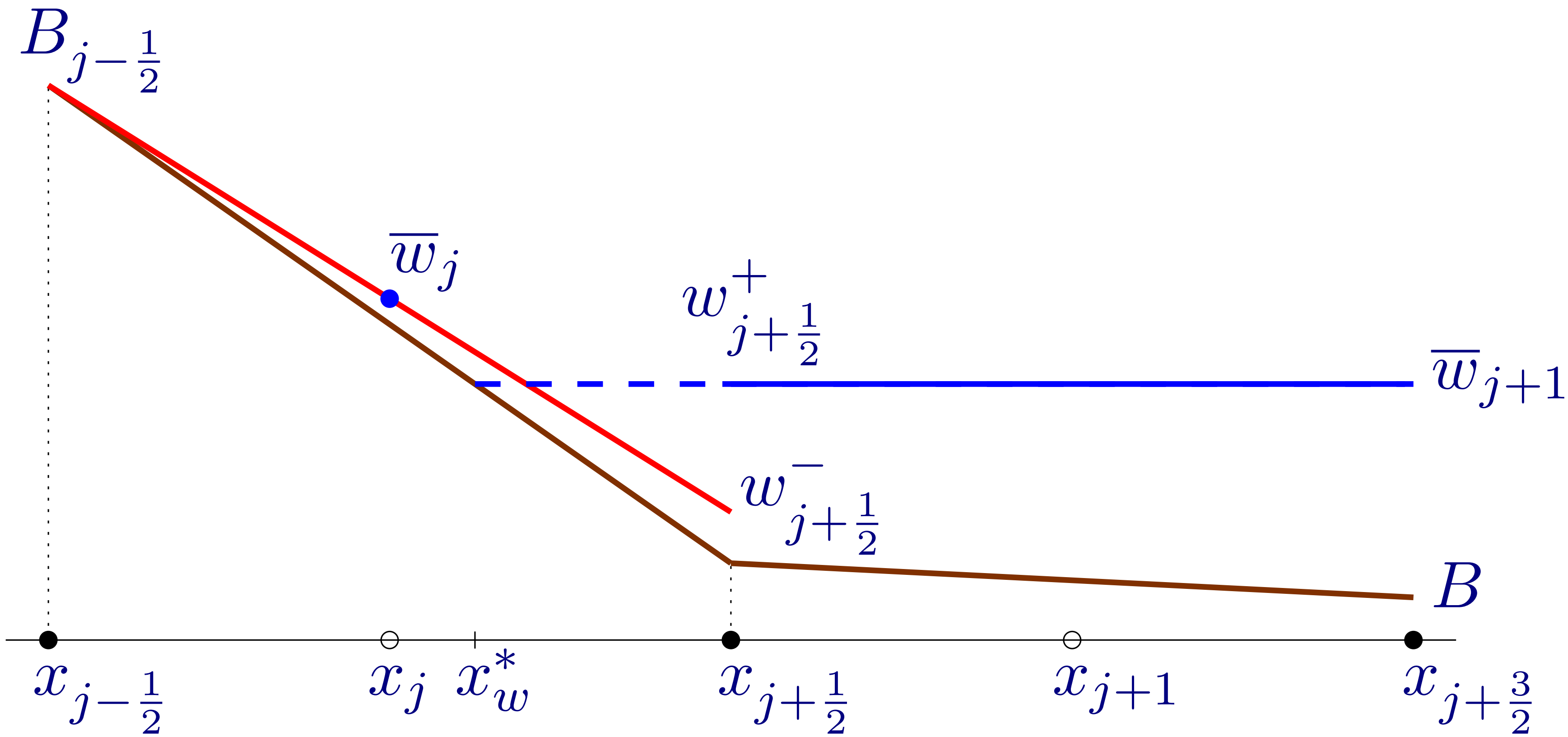
"Lake at rest" steady state combined with dry boundaries

Left: Real situation

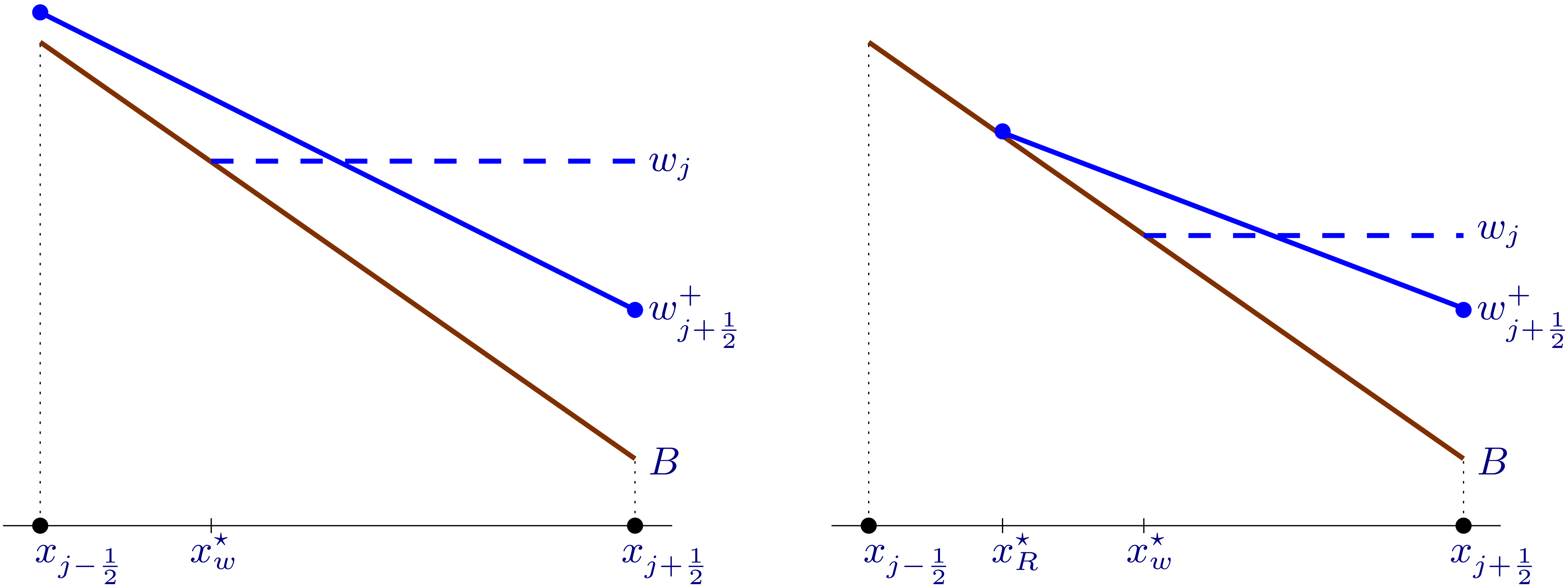
Right: Desired reconstruction



1-order piecewise constant reconstruction is wrong



Conservative, positivity preserving, but NOT well-balanced piecewise linear reconstruction from [Kurganov, Petrova; 2007]



w_j
$w^+_{j+\frac{1}{2}}$
B
$x_{j-\frac{1}{2}}$
$x^\star_w$
$x_{j+\frac{1}{2}}$
w_j
$w^+_{j+\frac{1}{2}}$
B
$x_{j-\frac{1}{2}}$
$x^\star_R$
$x^\star_w$
$x_{j+\frac{1}{2}}$

Efficiency of the Well-Balanced Positivity Preserving Scheme

Switch from $\mathbf{U} = (w, hu)^T$ back to $\mathbf{V} := (h, hu)^T$

Consider the forward Euler discretization:

$$\overline{\mathbf{V}}_j^{n+1} = \overline{\mathbf{V}}_j^n - \Delta t \left(\frac{\mathbf{H}_{j+\frac{1}{2}} - \mathbf{H}_{j-\frac{1}{2}}}{\Delta x} + \overline{\mathbf{S}}_j(t) \right) \qquad (1)$$

We need to ensure that the first component of (1) satisfies

$$\overline{h}_j^{n+1} = \overline{h}_j^n - \Delta t \, \frac{\mathbf{H}^{(1)}_{j+\frac{1}{2}} - \mathbf{H}^{(1)}_{j-\frac{1}{2}}}{\Delta x} \geq 0 \qquad (2)$$

We introduce the draining time step

$$\Delta t_{j,\text{drain}} = \frac{\Delta x \overline{h}_j^n}{\max(0, \mathbf{H}^{(1)}_{j+\frac{1}{2}}) + \max(0, -\mathbf{H}^{(1)}_{j-\frac{1}{2}})}$$

which describes the time when the water contained in cell j in the beginning of the time step has left via the outflow fluxes.

We then replace the evolution step (2) with

$$\overline{h}_j^{n+1} = \overline{h}_j^n - \frac{\Delta t_{j+\frac{1}{2}} \mathbf{H}^{(1)}_{j+\frac{1}{2}} - \Delta t_{j-\frac{1}{2}} \mathbf{H}^{(1)}_{j-\frac{1}{2}}}{\Delta x}$$

where we set

$$\Delta t_{j+\frac{1}{2}} = \min(\Delta t, \Delta t_{i,\text{drain}}), \qquad i = j + \frac{1 - \text{sgn}\left(\mathbf{H}^{(1)}_{j+\frac{1}{2}}\right)}{2}$$

Definition of i selects the cell in upwind direction of the edge.

Choice of $\Delta t_{j+\frac{1}{2}}$ guarantees the positivity of the water depth.

Remark: The local modification of the time step only corresponds to the fact that the flux out of an empty cell vanishes, thus the physical flow is accurately represented here.

To ensure the well-balancing, we have to ensure that the gravity driven part of the flux is multiplied by the same time step as the source term.

We split the flux in its advective and gravity driven parts:

$$\mathbf{F}^{\mathrm{a}}(\mathbf{V}) := \left(hu, \frac{(hu)^2}{h}\right)^T \quad \text{and} \quad \mathbf{F}^{\mathrm{g}}(\mathbf{V}) := \left(0, \frac{g}{2}h^2\right)^T$$

The corresponding central-upwind fluxes are:

$$\mathbf{H}^{\mathrm{g}}_{j+\frac{1}{2}} = \frac{a^{+}_{j+\frac{1}{2}}\mathbf{F}^{\mathrm{g}}(\mathbf{V}^{-}_{j+\frac{1}{2}}) - a^{-}_{j+\frac{1}{2}}\mathbf{F}^{\mathrm{g}}(\mathbf{V}^{+}_{j+\frac{1}{2}})}{a^{+}_{j+\frac{1}{2}} - a^{-}_{j+\frac{1}{2}}} + \frac{a^{+}_{j+\frac{1}{2}}a^{-}_{j+\frac{1}{2}}}{a^{+}_{j+\frac{1}{2}} - a^{-}_{j+\frac{1}{2}}}\left[\mathbf{V}^{+}_{j+\frac{1}{2}} - \mathbf{V}^{-}_{j+\frac{1}{2}}\right]$$

$$\mathbf{H}^{\mathrm{a}}_{j+\frac{1}{2}} = \frac{a^{+}_{j+\frac{1}{2}}\mathbf{F}^{\mathrm{a}}(\mathbf{V}^{-}_{j+\frac{1}{2}}) - a^{-}_{j+\frac{1}{2}}\mathbf{F}^{\mathrm{a}}(\mathbf{V}^{+}_{j+\frac{1}{2}})}{a^{+}_{j+\frac{1}{2}} - a^{-}_{j+\frac{1}{2}}}$$

The above fluxes adds up to the modified finite volume update:

$$\overline{\mathbf{V}}_j^{n+1} = \overline{\mathbf{V}}_j^n - \left(\frac{\Delta t_{j+\frac{1}{2}} \mathbf{H}^{\mathsf{a}}_{j+\frac{1}{2}} - \Delta t_{j-\frac{1}{2}} \mathbf{H}^{\mathsf{a}}_{j-\frac{1}{2}}}{\Delta x} \right) - \Delta t \left(\frac{\mathbf{H}^{\mathsf{g}}_{j+\frac{1}{2}} - \mathbf{H}^{\mathsf{g}}_{j-\frac{1}{2}}}{\Delta x} + \overline{\mathbf{S}}_j^n \right)$$

This modification ensures both efficiency and well-balancing property of the positivity preserving scheme at (near) dry areas

Example — Oscillating Lake

This test describes a lake at rest situation with a sinusoidal perturbation over a smooth basin with dry boundaries.

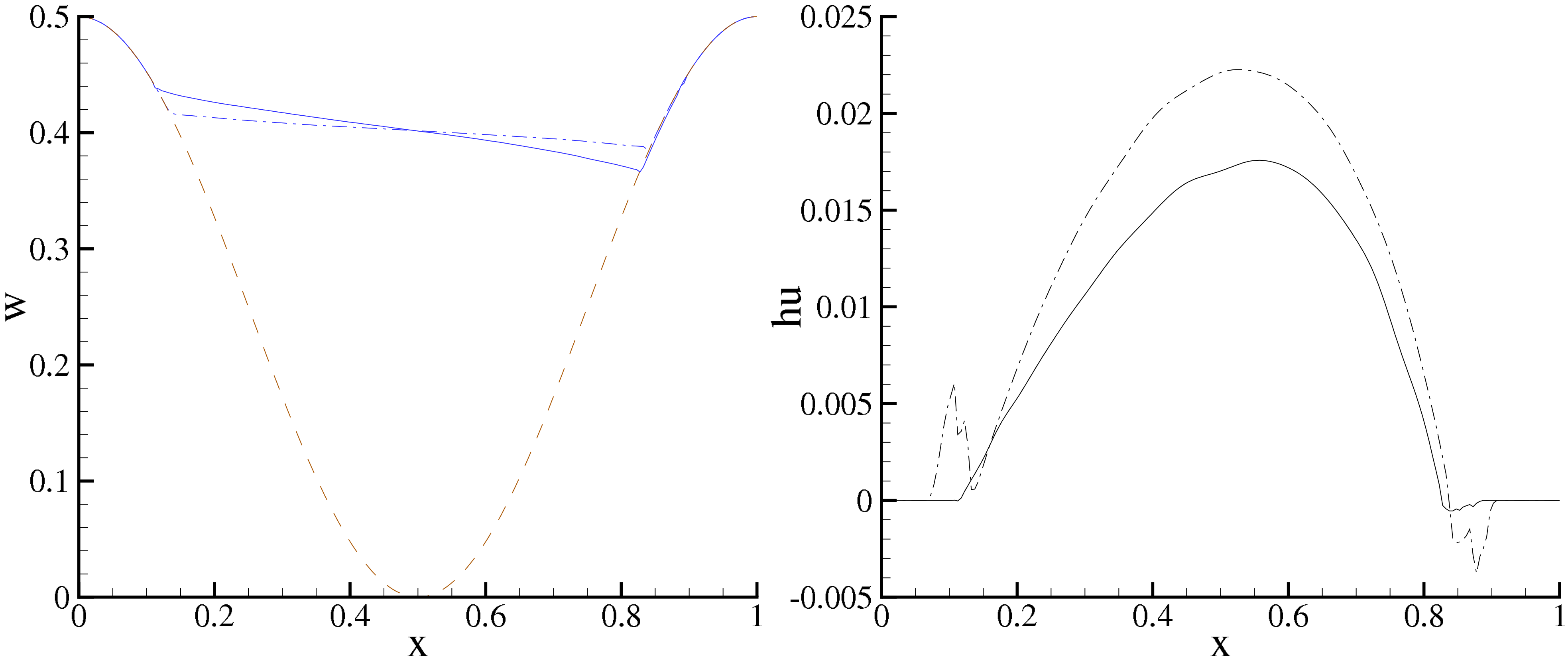
The bottom topography is

$$B(x) = \frac{1}{4} - \frac{1}{4}\cos((2x-1)\pi)$$

and the initial water depth is

$$h(x,0) = \max\left(0, 0.4 + \frac{\sin\left(4x - 2 - \max(0, -0.4 + B(x))\right)}{25} - B(x)\right)$$

The final time is $T = 19.87$, at which the wave has its maximal height at the left shore after some oscillations.

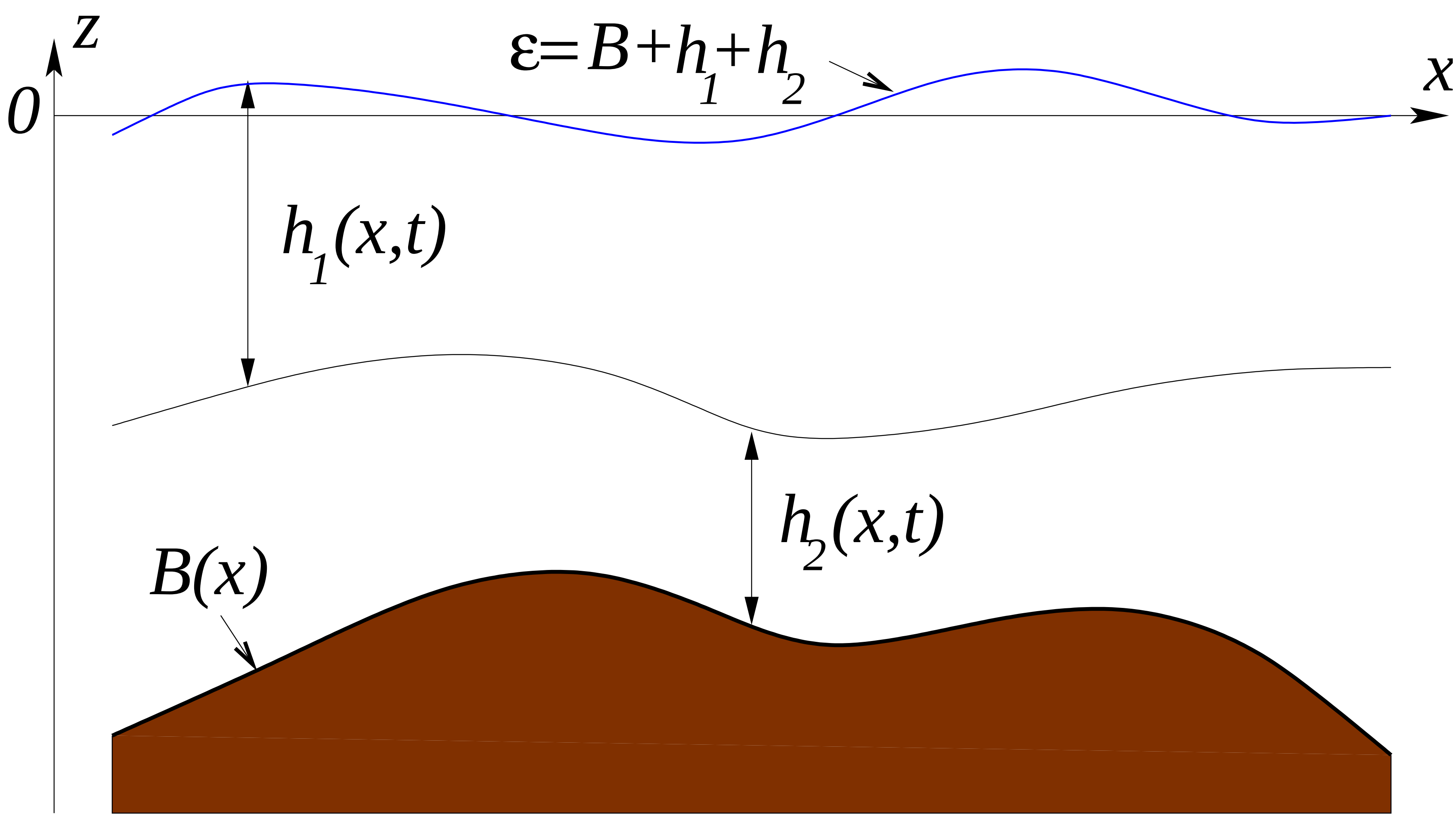


Left: Free surface and bottom topography. Right: Discharge.
BKN scheme (solid line) and KP scheme (dash-dotted line)

BKN scheme produces a significantly higher run-up on the shore.

For the discharge, the BKN scheme produces no oscillations at the boundary. As the overshoots of the KP scheme are directed downslope, this could be the reason for the reduced run-up.

Two-Layer Shallow Water Equations



$$
\begin{cases}
(h_1)_t + (q_1)_x = 0 \\
(q_1)_t + \left(h_1 u_1^2 + \frac{g}{2} h_1^2\right)_x = -g h_1 B_x - g h_1 (h_2)_x \\
(h_2)_t + (q_2)_x = 0 \\
(q_2)_t + \left(h_2 u_2^2 + \frac{g}{2} h_2^2\right)_x = -g h_2 B_x - g h_2 (\widehat{h}_1)_x
\end{cases}
$$

u_1, u_2: velocities

$q_1 := h_1 u_1$, $q_2 := h_2 u_2$: discharges

g: gravitational constant

$r := \frac{\rho_1}{\rho_2}$: constant density ratio

$\widehat{h}_1 := r h_1$.

$$\begin{cases} (h_1)_t + (q_1)_x = 0 \\ (q_1)_t + \left(h_1 u_1^2 + \frac{g}{2} h_1^2\right)_x = -g h_1 B_x - g h_1 (h_2)_x \\ (h_2)_t + (q_2)_x = 0 \\ (q_2)_t + \left(h_2 u_2^2 + \frac{g}{2} h_2^2\right)_x = -g h_2 B_x - g h_2 (\widehat{h}_1)_x \end{cases}$$

Difficulties:

- Nonlinear hyperbolic system $\Longrightarrow$

  shock waves

  rarefaction waves

  contact waves (if B is discontinuous)

- Geometric source terms (nonflat bottom topography)

- Nonconservative products (interlayer exchange)

Some References

[Farmer, Armi; 1986]

[Macías, Pares, Castro; 1999]

[Castro, Macías, Parés; 2001]

[Castro, Macías, Parés, García-Rodríguez, Vázquez-Cendón; 2004]

[Pares, Castro; 2004]

[Castro, Chacón Rebollo, Fernández-Nieto, Pares; 2007]

[Bouchut, Morales de Luna; 2008]

[Abgrall, Karni; 2009]

$$
\begin{cases}
(h_1)_t + (q_1)_x = 0 \\
(q_1)_t + \left(h_1 u_1^2 + \frac{g}{2} h_1^2\right)_x = -g h_1 B_x - g h_1 (h_2)_x \\
(h_2)_t + (q_2)_x = 0 \\
(q_2)_t + \left(h_2 u_2^2 + \frac{g}{2} h_2^2\right)_x = -g h_2 B_x - g h_2 (\widehat{h}_1)_x
\end{cases}
$$

$$
\Downarrow \qquad (h_1, q_1, h_2, q_2) \to (h_1, q_1, w := h_2 + B, q_2)
$$

$$
\begin{cases}
(h_1)_t + (q_1)_x = 0 \\
(q_1)_t + \left(\dfrac{q_1^2}{h_1} + g\varepsilon h_1\right)_x = g\varepsilon (h_1)_x \\
w_t + (q_2)_x = 0 \\
(q_2)_t + \left(\dfrac{q_2^2}{w - B} + \dfrac{g}{2} w^2 - \dfrac{g}{2} r h_1^2 - g B \widehat{\varepsilon}\right)_x = -g \widehat{\varepsilon} B_x - g\varepsilon (\widehat{h}_1)_x
\end{cases}
$$

$$
\varepsilon = h_1 + h_2 + B = h_1 + w, \qquad \widehat{\varepsilon} := \widehat{h}_1 + w
$$

The new system is preferable for numerical computations thanks to the following two reasons:

(i) the stationary steady state is given by $\mathbf{U} \equiv (C_1, 0, C_2, 0)^T$, with C_1, C_2 constants

(ii) the coefficients in the nonconservative products $g\varepsilon(h_1)_x$ and $g\varepsilon(\widehat{h}_1)_x$ are proportional to ε, which vanishes at stationary steady states, and, what is even more important, in most oceanographic applications remains very small.

Note that when $r \sim 1$, the variable $\widehat{\varepsilon}$ is also expected to be small in typical oceanographic applications

As in the single-layer case, we:

- replace the bottom with its piecewise linear approximation
- correct reconstruction of w to ensure that $w > B$
- regularize computed velocities to avoid division by $h_i \sim 0$
- use well-balanced quadrature for the geometric source term

The e-values are given implicitly by

$$(\lambda^2 - 2u_1\lambda + u_1^2 - gh_1)(\lambda^2 - 2u_2\lambda + u_2^2 - gh_2) = g^2\widehat{h}_1 h_2$$

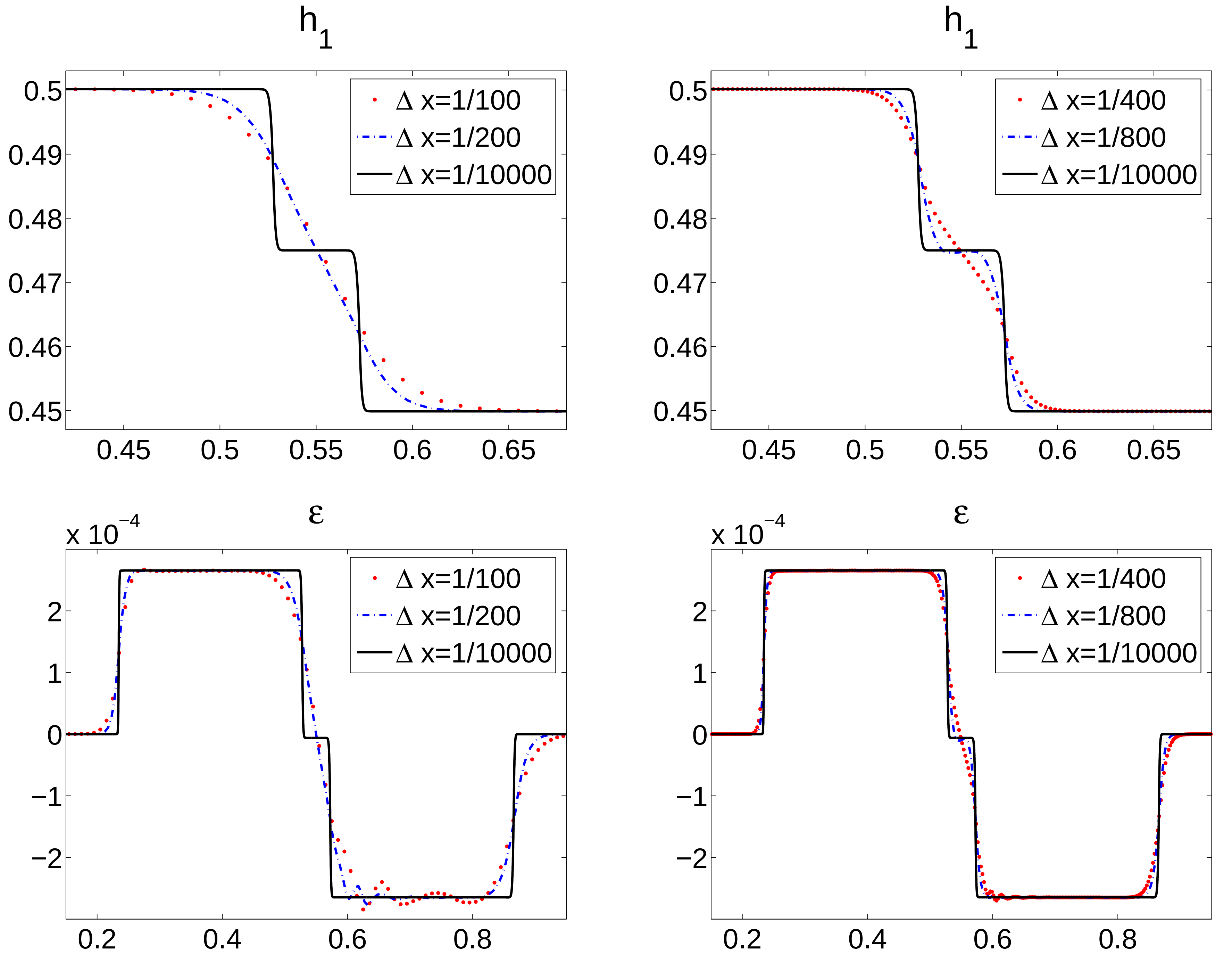
We either:

- use the first-order approximation of the e-values (when $r \sim 1$ and $u_1 \sim u_2$)
- use their upper/lower bounds to enforce stability (if the system is still in the hyperbolic regime)

Example — Interface Propagation

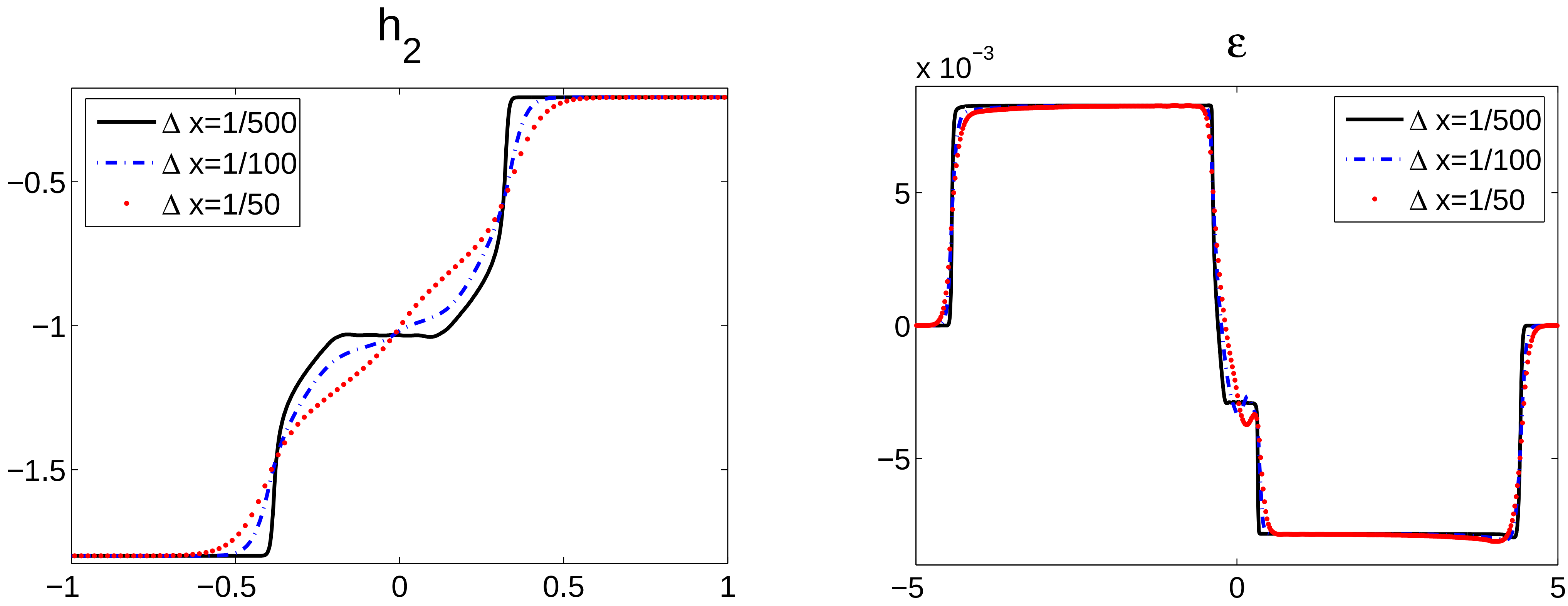
$$(h_1, q_1, h_2, q_2)(x, 0) = \begin{cases} (0.50, 1.250, 0.50, 1.250), & x < 0.3 \\ (0.45, 1.125, 0.55, 1.375), & x > 0.3 \end{cases}$$

$B \equiv -1$, $r = 0.98$



h1
0.5
0.49
0.48
0.47
0.46
0.45
0.45
0.5
0.55
0.6
0.65
Δ x=1/100
Δ x=1/200
Δ x=1/10000
Δ x=1/400
Δ x=1/800
ε
x 10^−4
2
1
0
−1
−2
0.2
0.4
0.6
0.8

$$(h_1, q_1, h_2, q_2)(x, 0) = \begin{cases} (1.8, 0, 0.2, 0), & x < 0, \\ (0.2, 0, 1.8, 0), & x > 0, \end{cases} \quad B \equiv -2, \;\; r = 0.98$$

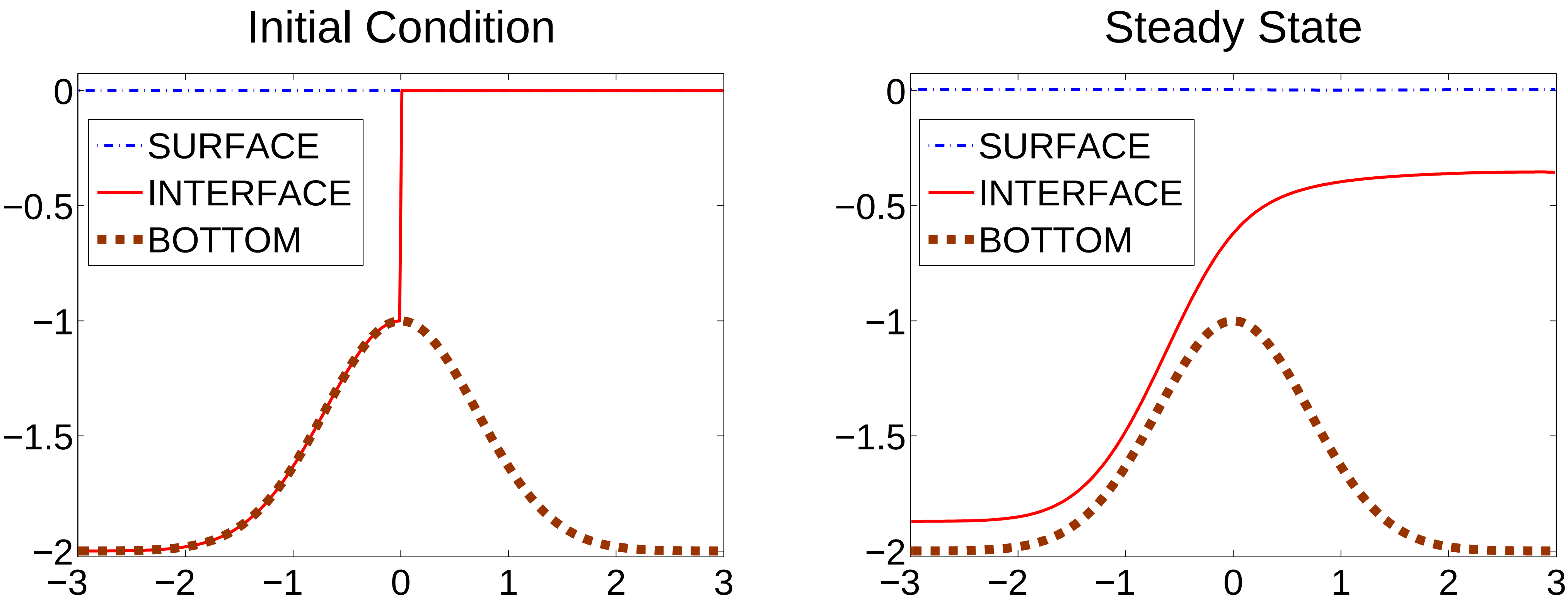


Example — Lock Exchange Problem

$$(h_1, q_1, h_2, q_2)(x, 0) = \begin{cases} (-B(x), 0, 0, 0), & x < 0 \\ (0, 0, -B(x), 0), & x > 0 \end{cases}$$

$B(x) = e^{-x^2} - 2, \quad r = 0.98$

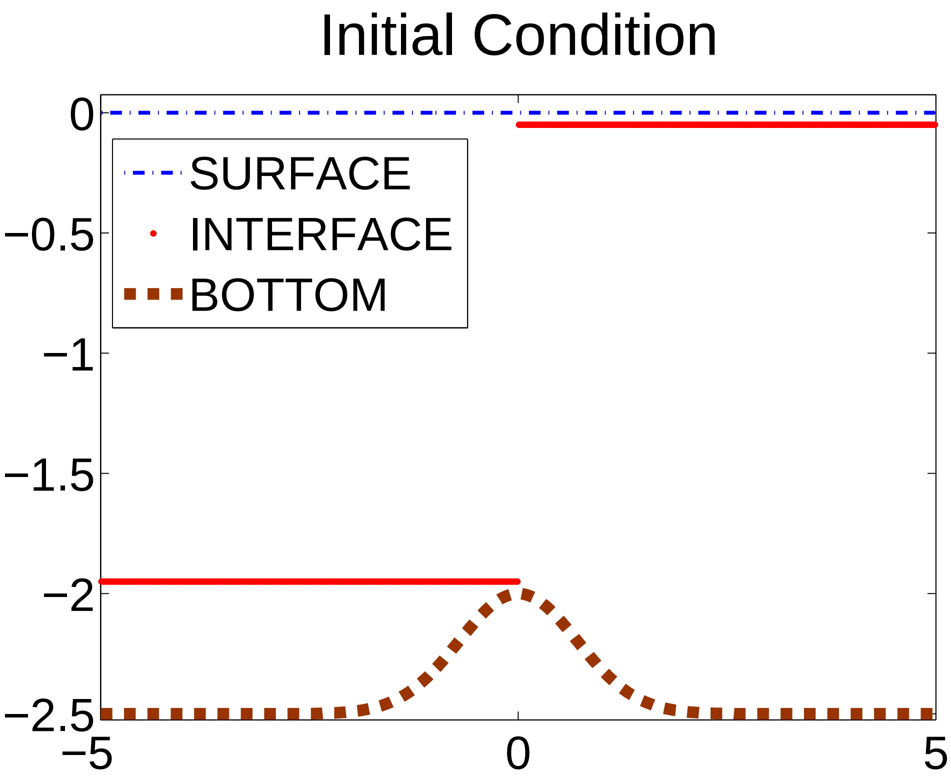
The computational domain is $[-3, 3]$ and the boundary conditions are $q_1 = -q_2$ at each end of the interval


Initial Condition
Steady State
SURFACE
INTERFACE
BOTTOM
SURFACE
INTERFACE
BOTTOM
0
−0.5
−1
−1.5
−2
−3
−2
−1
0
1
2
3

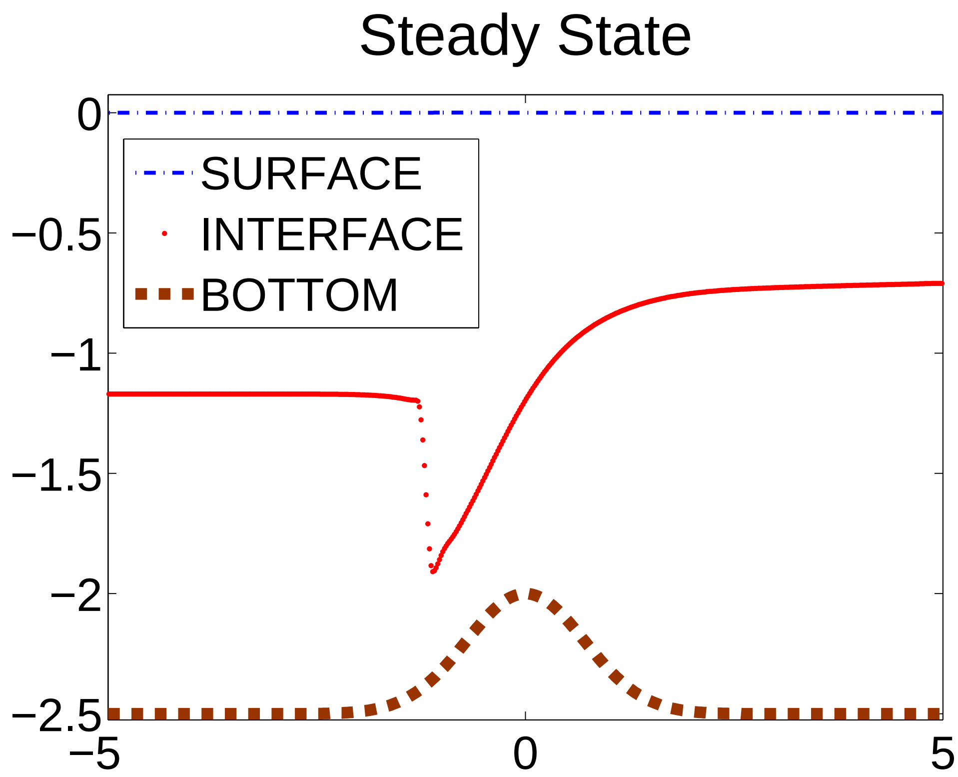
Example — Internal Dam Break

$$(h_1, q_1, h_2, q_2)(x, 0) = \begin{cases} (1.95, 0, -1.95 - B(x), 0), & x < 0 \\ (0.05, 0, -0.05 - B(x), 0), & x > 0 \end{cases}$$

$B(x) = 0.5e^{-x^2} - 2.5, \quad r = 0.998$



Initial Condition
0
−0.5
−1
−1.5
−2
−2.5
−5
0
5
SURFACE
INTERFACE
BOTTOM



Steady State
0
−0.5
−1
−1.5
−2
−2.5
−5
0
5
SURFACE
INTERFACE
BOTTOM

Example — 2-D Interface Propagation

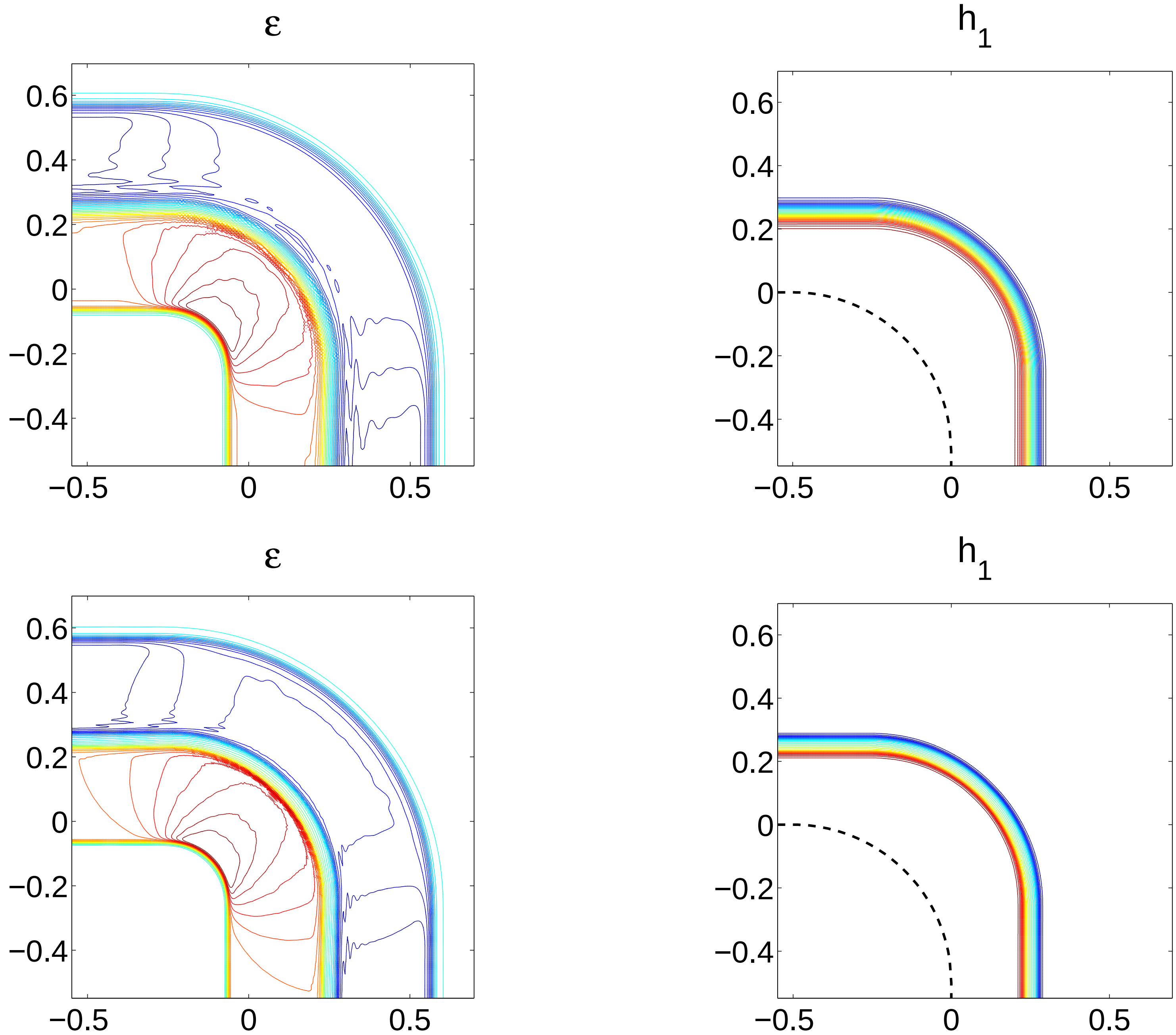
$$(h_1, q_1, p_1, h_2, q_2, p_2)(x, y, 0)$$
$$= \begin{cases} (0.50, 1.250, 1.250, 0.50, 1.250, 1.250), & \text{if } (x, y) \in \Omega \\ (0.45, 1.125, 1.125, 0.55, 1.375, 1.375), & \text{otherwise} \end{cases}$$

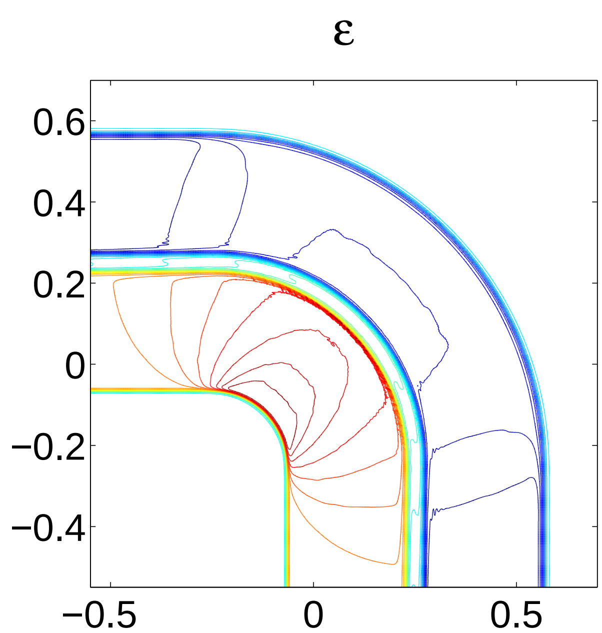
where

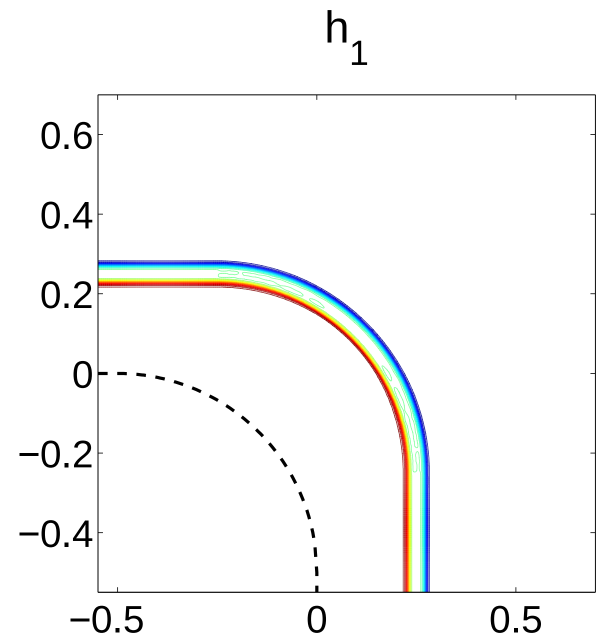
$$\Omega = \{x < -0.5, y < 0\} \cup \left\{(x + 0.5)^2 + (y + 0.5)^2 < 0.25\right\} \cup \{x < 0, y < -0.5\}.$$

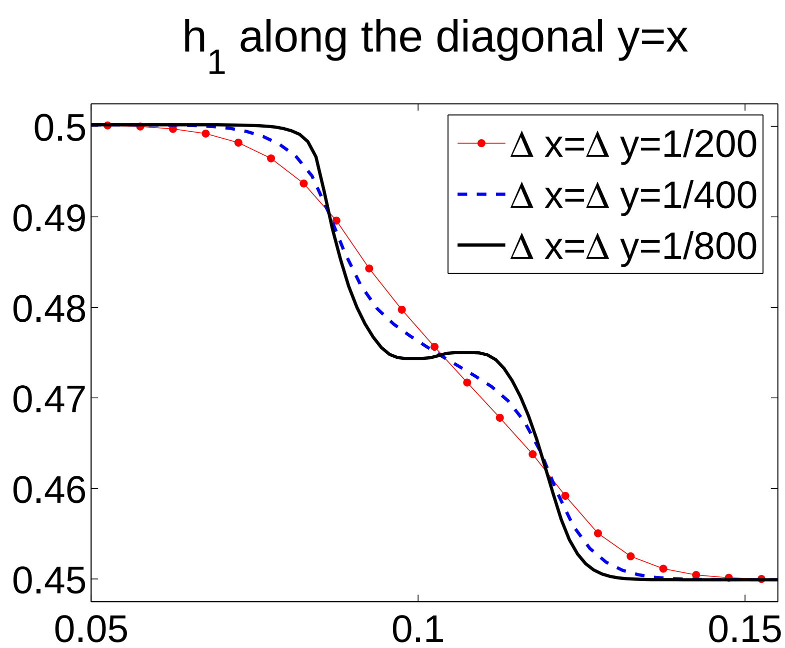
The initial location of the interface is shown by the dashed line.

$B(x, y) \equiv -1$


ε
0.6
0.4
0.2
0
−0.2
−0.4
−0.5
0
0.5
h_1
0.6
0.4
0.2
0
−0.2
−0.4
−0.5
0
0.5
ε
0.6
0.4
0.2
0
−0.2
−0.4
−0.5
0
0.5
h_1
0.6
0.4
0.2
0
−0.2
−0.4
−0.5
0
0.5


ε
0.6
0.4
0.2
0
−0.2
−0.4
−0.5
0
0.5

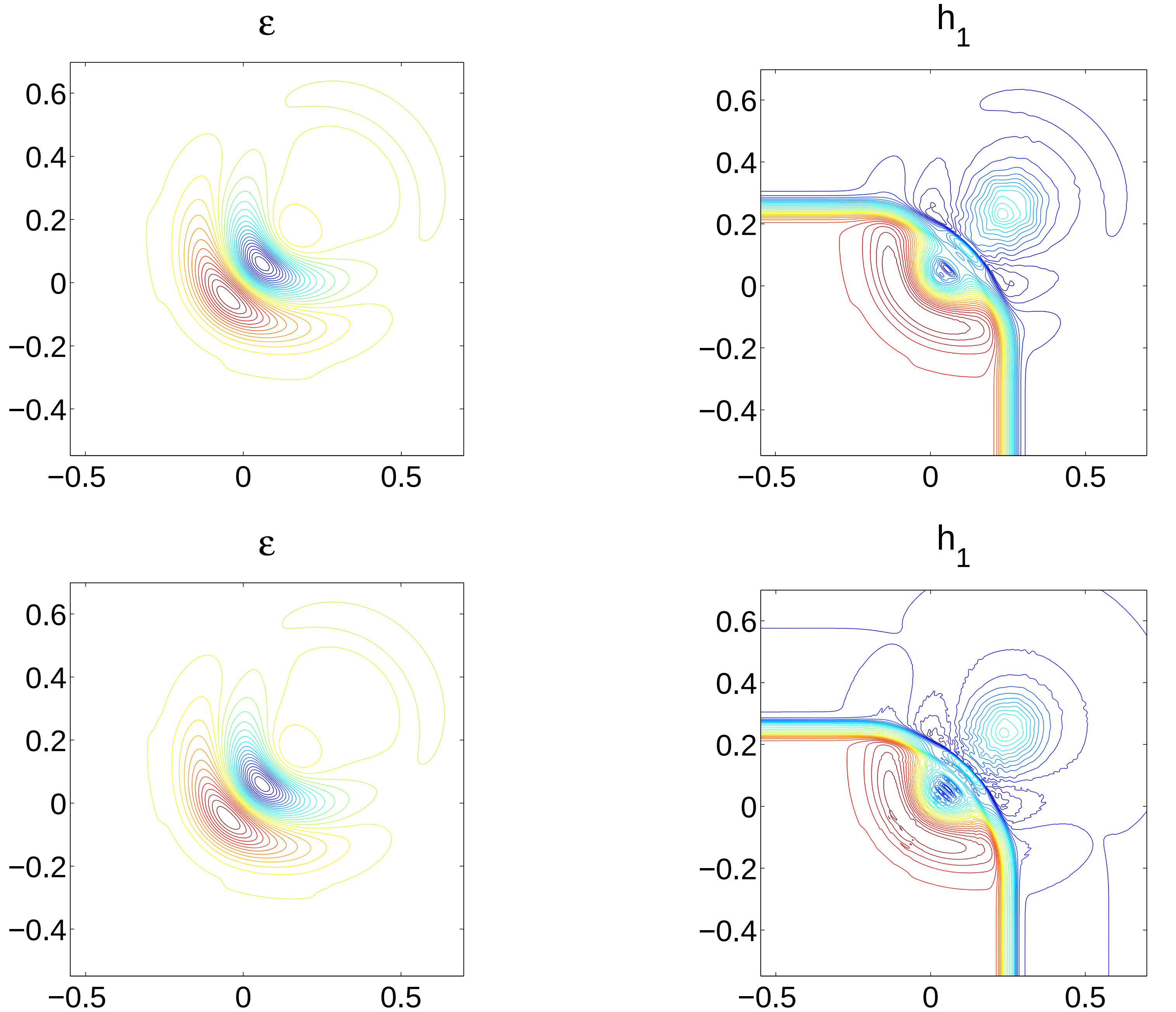

h_1
0.6
0.4
0.2
0
−0.2
−0.4
−0.5
0
0.5

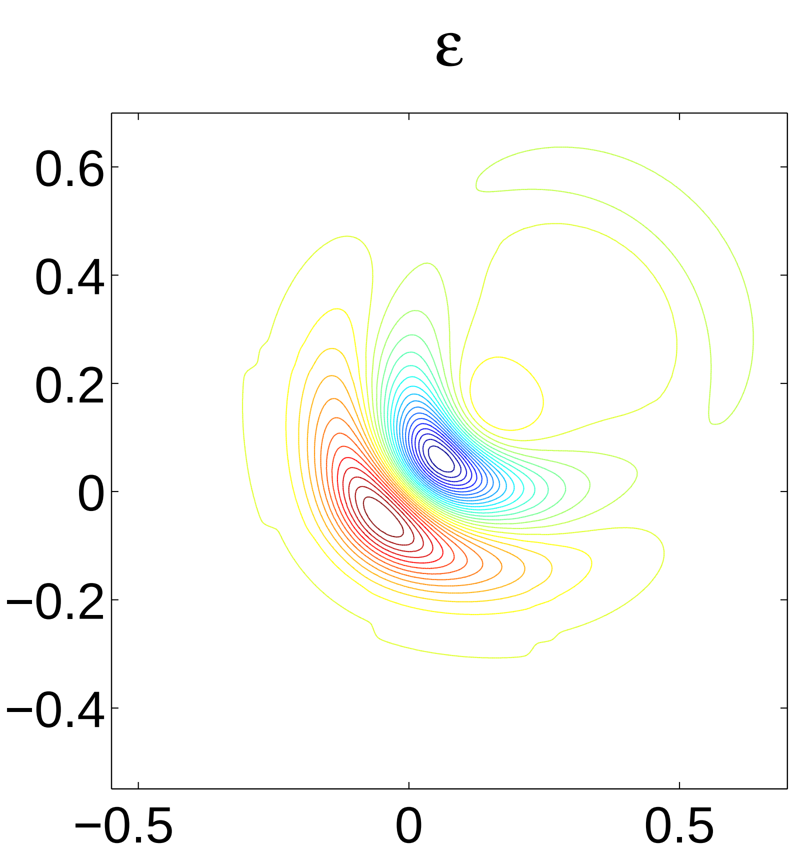

h_1 along the diagonal y=x
Δ x=Δ y=1/200
Δ x=Δ y=1/400
Δ x=Δ y=1/800
0.5
0.49
0.48
0.47
0.46
0.45
0.05
0.1
0.15

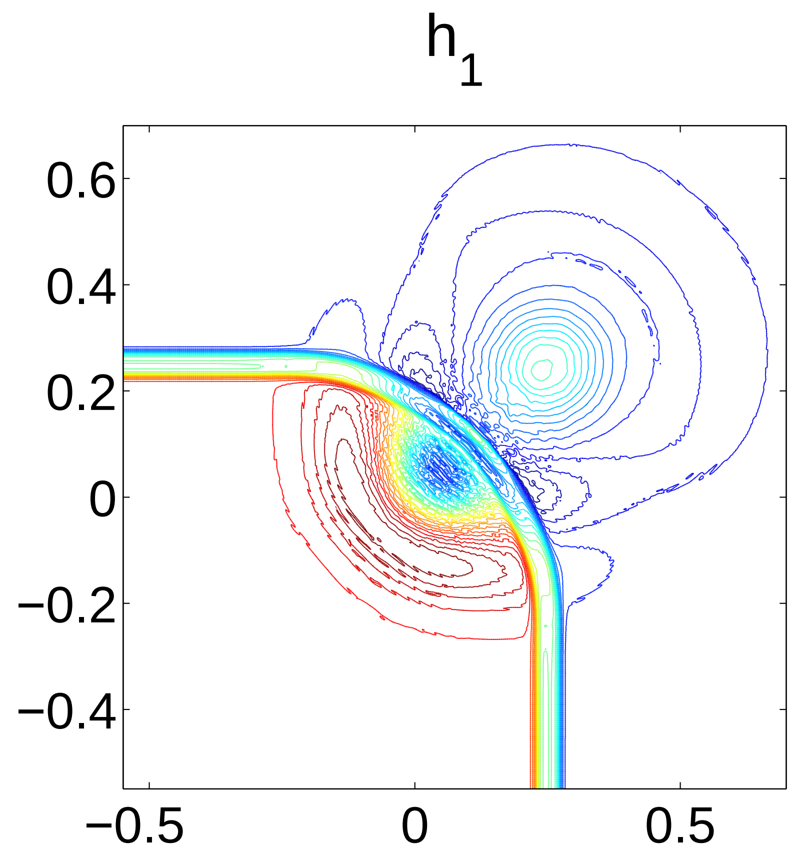
Example — 2-D Interface over a Nonflat Bottom

$$B(x, y) = 0.05e^{-100(x^2+y^2)} - 1$$

$$(h_1, u_1, v_1, w, u_2, v_2) = \begin{cases} (0.50, 2.5, 2.5, -0.50, 2.5, 2.5), & (x, y) \in \Omega \\ (0.45, 2.5, 2.5, -0.45, 2.5, 2.5), & \text{otherwise} \end{cases}$$


ε
h1
ε
h1
0.6
0.4
0.2
0
−0.2
−0.4
−0.5
0
0.5


ε
0.6
0.4
0.2
0
−0.2
−0.4
−0.5
0
0.5


h_1
0.6
0.4
0.2
0
−0.2
−0.4
−0.5
0
0.5