LOCAL HILBERT EXPANSION FOR THE BOLTZMANN EQUATION

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Abstract. We revisit the classical work of Caflisch [C] for compressible Euler limit of the Boltzmann equation. By using a new L^2-L^∞ method, we prove the validity of the Hilbert expansion before shock formathions in the Euler system with moderate temperature variation.

1. Introduction

We study the Boltzmann equation

$$\partial_t F^\varepsilon + v\cdot\nabla_x F^\varepsilon = \frac{1}{\varepsilon}\mathcal{Q}(F^\varepsilon, F^\varepsilon) \tag{1}$$

where $F^\varepsilon(t,x,v)\geq 0$ is the density of particles of velocity $v\in\mathbf{R}^3$, and position $x\in\Omega=\mathbf{R}^3$ or $\mathbf{T}^3$, a periodic box. Throughout this paper, the collision operator takes the form

$$\begin{aligned}\mathcal{Q}(F_1\,,F_2) = &\int_{\mathbf{R}^3}\int_{\mathbf{S}^2}|v-u|^\gamma F_1(u')F_2(v')q_0(\theta)\,d\mu\,du\\ &-\int_{\mathbf{R}^3}\int_{\mathbf{S}^2}|v-u|^\gamma F_1(u)F_2(v)q_0(\theta)\,d\mu\,du\,,\end{aligned} \tag{2}$$

where $u'=u+(v-u)\cdot\omega$, $v'=v-(v-u)\cdot\omega$, $\cos\theta=(u-v)\cdot\omega/|v-u|$, $0\leq\gamma\leq 1$ (hard potential) and $0\leq q_0(\theta)\leq C|\cos(\theta)|$ (angular cutoff). We assume hard-sphere interaction for $\mathcal{Q}$ in this paper, i.e. $\gamma=1$. We believe the result could be generalized to broader class of the collision kernels. We define a special distribution function μ, the *local Maxwellians* by

$$\mu(t\,,,\,,v)=\frac{\rho(t,x)}{[2\pi T(t,x)]^{3/2}}\exp\left\{-\frac{[v-u(t,x)]^2}{2T(t,x)}\right\} \tag{3}$$

which are in equilibrium with the collision process, i.e.

$$\mathcal{Q}(\mu\,,\mu)=0\,. \tag{4}$$

$\rho\,,u\,,T$ are the macroscopic density, bulk velocity and temperature, respectively. If ρ,u,T are constant in x and t, μ is called a *global Maxwellian.*

The fluid dynamics description of a gas is given by the compressible Euler equations:

$$\begin{aligned}\partial_t\rho+\nabla_x\cdot(\rho u) &= 0,\\ \partial_t(\rho u)+\nabla_x\cdot(\rho u\otimes u)+\nabla_x p &= 0,\\ \partial_t\left[\rho(e+\tfrac12|u|^2)\right]+\nabla_x\cdot\left[\rho u(e+\tfrac12|u|^2)\right]+\nabla_x\cdot(pu) &= 0\end{aligned} \tag{5}$$

with the equation of state $p=\rho RT=\frac23\rho e$. These are the local conservation laws of mass, momentum, and energy.

In [C], Caflisch showed that any smooth solution to the compressible Euler system (5) can be used to construct the corresponding solution to the Boltzmann equation (1).

The solution was founded as a Hilbert expansion with remainder which was decomposed into low and high velocity components. After the linearized version of the decomposed remainder equation was estimated, the nonlinear equations are solved by iteration.

However, a crucial but undesirable assumption was made in [C] that the remainder vanishes initially. The truncated Hilbert expansion with such a remainder term can lead to physically unreasonable negative data. This assumption was essentially needed in the linear estimates on the remainder equation.

In this paper, we basically follow Caflisch's approach, give a new *a priori* estimate. Applying the so-called nonlinear energy method developed by the first author, especially some new L^2-L^∞ interplay estimates in [G2], we can remove the assumption on the initial data of the remainder in [C]. Furthermore, we establish the uniform L^2-L^∞ estimate for the remainder. This key improvement allows us to apply the result of the current paper to the hydrodynamic limits of the Boltzmann equation, for example, acoustic limits. This work is under preperation currently.

The paper is organized as follows: the next section contains the statement of the main theorem and some key lemmas. Section 3 is devoted to the proof of L^2 estimate. In Section 4, we establish the L^∞ estimate for the high velocity part.

2. The Main Theorem

As in [C], we take the truncated Hilbert expansion with the form

$$F^\varepsilon = \sum_{n=0}^{6} \varepsilon^n F_n + \varepsilon^3 F_R^\varepsilon,$$

where $F_0, ..., F_6$ are the first 6 terms of the Hilbert expansion, independent of ε, which solve the equations

$$\begin{aligned}
0 &= \mathcal{Q}(F_0, F_0),\\
\{\partial_t + v\cdot\nabla_x\} F_0 &= \mathcal{Q}(F_0, F_1) + \mathcal{Q}(F_1, F_0),\\
\{\partial_t + v\cdot\nabla_x\} F_1 &= \mathcal{Q}(F_0, F_2) + \mathcal{Q}(F_2, F_0) + \mathcal{Q}(F_1, F_1),\\
&\ldots\\
\{\partial_t + v\cdot\nabla_x\} F_5 &= \mathcal{Q}(F_0, F_6) + \mathcal{Q}(F_6, F_0) + \sum_{\substack{i+j=6\\ 1\le i\le 5, 1\le j\le 5}} \mathcal{Q}(F_i, F_j).
\end{aligned}$$

Let $[\rho(t,x), u(t,x), T(t,x)]$ be a smooth solution of the Euler equations (5) for $t\in[0,\tau]$, $x\in\Omega$ and let

$$F_0 = \mu(t,x,v)$$

from the local Maxwellian $\mu(t,x,v)$ from ρ, u and T as in (3). We further construct smooth $F_1(t,x,v), F_2(t,x,v), ...F_6(t,x,v)$ for $0\le t\le\tau$. For more detailed discussion, see [C]. Now we put $F^\varepsilon = \sum_{n=0}^5 \varepsilon^n F_n + \varepsilon^3 F_R^\varepsilon$ (notice we drop F_6) into the Boltzmann equation (1) to derive the equation of the remainder. Recall that in [C], the remainder F_R^ε satisfies

$$\begin{aligned}
(6)\qquad & \partial_t F_R^\varepsilon + v\cdot\nabla_x F_R^\varepsilon - \frac{1}{\varepsilon}\{\mathcal{Q}(\mu, F_R^\varepsilon) + \mathcal{Q}(F_R^\varepsilon, \mu)\}\\
=\ & \varepsilon^2 \mathcal{Q}(F_R^\varepsilon, F_R^\varepsilon) + \{\mathcal{Q}(F_1 + \varepsilon F_2 + \varepsilon^2 F_3, F_R^\varepsilon) + \mathcal{Q}(F_R^\varepsilon, F_1 + \varepsilon F_2 + \varepsilon^2 F_3)\} + \varepsilon^2 A
\end{aligned}$$

where

$$A = -\{\partial_t + v \cdot \nabla_x\}F_5 + \sum_{i+j\geq 6, 1\leq i,j\leq 5} \varepsilon^{i+j-6}\mathcal{Q}(F_i, F_j). \tag{7}$$

Note that in the above equation, we drop out the higher order term $\varepsilon^3\{\mathcal{Q}(F_4+\varepsilon F_5\,, F_R^\varepsilon)+\mathcal{Q}(F_R^\varepsilon\,, F_4+\varepsilon F_5)\}$. We define the linearized Boltzmann operator at μ as

$$\begin{aligned}\mathcal{L}g &= -\frac{1}{\sqrt{\mu}}\{\mathcal{Q}(\sqrt{\mu}g,\mu)+\mathcal{Q}(\mu,\sqrt{\mu}g)\} = \nu(\mu)g + K_\mu g,\\ \Gamma(g_1,g_2) &= \frac{1}{\sqrt{\mu}}\mathcal{Q}(\sqrt{\mu}g_1,\sqrt{\mu}g_2).\end{aligned}$$

We use $\langle\cdot\,,\cdot\rangle$ to denote the standard L^2 inner product in $\mathbf{R}^3_v$, while we use $(\cdot\,,\cdot)$ to denote L^2 inner product in $\Omega\times\mathbf{R}^3$ with corresponding L^2 norm $\|\cdot\|_2$. We denote the standard L^∞ norm in $\Omega\times\mathbf{R}^3$ by $\|\cdot\|_\infty$. We also define a weighted L^2 norm

$$\|g\|_\nu^2 = \int_{\Omega\times\mathbf{R}^3} g^2(x\,,v)\nu(v)\,dx\,dv\,,$$

where the collision frequency $\nu(v)\equiv\nu(\mu)(v)$ is defined as

$$\nu(\mu) = \int_{\mathbf{R}^3}|v-v'|\mu(v')\,dv'\,.$$

Theorem 1. *Assume that the solution to the Euler equations $(\rho(t,x),u(t,x),T(t,x))$ is smooth and $\rho(t,x)$ has a positive lower bound for $0\leq t\leq\tau$. Furthermore, assume that the temperature $T(t,x)$ satisfies the condition:*

$$T_M < \max_{t\in[0,\tau],x\in\Omega} T(t,x) < 2T_M\,, \tag{8}$$

where $T_M = \min_{t\in[0,\tau],x\in\Omega} T(t,x)$. Let

$$F^\varepsilon(0,x,v) = \mu(0,x,v) + \sum_{n=1}^{5}\varepsilon^n F_n(0,x,v) + \varepsilon^3 F_R^\varepsilon(0,x,v) \geq 0.$$

Then there is an $\varepsilon_0>0$ such that for $0\leq\varepsilon\leq\varepsilon_0$, and for any $\beta\geq\frac{7}{2}$, there exists a constant $C_\tau(\mu,F_0,F_1,..F_6)$ such that

$$\begin{aligned}&\sup_{0\leq t\leq\tau}\varepsilon^{\frac{3}{2}}\left\|\sqrt{\mu}^{-1}(1+|v|^2)^\beta F_R^\varepsilon(t)\right\|_\infty + \sup_{0\leq t\leq\tau}\left\|\sqrt{\mu}^{-1}F_R^\varepsilon(t)\right\|_2\\ \leq\;& C_\tau\left\{\varepsilon^{\frac{3}{2}}\left\|\sqrt{\mu}^{-1}(1+|v|^2)^\beta F_R^\varepsilon(0)\right\|_\infty + \left\|\sqrt{\mu}^{-1}F_R^\varepsilon(0)\right\|_2 + 1\right\}.\end{aligned}$$

We give a few remarks on the Theorem 1: *First*, based on the *a priori* estimates given in Theorem 1, following the arguments in [C], we can immediately derive the compressible Euler limit as well as the existence of the solutions to the Boltzmann equation (1). We skip the details here. *Second*, we make the assumption (8) on the temperature, which seems restrictive. However, one of the main applications of this new uniform L^2-L^∞ interplay estimate is to derive the hydrodynamic limits from the Boltzmann equation to fluid dynamics which is in the regime close to constant states. For those cases, the condition (8) is easy to be achieved. *Third*, as we mentioned in the introduction, the main new ingredient of the theorem is the removal of the crucial but undesirable assumption $F_R^\varepsilon(0,x,v)\equiv 0$ in [C].

The solutions to the Boltzmann equation are constructed near the local Maxwellian of the compressible Euler system. So it is natural to rewrite the remainder

$$F_R^\varepsilon = \sqrt{\mu} f^\varepsilon. \tag{9}$$

Because μ is a local Maxwellian, the equation of the remainder includes the new term $\sqrt{\mu}^{-1}(\partial_t + v\cdot\nabla_x)\sqrt{\mu} f^\varepsilon$, thus at large velocities, the distribution functions may be growing rapidly due to streaming. To remedy this difficulty, following Caflisch, we introduce a global Maxwellian

$$\mu_M = \frac{1}{(2\pi T_M)^{3/2}} \exp\left\{ -\frac{|v|^2}{2T_M} \right\}.$$

Note that under the assumption (8), there exist two constants c_1, c_2 such that for all $(t, x, v) \in [0,\tau] \times \Omega \times \mathbf{R}^3$, the following holds

$$c_1\mu \le \mu_M \le c_2 \mu.$$

We further define

$$F_R^\varepsilon = \{1+|v|^2\}^{-\beta}\sqrt{\mu_M} h^\varepsilon \equiv \frac{1}{w(v)}\sqrt{\mu_M} h^\varepsilon \tag{10}$$

for some $\beta \ge 7/2$. It then suffices to estimate $\|f^\varepsilon(t)\|_2$ and $\|h^\varepsilon(t)\|_\infty$ to conclude the theorem.

Let $\mathbf{P}g$ be the L_v^2 projection with respect to $[\sqrt{\mu}, v\sqrt{\mu}, |v|^2\sqrt{\mu}]$. We have that there exists a positive number $\delta_0 > 0$ such that

$$\langle \mathcal{L}g, g\rangle \ge \delta_0 \|\{\,\mathbf{I}-\mathbf{P}\}g\|_\nu^2\,. \tag{11}$$

The proof of Theorem 1 relies on an interplay between L^2 and L^∞ estimates for the Boltzmann equation [G2]: L^2 norm of f^ε is controlled by the L^∞ norm of the high velocity part and vice versa. These uniform L^2-L^∞ estimates are stated in the following two lemmas:

Lemma 2. (L^2-Estimate): *Let $f^\varepsilon, h^\varepsilon$ be defined in* (9) *and* (10)*, and $\delta_0 > 0$ be as in the coercivity estimate* (11)*. Then there exists $\varepsilon_0 > 0$ and a positive constant $C = C(\mu, F_0, F_1, \cdots, F_6) > 0$, such that for all $\varepsilon < \varepsilon_0$*

$$\frac{d}{dt}\|f^\varepsilon\|_2^2 + \frac{\delta_0}{2\varepsilon}\|\{\mathbf{I}-\mathbf{P}\}f^\varepsilon\|_\nu^2 \le C\{\sqrt{\varepsilon}\|\varepsilon^{3/2}h^\varepsilon\|_\infty + 1\}(\|f^\varepsilon\|_2^2 + \|f^\varepsilon\|_2)\,. \tag{12}$$

Lemma 3. (L^∞-Estimate): *Let $f^\varepsilon, h^\varepsilon$ and $\delta_0 > 0$ be the same as in Lemma 2. Then there exist $\varepsilon_0 > 0$ and a positive constant $C = C(\mu, F_0, F_1, \cdots, F_6) > 0$, such that for all $\varepsilon < \varepsilon_0$*

$$\sup_{0\le s\le\tau}\{\varepsilon^{3/2}\|h^\varepsilon(s)\|_\infty\} \le C\{\|\varepsilon^{3/2}h_0\|_\infty + \sup_{0\le s\le\tau}\|f^\varepsilon(s)\|_2 + \varepsilon^{7/2}\}. \tag{13}$$

The proof of Theorem 1 is an easy consequence of Lemmas 2 and 3.

Proof. **of Theorem 1:**

$$\begin{aligned}
&\frac{d}{dt}\|f^\varepsilon\|_2^2 + \frac{\delta_0}{2\varepsilon}\|\{\mathbf{I}-\mathbf{P}\}f^\varepsilon\|_\nu^2 \\
\le & C\left\{\sqrt{\varepsilon}\left[\|\varepsilon^{3/2}h_0\|_\infty + \sup_{0\le s\le\tau}\|f^\varepsilon(s)\|_2 + \varepsilon^{7/2}\right] + 1\right\}\left(\|f^\varepsilon\|_2^2 + \|f^\varepsilon\|_2\right).
\end{aligned}$$

A simple Gronwall inequality yields

$$\|f^\varepsilon(t)\|_2 + 1 \le (\|f^\varepsilon(0)\|_2 + 1)e^{Ct\{2+\sqrt{\varepsilon}\|\varepsilon^{3/2}h_0\|_\infty + \sqrt{\varepsilon}\sup_{0\le s\le\tau}\|f^\varepsilon(s)\|_2\}}.$$

For ε small, using the Taylor expansion of the exponential function in the above inequality, we have

$$\|f^\varepsilon\|_2 \le C_1(\|f^\varepsilon(0)\|_2 + 1)\left\{1 + \sqrt{\varepsilon}\|\varepsilon^{3/2}h_0\|_\infty + \sqrt{\varepsilon}\sup_{0\le s\le\tau}\|f^\varepsilon(s)\|_2\right\}. \tag{14}$$

For $t \le \tau$, letting ε small, we conclude the proof of our main theorem as:

$$\sup_{0\le t\le\tau}\|f^\varepsilon(t)\|_2 \le C_\tau\{1 + \|f^\varepsilon(0)\|_2 + \|\varepsilon^{3/2}h_0\|_\infty\}.$$

□

3. L^2 Estimate For f^ε

Proof. **of Lemma** 2: In terms of f^ε, we obtain

$$\begin{aligned}
&\partial_t f^\varepsilon + v\cdot\nabla_x f^\varepsilon + \frac{1}{\varepsilon}\mathcal{L}f^\varepsilon \\
=\ & \frac{\{\partial_t + v\cdot\nabla_x\}\sqrt{\mu}}{\sqrt{\mu}}f^\varepsilon + \varepsilon^2\Gamma(f^\varepsilon, f^\varepsilon) + \Gamma(\frac{F_1+\varepsilon F_2+\varepsilon^2 F_3}{\sqrt{\mu}}, f^\varepsilon) \\
+\ & \Gamma(f^\varepsilon, \frac{F_1+\varepsilon F_2+\varepsilon^2 F_3}{\sqrt{\mu}}) + \varepsilon^2\bar{A}
\end{aligned}$$

where $\bar{A} = -\frac{\{\partial_t+v\cdot\nabla_x\}F_5}{\sqrt{\mu}} + \sum_{i+j\ge 6, i\le 5, j\le 5}\varepsilon^{i+j-6}\Gamma(\frac{F_i}{\sqrt{\mu}}, \frac{F_j}{\sqrt{\mu}})$.

Taking L^2 inner product with f^ε on both sides, since $\frac{\{\partial_t+v\cdot\nabla_x\}\sqrt{\mu}}{\sqrt{\mu}}$ is a cubic polynomial in v, we have, for any $\kappa > 0$,

$$\begin{aligned}
&\left\langle \frac{\{\partial_t + v\cdot\nabla_x\}\sqrt{\mu}}{\sqrt{\mu}}f^\varepsilon, f^\varepsilon\right\rangle \\
=\ & \int_{|v|\ge\frac{\kappa}{\sqrt{\varepsilon}}} + \int_{|v|\le\frac{\kappa}{\sqrt{\varepsilon}}} \\
\le\ & \{\|\nabla_x\rho\|_2 + \|\nabla_x u\|_2 + \|\nabla_x T\|_2\} \times \|\{1+|v|^2\}^{3/2}f^\varepsilon\mathbf{1}_{|v|\ge\frac{\kappa}{\sqrt{\varepsilon}}}\|_\infty \times \|f^\varepsilon\|_2 \\
& + \{\|\nabla_x\rho\|_\infty + \|\nabla_x u\|_\infty + \|\nabla_x T\|_\infty\} \times \|\{1+|v|^2\}^{3/4}f^\varepsilon\mathbf{1}_{|v|\le\frac{\kappa}{\sqrt{\varepsilon}}}\|_2^2 \\
\le\ & C_\kappa\varepsilon^2\|h^\varepsilon\|_\infty\|f^\varepsilon\|_2 + C\|\{1+|v|^2\}^{3/4}\mathbf{P}f^\varepsilon\mathbf{1}_{|v|\le\frac{\kappa}{\sqrt{\varepsilon}}}\|_2^2 + C\|\{1+|v|^2\}^{3/4}\{\mathbf{I}-\mathbf{P}\}f^\varepsilon\mathbf{1}_{|v|\le\frac{\kappa}{\sqrt{\varepsilon}}}\|_2^2 \\
\le\ & C_\kappa\varepsilon^2\|h^\varepsilon\|_\infty\|f^\varepsilon\|_2 + C\|f^\varepsilon\|_2^2 + \frac{C\kappa^2}{\varepsilon}\|\{\mathbf{I}-\mathbf{P}\}f^\varepsilon\|_\nu^2.
\end{aligned}$$

Here we have used the fact $\{1+|v|^2\}^{3/2}f^\varepsilon \le \{1+|v|^2\}^{-2}h^\varepsilon$, for $\beta \ge 7/2$ in (10), and the fact $\mu_M < C\mu$, under the assumption (8).

By Lemma 2.3 in [G1] and (10):

$$\varepsilon^2\langle\Gamma(f^\varepsilon, f^\varepsilon), f^\varepsilon\rangle \le C\varepsilon^2\{\|\nu(\mu)f^\varepsilon\|_\infty\}\|f^\varepsilon\|_2^2 \le C\sqrt{\varepsilon}\|\varepsilon^{3/2}h^\varepsilon\|_\infty\|f^\varepsilon\|_2^2.$$

Similarly, by Lemma 2.3 in [G1] and (10):,

$$
\begin{aligned}
&\langle\Gamma(\frac{F_1+\varepsilon F_2+\varepsilon^2F_3}{\sqrt{\mu}}, f^\varepsilon), f^\varepsilon\rangle + \langle\Gamma(f^\varepsilon, \frac{F_1+\varepsilon F_2+\varepsilon^2F_3}{\sqrt{\mu}}), f^\varepsilon\rangle \\
\leq\ & C\|f^\varepsilon\|_\nu^2 \| \int_{\mathbf{R}^3} \frac{F_1+\varepsilon F_2+\varepsilon^2F_3}{\sqrt{\mu}} dv\|_\infty \\
\leq\ & C\{\|\mathbf{P}f^\varepsilon\|_\nu^2 + \|\{\mathbf{I}-\mathbf{P}\}f^\varepsilon\|_\nu^2\} \\
\leq\ & C\{\|f^\varepsilon\|_2^2 + \|\{\mathbf{I}-\mathbf{P}\}f^\varepsilon\|_\nu^2\}.
\end{aligned}
$$

Clearly, $\langle\varepsilon^2\bar{A}, f^\varepsilon\rangle \leq C\|f^\varepsilon\|_2$. We therefore conclude our lemma by choosing κ small. $\square$

4. L^∞ Estimate For h^ε

Proof. **of Lemma** 3: As in [C], we define

$$
\mathcal{L}_M g = -\frac{1}{\sqrt{\mu_M}}\{\mathcal{Q}(\mu, \sqrt{\mu_M}g) + \mathcal{Q}(\sqrt{\mu_M}g, \mu)\} = \{\nu(\mu) + K_M\}g.
$$

Letting $K_{M,w}g \equiv wK_M(\frac{g}{w})$, from (6) and (10), we obtain

$$
\begin{aligned}
&\partial_t h^\varepsilon + v\cdot\nabla_x h^\varepsilon + \frac{\nu(\mu)}{\varepsilon}h^\varepsilon + \frac{1}{\varepsilon}K_{M,w}h^\varepsilon \\
=\ & \frac{\varepsilon^2 w}{\sqrt{\mu_M}}\mathcal{Q}(\frac{h^\varepsilon\sqrt{\mu_M}}{w}, \frac{h^\varepsilon\sqrt{\mu_M}}{w}) + \sum_{i=1}^5 \varepsilon^{i-1}\frac{w}{\sqrt{\mu_M}}\{\mathcal{Q}(F_i, \frac{h^\varepsilon\sqrt{\mu_M}}{w}) + \mathcal{Q}(\frac{h^\varepsilon\sqrt{\mu_M}}{w}, F_i)\} + \varepsilon^2\tilde{A},
\end{aligned}
$$

where $\tilde{A} = -\frac{w\{\partial_t + v\cdot\nabla_x\}F_5}{\sqrt{\mu_M}} + \sum_{i+j\geq 6, i\leq 5, j\leq 5}\varepsilon^{i+j-6}\frac{w}{\sqrt{\mu_M}}\mathcal{Q}(F_i, F_j)$.

By Duhamel's principle, we have $h^\varepsilon(t,x,v) =$

$$
\begin{aligned}
&\exp\{-\frac{\nu t}{\varepsilon}\}h^\varepsilon(0, x-vt, v) - \int_0^t \exp\{-\frac{\nu(t-s)}{\varepsilon}\}\left(\frac{1}{\varepsilon}K_{M,w}h^\varepsilon\right)(s, x-v(t-s), v)ds \\
&+ \int_0^t \exp\{-\frac{\nu(t-s)}{\varepsilon}\}\left(\frac{\varepsilon^2 w}{\sqrt{\mu_M}}\mathcal{Q}(\frac{h^\varepsilon\sqrt{\mu_M}}{w}, \frac{h^\varepsilon\sqrt{\mu_M}}{w})\right)(s, x-v(t-s), v)ds \\
&+ \int_0^t \exp\{-\frac{\nu(t-s)}{\varepsilon}\}\left(\sum_{i=1}^5\varepsilon^{i-1}\frac{w}{\sqrt{\mu_M}}\mathcal{Q}(F_i, \frac{h^\varepsilon\sqrt{\mu_M}}{w})\right)(s, x-v(t-s), v)ds \\
&+ \int_0^t \exp\{-\frac{\nu(t-s)}{\varepsilon}\}\left(\sum_{i=1}^5\varepsilon^{i-1}\frac{w}{\sqrt{\mu_M}}\mathcal{Q}(\frac{h^\varepsilon\sqrt{\mu_M}}{w}, F_i)\right)(s, x-v(t-s), v)ds \\
&+ \int_0^t \exp\{-\frac{\nu(t-s)}{\varepsilon}\}\varepsilon^2\tilde{A}(s, x-v(t-s), v)ds. \qquad (15)
\end{aligned}
$$

Since $|\frac{w}{\sqrt{\mu_M}}\mathcal{Q}(\frac{h^\varepsilon\sqrt{\mu_M}}{w}, \frac{h^\varepsilon\sqrt{\mu_M}}{w})| \leq C\{\nu(\mu)|h^\varepsilon(v)| + \|h^\varepsilon\|_\infty\}\|h^\varepsilon\|_\infty$ from Lemma 9 of [G2], and since

$$
\begin{aligned}
\nu(\mu) &= c\int |v-u|\mu du \backsim |v|\rho(t,x) \backsim \nu_M(v), \\
\int_0^t \exp\{-\frac{\nu(\mu)(t-s)}{\varepsilon}\}\nu(\mu)ds &\leq c\int_0^t \exp\{-\frac{c\nu_M(t-s)}{\varepsilon}\}\nu_M ds = O(\varepsilon),
\end{aligned}
$$

the second line in (15) is bounded by

$$
\begin{aligned}
(16)\quad & C\varepsilon^2\int_0^t \exp\{-\frac{\nu(\mu)(t-s)}{\varepsilon}\}C\{\nu(\mu)|h^\varepsilon(s,x-v(t-s),v)|+\|h^\varepsilon(s)\|_\infty\}\|h^\varepsilon(s)\|_\infty ds\\
\leq\ & C\varepsilon^3\sup_{0\leq s\leq t}\|h^\varepsilon(s)\|_\infty^2.
\end{aligned}
$$

From Lemma 9 from [G2] again,

$$\sum_{i=1}^5\varepsilon^{i-1}\frac{w}{\sqrt{\mu_M}}\{\mathcal{Q}(F_i,\frac{h^\varepsilon\sqrt{\mu_M}}{w})+\mathcal{Q}(\frac{h^\varepsilon\sqrt{\mu_M}}{w},F_i)\}\leq\nu_M(v)\|h^\varepsilon\|_\infty\|\frac{w}{\sqrt{\mu_M}}\sum_{i=1}^5\varepsilon^{i-1}F_i\|_\infty,$$

so that the third and fourth lines in (15) are bounded by

$$(17)\qquad C\int_0^t\exp\{-\frac{\nu(\mu)(t-s)}{\varepsilon}\}\nu_M(v)\|h^\varepsilon(s)\|_\infty ds\leq C\varepsilon\sup_{0\leq s\leq t}\|h^\varepsilon(s)\|_\infty.$$

The last line in (15) is clearly bounded by $C\varepsilon^3$.

We shall mainly concentrate on the second term in the right hand side of (15). Let $l_M(v,v')$ be the correspoding kernel associated with K_M in [C]. We have

$$(18)\qquad |l_M(v,v')|\leq C\{|v-v'|+\frac{1}{|v-v'|}\}\exp\{-c|v-v'|^2-c\frac{\||v|^2-|v'|^2|^2}{|v-v'|^2}\}.$$

Since $\nu(\mu)\backsim\nu_M$, we bound the second term by

$$\frac{1}{\varepsilon}\int_0^t\exp\{-\frac{\nu(t-s)}{\varepsilon}\}\int_{\mathbf{R}^3}|l_{M,w}(v,v')h^\varepsilon(s,x-v(t-s),v')|dv'ds,$$

where $l_{M,w}=\frac{w(v)}{w(v')}l_M$. We now use (15) again to evaluate h^ε. By (16) and (17), we can bound the above by

$$
\begin{aligned}
&\frac{1}{\varepsilon}\int_0^t\exp\{-\frac{\nu(t-s)}{\varepsilon}\}\sup_v\int_{\mathbf{R}^3}|l_{M,w}(v,v')|dv'\exp\{-\frac{\nu s}{\varepsilon}\}h^\varepsilon(0,x-v(t-s)-v's,v')|ds\\
&+\frac{1}{\varepsilon^2}\int_0^t\exp\{-\frac{\nu(t-s)}{\varepsilon}\}\int_{\mathbf{R}^3\times\mathbf{R}^3}|l_{M,w}(v,v')l_{M,w}(v',v'')\\
&\times|\int_0^s\exp\{-\frac{\nu(v')(s-s_1)}{\varepsilon}\}h^\varepsilon(s_1,x-v(t-s)-v'(s-s_1),v'')|dv'dv''ds_1ds\\
&+\frac{C}{\varepsilon}\int_0^t\exp\{-\frac{\nu(t-s)}{\varepsilon}\}ds\times\sup_v\int_{\mathbf{R}^3}|l_{M,w}(v,v')|dv'\times\{\varepsilon^3\sup_{0\leq s\leq t}\|h^\varepsilon(s)\|_\infty^2\}\\
&+\frac{C}{\varepsilon}\int_0^t\exp\{-\frac{\nu(t-s)}{\varepsilon}\}ds\times\sup_v\int_{\mathbf{R}^3}|l_{M,w}(v,v')|dv'\times\{\varepsilon\sup_{0\leq s\leq t}\|h^\varepsilon(s)\|_\infty\}\\
(19)&+\frac{C}{\varepsilon}\int_0^t\exp\{-\frac{\nu(t-s)}{\varepsilon}\}ds\times\sup_v\int_{\mathbf{R}^3}|l_{M,w}(v,v')|dv'\times\{\varepsilon^2\sup_{0\leq s\leq t}\|\tilde{A}\|_\infty\}.
\end{aligned}
$$

Since $\sup_v\int_{\mathbf{R}^3}|l_{M,w}(v,v')|dv'<+\infty$ from Lemma 7 of [G2], there is an upper bound except for the seond term as

$$C\{\|h^\varepsilon(0)\|_\infty+\varepsilon^3\sup_{0\leq s\leq t}\|h^\varepsilon(s)\|_\infty^2+\varepsilon\sup_{0\leq s\leq t}\|h^\varepsilon(s)\|_\infty+C\varepsilon^3\}.$$

We now concentrate on the second term in (19), which will be estimated as in the proof of Theorem 20 in [G2].

CASE 1: For $|v|\geq N$. By Lemma 7 in [G2],

$$\int\int l_{M,w}(v,v')l_{M,w}(v',v'')dv'dv''\leq\frac{C}{1+|v|}\leq\frac{C}{N},$$

and thus we have the following bound

$$\frac{C}{\varepsilon^2 N}\int_0^t\int_0^s\exp\{-\frac{\nu(v)(t-s)}{\varepsilon}\}\exp\{-\frac{\nu(v')(s-s_1)}{\varepsilon}\}\|h^\varepsilon(s_1)\|_\infty ds_1ds\leq\frac{C}{N}\sup_{0\leq s\leq t}\|h^\varepsilon(s)\|_\infty.$$

CASE 2: For $|v|\leq N$, $|v'|\geq 2N$, or $|v'|\leq 2N$, $|v''|\geq 3N$. Notice that we have either $|v'-v|\geq N$ or $|v'-v''|\geq N$, and either one of the following is valid correspondingly for some $\eta>0$:

(20)
$$|l_{M,w}(v,v')|\leq e^{-\frac{\eta}{8}N^2}|l_{M,w}(v,v')e^{\frac{\eta}{8}|v-v'|^2}|,\qquad |l_{M,w}(v',v'')|\leq e^{-\frac{\eta}{8}N^2}|l_{M,w}(v',v'')e^{\frac{\eta}{8}|v'-v''|^2}|.$$

From Lemma 7 in [G2], both $\int|l_{M,w}(v,v')e^{\frac{\eta}{8}|v-v'|^2}|$ and $\int|l_{M,w}(v',v'')e^{\frac{\eta}{8}|v'-v''|^2}|$ are still finite. We use (20) to combine the cases of $|v'-v|\geq N$ or $|v'-v''|\geq N$ as:

$$\begin{aligned}
&\int_0^t\int_0^s\left\{\int_{|v|\leq N,|v'|\geq 2N,}+\int_{|v'|\leq 2N,|v''|\geq 3N}\right\}\\
\leq\ & C\int_0^t\int_0^s\left\{\int_{|v|\leq N,|v'|\geq 2N,}|l_{M,w}(v,v')|dv'+\sup_{v'}\int_{|v'|\leq 2N,|v''|\geq 3N}|l_{M,w}(v',v'')|dv''\right\}\\
\leq\ & \frac{C_\eta}{\varepsilon^2}e^{-\frac{\eta}{8}N^2}\int_0^t\int_0^s\exp\{-\frac{\nu(v)(t-s)}{\varepsilon}\}\exp\{-\frac{\nu(v')(s-s_1)}{\varepsilon}\}\|h^\varepsilon(s_1)\|_\infty ds_1ds\\
\leq\ & C_\eta e^{-\frac{\eta}{8}N^2}\sup_{0\leq s\leq t}\{\|h^\varepsilon(s)\|_\infty\}. \qquad (21)
\end{aligned}$$

CASE 3: $s-s_1\leq\varepsilon\kappa$, for $\kappa>0$ small. We bound the last term in (19) by

$$\begin{aligned}
&\frac{1}{\varepsilon^2}\int_0^t\int_{s-\varepsilon\kappa}^s C\exp\{-\frac{\nu(v)(t-s)}{\varepsilon}\}\exp\{-\frac{\nu(v')(s-s_1)}{\varepsilon}\}\|h^\varepsilon(s_1)\|_\infty ds_1ds\\
\leq\ & C\sup_{0\leq s\leq t}\{\|h^\varepsilon(s)\|_\infty\}\times\frac{1}{\varepsilon}\int_0^t\exp\{-\frac{\nu(v)(t-s)}{\varepsilon}\}ds\times\int_{s-\varepsilon\kappa}^s\frac{1}{\varepsilon}ds_1\\
\leq\ & \kappa C\sup_{0\leq s\leq t}\{\|h^\varepsilon(s)\|_\infty\}. \qquad (22)
\end{aligned}$$

CASE 4. $s-s_1\geq\varepsilon$, and $|v|\leq N$, $|v'|\leq 2N, |v''|\leq 3N$. This is the last remaining case because if $|v'|>2N$, it is included in Case 2; while if $|v''|>3N$, either $|v'|\leq 2N$ or $|v'|\geq 2N$ are also included in Case 2. We now can bound the second term in (19) by

$$C\int_0^t\int_B\int_0^{s-\varepsilon\kappa}e^{-\frac{\nu(v)(t-s)}{\varepsilon}}e^{-\frac{\nu(v')(s-s_1)}{\varepsilon}}|l_{M,w}(v,v')l_{M,w}(v',v'')h^\varepsilon(s_1,x_1-(s-s_1)v',v'')|$$

where $B=\{|v'|\leq 2N,\ |v''|\leq 3N\}$. By (18), $l_{M,w}(v,v')$ has possible integrable singularity of $\frac{1}{|v-v'|}$, we can choose $l_N(v,v')$ smooth with compact support such that

$$\sup_{|p|\leq 3N}\int_{|v'|\leq 3N}|l_N(p,v')-l_{M,w}(p,v')|dv'\leq\frac{1}{N}. \qquad (23)$$

Splitting

$$\begin{aligned} l_{M,w}(v,v')l_{M,w}(v',v'') &= \{l_{M,w}(v,v')-l_N(v,v')\}l_{M,w}(v',v'') \\ &\quad +\{l_{M,w}(v',v'')-l_N(v',v'')\}l_N(v,v')+l_N(v,v')l_N(v',v''), \end{aligned}$$

we can use such an approximation (23) to bound the above s_1, s integration by

$$\begin{aligned} (24)\quad & \frac{C}{N}\sup_{0\le s\le t}\{\|h^\varepsilon(s)\|_\infty\}\times\left\{\sup_{|v'|\le 2N}\int |l_{M,w}(v',v'')|dv''+\sup_{|v|\le 2N}\int |l_N(v,v')|dv'\right\} \\ & +C\int_0^t\int_B\int_0^{s-\varepsilon\kappa} e^{-\frac{\nu(v)(t-s)}{\varepsilon}}e^{-\frac{\nu(v')(s-s_1)}{\varepsilon}}|l_N(v,v')l_N(v',v'')|h^\varepsilon(s_1,x_1-(s-s_1)v',v'')|. \end{aligned}$$

Since $l_N(v,v')l_N(v',v'')$ is bounded, we first integrate over v' to get

$$\begin{aligned} & C_N\int_{|v'|\le 2N}|h^\varepsilon(s_1,x_1-(s-s_1)v',v'')|dv' \\ \le\ & C_N\left\{\int_{|v'|\le 2N}\mathbf{1}_\Omega(x_1-(s-s_1)v')|h^\varepsilon(s_1,x_1-(s-s_1)v',v'')|^2dv'\right\}^{1/2} \\ \le\ & \frac{C_N}{\kappa^{3/2}\varepsilon^{3/2}}\left\{\int_{|y-x_1|\le(s-s_1)3N}|h^\varepsilon(s_1,y,v'')|^2dy\right\}^{1/2} \\ \le\ & \frac{C_N\{(s-s_1)^{3/2}+1\}}{\kappa^{3/2}\varepsilon^{3/2}}\left\{\int_{\mathbf{\Omega}}|h^\varepsilon(s_1,y,v'')|^2dy\right\}^{1/2}. \end{aligned}$$

Here we have made a change of variable $y=x_1-(s-s_1)v'$, and for $s-s_1\ge\kappa\varepsilon$, $|\frac{dy}{dv'}|\ge\frac{1}{\kappa^3\varepsilon^3}$. In the case of $\Omega=\mathbf{R}^3$, the factor $\{(s-s_1)^{3/2}+1\}$ is not needed. By (10) and (9), we then further control the last term in (24) by:

$$\begin{aligned} & \frac{C_{N,\kappa}}{\varepsilon^{7/2}}\int_0^t\int_0^{s-\kappa\varepsilon} e^{-\frac{\nu(v)(t-s)}{\varepsilon}}e^{-\frac{\nu(v')(s-s_1)}{\varepsilon}}\{(s-s_1)^{3/2}+1\}\int_{|v''|\le 3N}\left\{\int_{\mathbf{\Omega}}|h^\varepsilon(s_1,y,v'')|^2dy\right\}^{1/2}dv''ds_1ds \\ \le\ & \frac{C_{N,\kappa}}{\varepsilon^{7/2}}\int_0^t\int_0^{s-\kappa\varepsilon} e^{-\frac{\nu(v)(t-s)}{\varepsilon}}e^{-\frac{\nu(v')(s-s_1)}{\varepsilon}}\{(s-s_1)^{3/2}+1\}\left\{\int_{|v''|\le 3N}\int_{\Omega}|f^\varepsilon(s_1,y,v'')|^2dydv''\right\}^{1/2}ds_1ds \\ \le\ & \frac{C_{N,\kappa}}{\varepsilon^{3/2}}\sup_{0\le s\le t}\|f^\varepsilon(s)\|_2. \end{aligned}$$

In summary, we have established, for any $\kappa>0$ and large $N>0$,

$$\begin{aligned} \sup_{0\le s\le t}\{\varepsilon^{3/2}\|h^\varepsilon(s)\|_\infty\} &\le \{\kappa+\frac{C_\kappa}{N}\}\sup_{0\le s\le t}\{\varepsilon^{3/2}\|h^\varepsilon(s)\|_\infty\}+\varepsilon^{7/2}C+C_{\varepsilon,N}\|\varepsilon^{3/2}h_0\|_\infty \\ &\quad +\sqrt{\varepsilon}C\sup_{0\le s\le t}\{\varepsilon^{3/2}\|h^\varepsilon(s)\|_\infty\}^2+C_{N,\kappa}\sup_{0\le s\le t}\|f^\varepsilon(s)\|_2. \end{aligned}$$

For sufficiently small $\varepsilon>0$, first choosing κ small, then N sufficiently large so that $\{\kappa+\frac{C_\kappa}{N}\}<\frac{1}{2}$,

$$\sup_{0\le s\le\tau}\{\varepsilon^{3/2}\|h^\varepsilon(s)\|_\infty\}\le C\{\|\varepsilon^{3/2}h_0\|_\infty+\sup_{0\le s\le\tau}\|f^\varepsilon(s)\|_2+\varepsilon^{7/2}\}.$$

and we conclude our proof. $\square$

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