International Journal of Computer Mathematics
Vol. 00, No. 00, Month 200x, 1–20

A Hybrid Meshless-Collocation/Spectral-Element Method for Elliptic Partial Differential Equations

Christopher D. Blakely

University of Maryland, College Park, Center for Scientific Computation and Mathematical Modeling
cblakely@cscamm.umd.edu

(*July 2006*)

We present a hybrid numerical technique that couples spectral element approximation methods with meshless collocation methods for use in solving elliptic boundary value partial differential equations. After briefly reviewing the empirical Backus-Gilbert meshless collocation method by Blakely in [4], we introduce a domain decomposition procedure which effectively couples nodal spectral element approximations with the meshless collocation method. This domain decomposition approach is an adaptation of the three-field variational formulation by Brezzi et al. [6] which uses two additional functional spaces along the interface between the different approximations. Sufficient and necessary conditions for the hybrid numerical approach to yield stable approximations will be discussed followed by numerical examples using the Helmholtz equation and different choices of discrete three-field spaces.

Keywords: Meshless Collocation; Three-Field Formulation; Domain Decomposition; Elliptic PDEs; Spectral Element Method

AMS Subject Classifications: 33F05

1. Introduction and Motivations

The purpose of this paper is to present a hybrid numerical technique that couples spectral element approximation methods with meshless collocation methods for use in solving elliptic boundary value partial differential equations. The motivation for constructing such a hybrid scheme is ultimately to reap the advantages of each method, which is namely to couple the high-order approximation ability of the spectral-element method with geometrically versatile meshless collocation schemes. The main concern and difficulty that comes into mind when building such a hybrid method is in choosing the appropriate domain decomposition method for effectively coupling the two methods. We propose to tackle such a task using an innovative domain decomposition technique developed by Brezzi et al. in [6] called the three-field formulation.

Coupled with the nodal Lagrangian spectral element method is the empirical Backus-Gilbert reproducing kernel meshless collocation method developed by Blakely in [4] which was shown to be quite versatile in choosing the approximation space of the method while endowed with high-order accuracy in approximation. However, as with other meshless collocation methods, high collocation node counts in the computational domain renders the method ill-conditioned thus making it computationally infeasible for large-scale problems such as geophysical fluid dynamics on the sphere. This paper provides a computational procedure in which meshless collocation can perhaps be utilized in large-scale problems without succumbing to its ill-conditioned properties thanks to a coupling with the spectral element method.

The first section reviews the empirical Backus-Gilbert meshless collocation technique introduced in [4] which constructs in an empirical manner a discrete reproducing kernel on a given bounded domain. The second part of this paper deals with the domain decomposition technique called the three-field formulation for coupling this meshless collocation method with spectral element approximations. The continuous

International Journal of Computer Mathematics
ISSN 0020-7160 print/ISSN 1029-0265 online
http://www.tandf.co.uk/journals
DOI: 10.1080/0020716YYxxxxxxxx

version of the method will be reviewed followed by the hybrid discretization using spectral element and reproducing kernel spaces. Finally, we discuss stability issues of the method along with numerical examples using a Helmholtz elliptic equation.

2. Empirical Backus-Gilbert Reproducing Kernel Method

The empirical Backus-Gilbert Reproducing Kernel Method (EBGRK) was originally introduced in [4] as an attractive alternative to the original Backus-Gilbert moving least squares method discussed in papers such as [10] and [5]. As demonstrated in [4], the EBGRK utilizes about the same amount of computational time in building an approximation, however resulting collocation and differentiation matrices are much better conditioned than in the standard Backus-Gilbert method. Furthermore, the new empirical approach retains the versatile property of Backus-Gilbert moving least-squares approximation in that one can build an approximation that reproduces any given discrete space of functions, normally allocated to either a set of polynomials or radial basis functions. For the proposed hybrid method in this paper, we use the latter. set of functions. In this section, we briefly review the EBGRK method which begins with the standard Backus-Gilbert approximation approach which is then followed by the discretization of differential operators using the empirical discrete reproducing kernels.

Before explaining the EBGRK method, we begin by reviewing the standard Backus-Gilbert approximation scheme which provides the motivation for the EBGRK method. For details on the method, we refer the reader to [10] and relavent references therein. We consider a quasi-interpolant of the form

$$Pf(\mathbf{x}) = \sum_{i=1}^{N} f(\mathbf{x}_i)\Psi_i(\mathbf{x}) \tag{1}$$

where $\mathbf{f} = [f(\mathbf{x}_1), ..., f(\mathbf{x}_N)]^T$ represents the given data on a set of N distinct evaluation nodes $\mathcal{X} = \{\mathbf{x}_1, \ldots, \mathbf{x}_N\}$ on a bounded domain $\Omega \subset \mathbb{R}^2$ of some unknown function $f(\mathbf{x})$, and $Pf(\mathbf{x})$ is denoted as the approximation of f on Ω.. The finite set of nodes $\mathcal{X}$ is endowed with a *seperation distance* defined as

$$q_{\mathcal{X}} := \frac{1}{2} \min_{\mathbf{x}_j \neq \mathbf{x}_i} \|\mathbf{x}_i - \mathbf{x}_j\|_2.$$

The discrete reproducing kernel $\Psi_i(\mathbf{x})$, or quasi-interpolant in some literature, is constructed to be minimized in a discrete quadratic expression subject to some approximation space reproduction constraints. This is done by choosing an approximation space $\mathcal{U}_{\phi,\mathcal{X}}$ of dimension N built from radial functions translated on a set of distinct nodes $\Xi = \{\xi_1, \ldots, \xi_N\} \subset \Omega \cup \partial\Omega$. Namely $\mathcal{U}_{\phi,\mathcal{X}} := \text{span}\{\phi(||\cdot - \xi_j||),\ \xi_j \in \Xi\}$. Denoting $\phi_j(\mathbf{x}) := \phi(||\mathbf{x} - \xi_j||),\ j = 1, \ldots, N$, the approximation space reproduction constraints are written as

$$\sum_{i=0}^{N} \phi_j(\mathbf{x}_i)\Psi_i(\mathbf{x}) = \phi_j(\mathbf{x}), \quad \text{for all } \phi_j \in \mathcal{U}_{\phi,\mathcal{X}}, \tag{2}$$

or in matrix form as

$$A\Psi(\mathbf{x}) = \phi(\mathbf{x}),$$

with $A_{ji} = \phi_j(\mathbf{x}_i), i = 1, \ldots, N, j = 1, \ldots, N$. The objective is to minimize

$$\frac{1}{2} \sum_{i=0}^{N} \Psi_i^2(\mathbf{x}) W(\mathbf{x}, \mathbf{x}_i), \tag{3}$$

subject to the above reproducing approximation space constraints. $W(\mathbf{x}, \mathbf{x}_i)$ is a positive weight function for any pair $\mathbf{x} \in \mathbb{R}^2$ and $\mathbf{x}_i \in \mathcal{X}$.

To solve this moving-least squares problem, Lagrangian multipliers $\lambda_j(\mathbf{x}), j = 1, \ldots, N$ are introduced which leads to a system where we solve for both the Lagrangian multipliers and the reproducing kernel functions. This system can be written in block matrix form as

$$\begin{bmatrix} \mathbf{Q}(\mathbf{x}) & -\mathbf{A}^T \\ \mathbf{A} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \Psi(\mathbf{x}) \\ \lambda(\mathbf{x}) \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \phi(\mathbf{x}) \end{bmatrix}, \tag{4}$$

where $Q(\mathbf{x}) = \text{diag}\Big(W(\mathbf{x}, \mathbf{x}_1), \ldots, W(\mathbf{x}, \mathbf{x}_N)\Big)$ which is positive definite for any $\mathbf{x}$. Since we are assuming that the sets Ξ and $\mathcal{X}$ are comprised of distinct nodes in the domain of interest, $\mathbf{A}$ has full rank and we can apply Gaussian elimination to the block matrix and arrive at

$$\lambda(\mathbf{x}) = G(\mathbf{x})^{-1}\phi(\mathbf{x}), \tag{5}$$

where we define $G(\mathbf{x}) = AQ^{-1}(\mathbf{x})A^T$ as the Gram matrix. The explicit form of the reproducing kernels can then be given as

$$\Psi_i(\mathbf{x}) = \sum_{j=1}^{M} \lambda_j(\mathbf{x})\phi(\mathbf{x}_i, \xi_i)\frac{1}{W(\mathbf{x}, \mathbf{x}_i)}.$$

So for every evaluation node $\mathbf{x} \in \mathcal{X}$, an $N \times N$ system of equations must be solved in order to calculate $\Psi_i(\mathbf{x})$. We note here that the closeness of the generating functions $\Psi_i(\mathbf{x})$ to the pointwise cardinal functions, (i.e., $\Psi_i(\mathbf{x}_j) = \delta_{i,j}, i, j = 1, ...N$) determines how well the summation in (1) approximates the function $f(\mathbf{x})$.

The EBGRK method utilizes a similar construction. We first begin by setting the weight function $W(\cdot, \mathbf{x}_i) \equiv 1$. Suppose we have N distinct evaluation nodes in the domain of interest $\Omega \cup \partial\Omega$ which can either be uniformly or randomly distributed. We can thus write the $N \times N$ interpolation matrix based on these evaluation nodes as

$$\mathbf{A} = \begin{pmatrix} \phi_1(\mathbf{x}_1) & \phi_2(\mathbf{x}_1) & \cdots & \phi_N(\mathbf{x}_1) \\ \phi_1(\mathbf{x}_2) & \phi_2(\mathbf{x}_2) & \cdots & \phi_N(\mathbf{x}_2) \\ \vdots & & & \\ \phi_1(\mathbf{x}_N) & \phi_2(\mathbf{x}_N) & \cdots & \phi_N(\mathbf{x}_N) \end{pmatrix}. \tag{6}$$

Using this matrix, we consider the resulting Q matrix from a QR decompositon algorithm, namely $A = QR$. A new basis $\{\tilde{\phi}_j(\cdot)\}_{j=1}^N$ from the rows of Q is provided which is orthonormal with respect to the evalutation nodes in the sense that

$$\langle \tilde{\phi}_j, \tilde{\phi}_k \rangle = \sum_{i=1}^{N} \tilde{\phi}_j(\mathbf{x}_i)\tilde{\phi}_k(\mathbf{x}_i) = \delta_{j,k}.$$

Going back to the Backus-Gilbert formulation using this new basis, we have the reproduction constraints

written as

$$\begin{pmatrix} \langle \tilde{\phi}_1, \tilde{\phi}_1 \rangle & \langle \tilde{\phi}_1, \tilde{\phi}_2 \rangle & \cdots & \langle \tilde{\phi}_1, \tilde{\phi}_M \rangle \\ \langle \tilde{\phi}_2, \tilde{\phi}_1 \rangle & \langle \tilde{\phi}_2, \tilde{\phi}_2 \rangle & \cdots & \langle \tilde{\phi}_2, \tilde{\phi}_M \rangle \\ \vdots & & & \\ \langle \tilde{\phi}_M, \tilde{\phi}_1 \rangle & \langle \tilde{\phi}_M, \tilde{\phi}_2 \rangle & \cdots & \langle \tilde{\phi}_M, \tilde{\phi}_M \rangle \end{pmatrix} \begin{pmatrix} \lambda_1(\mathbf{x}) \\ \lambda_2(\mathbf{x}) \\ \vdots \\ \lambda_M(\mathbf{x}) \end{pmatrix} = \begin{pmatrix} \tilde{\phi}_1(\mathbf{x}) \\ \tilde{\phi}_2(\mathbf{x}) \\ \vdots \\ \tilde{\phi}_M(\mathbf{x}) \end{pmatrix}. \tag{7}$$

But since $\langle \tilde{\phi}_j, \tilde{\phi}_k \rangle = \delta_{j,k}$ this means that $\lambda_j(\mathbf{x}) = \tilde{\phi}_j(\mathbf{x})$ for $j = 1, \ldots, M$ and so the Backus-Gilbert formulation yields a discrete reproducing kernel

$$\Psi_i(\mathbf{x}_k) = \sum_{j=1}^{N} \lambda_j(\mathbf{x}_k)\tilde{\phi}_j(\mathbf{x}_i) = \sum_{j=1}^{N} \tilde{\phi}(||\mathbf{x}_k - \xi_j||)\tilde{\phi}(||\mathbf{x}_i - \xi_j||), \tag{8}$$

which also forms a partition of unity on the domain with respect to the node distribution $\mathcal{X}$. With the definition of the kernel now at hand, we define the Ψ reproducing kernel approximation space as $\mathcal{V}_{\Psi,\mathcal{X}} = \text{span}\{\Psi_i(\cdot),\ i = 1, \ldots, N\}$.

Now consider a differential operator L and the problem of approximating $Lf(\mathbf{x})$ for $\mathbf{x} \in \Omega$ from the given data $\mathbf{f} = \{f(\mathbf{x}_1), \ldots, f(\mathbf{x}_N)\}$. This amounts to constructing the approximation

$$Lf(\mathbf{x}_j) = \sum_{i=1}^{N} L\Psi_i(\mathbf{x}_j) f(\mathbf{x}_i)$$

given the reproducing kernel Ψ constructed from (8). The problem is that we do not know how L acts on Ψ_i at a given node $\mathbf{x}_j$. In order to solve this problem, we project $\tilde{\phi}_j(\mathbf{x})$ for each $j = 1, \ldots, N$ onto $\mathcal{U}_{\phi,\mathcal{X}}$ by calculating the vector $\mathbf{c}^j \in \mathbb{R}^N$ defined as

$$\begin{pmatrix} \phi_1(\mathbf{x}_1) & \phi_2(\mathbf{x}_1) & \cdots & \phi_N(\mathbf{x}_1) \\ \phi_1(\mathbf{x}_2) & \phi_2(\mathbf{x}_2) & \cdots & \phi_N(\mathbf{x}_2) \\ \vdots & & & \\ \phi_1(\mathbf{x}_N) & \phi_2(\mathbf{x}_N) & \cdots & \phi_N(\mathbf{x}_N) \end{pmatrix} \begin{pmatrix} c_1^j \\ c_2^j \\ \vdots \\ c_M^j \end{pmatrix} = \begin{pmatrix} \tilde{\phi}_j(\mathbf{x}_1) \\ \tilde{\phi}_j(\mathbf{x}_2) \\ \vdots \\ \tilde{\phi}_j(\mathbf{x}_N) \end{pmatrix}, \quad \text{for } \mathbf{x}_i \in \mathcal{X}. \tag{9}$$

We can then approximate $L\tilde{\phi}_j(\mathbf{x}_k)$ for some $\mathbf{x}_k \in \mathcal{X}$ as

$$L\tilde{\phi}_j(\mathbf{x}_k) = \sum_{i=1}^{N} c_i^j L\phi_i(\mathbf{x}_k)$$

These projections can be efficiently implemented by storing the LU decomposition of the collocation matrix A and then applying direct substition for each vector $\mathbf{c}^j \in \mathbb{R}^N$, $j = 1, \ldots, N$.

To formulate a reproducing kernel that reproduces $Lf(\mathbf{x})$ using the original Backus-Gilbert formulation, presented in the beginning of this section, we want to construct a kernel $\Upsilon_i(\mathbf{x})$ such that construct a kernel $\Upsilon_i(\mathbf{x})$ such that

$$Lf(\mathbf{x}) = \sum_{i=1}^{N} \Upsilon_i(\mathbf{x}) f(\mathbf{x}_i) \quad \mathbf{x} \in \mathcal{X}$$

via Backus-Gilbert approximation using the orthonormal basis. We can apply the reproducing constraints

to this kernel as

$$L\tilde{\phi}(\mathbf{x}) = \sum_{i=1}^{N} \tilde{\phi}(\mathbf{x}_i)\Upsilon_i(\mathbf{x}) \quad \mathbf{x} \in \mathcal{X},$$

while minimizing $\frac{1}{2}\sum_{i=1}\Upsilon_i^2(\mathbf{x})$. To solve for Υ we introduce Lagrangian multipliers $\tilde{\lambda}_j$, $j = 1\ldots, M$ to this constrained optimization problem which yields the system

$$\begin{pmatrix} \langle\tilde{\phi}_1,\tilde{\phi}_1\rangle & \langle\tilde{\phi}_1,\tilde{\phi}_2\rangle & \cdots & \langle\tilde{\phi}_1,\tilde{\phi}_M\rangle \\ \langle\tilde{\phi}_2,\tilde{\phi}_1\rangle & \langle\tilde{\phi}_2,\tilde{\phi}_2\rangle & \cdots & \langle\tilde{\phi}_2,\tilde{\phi}_M\rangle \\ \vdots & & & \\ \langle\tilde{\phi}_M,\tilde{\phi}_1\rangle & \langle\tilde{\phi}_M,\tilde{\phi}_2\rangle & \cdots & \langle\tilde{\phi}_M,\tilde{\phi}_M\rangle \end{pmatrix} \begin{pmatrix} \tilde{\lambda}_1(\mathbf{x}) \\ \tilde{\lambda}_2(\mathbf{x}) \\ \vdots \\ \tilde{\lambda}_M(\mathbf{x}) \end{pmatrix} = \begin{pmatrix} L\tilde{\phi}_1(\mathbf{x}) \\ L\tilde{\phi}_2(\mathbf{x}) \\ \vdots \\ L\tilde{\phi}_M(\mathbf{x}) \end{pmatrix}. \tag{10}$$

Using the approximation of $L\tilde{\phi}_j(\mathbf{x})$ from the original basis, the expression for the dicrete kernel then becomes

$$\Upsilon_i(\mathbf{x}) = \sum_{j=1}^{M} \tilde{\lambda}_j(\mathbf{x})\tilde{\phi}_j(\mathbf{x}_i) = \sum_{j=1}^{M} L\tilde{\phi}_j(\mathbf{x})\tilde{\phi}_j(\mathbf{x}_i) \quad \text{for } \mathbf{x} \in \mathcal{X}.$$

With this Backus-Gilbert construction, it is not hard to see that $\Upsilon_i(\mathbf{x}_i) = L\Psi_i(\mathbf{x}_i)$ for any $\mathbf{x}_i \in \mathcal{X}$. Thus

$$Lf(\mathbf{x}_j) = \sum_{i=1}^{N} \Upsilon_i(\mathbf{x}_j)f(\mathbf{x}_i) = \sum_{i=1}^{N} L\Psi_i(\mathbf{x}_j)f(\mathbf{x}_i).$$

With the meshless collocation approximation of differential operators at hand, the approximation of PDEs can now be handled for an arbitrary domain Ω without the use of a mesh by constructing the space $\mathcal{V}_{\Psi,\mathcal{X}}$.

3. The Coupling Procedure

Because the meshless approximation utilizes collocation in strong form of PDEs, certain transition conditions are needed on the boundary of the subdomain connecting the meshless and spectral-element approximations in order to satisfy continuity and flux conditions of the solution along with the artificial fluxes of the field variables. In 1994, Brezzi and Marini (see [6]) developed a method termed the *three-field formulation* for *hybrid* finite-element formulations where the goal was to give the possibility of coupling different finite-element approximations using different meshes and basis functions from one subdomain to another. The method was shown to successfully couple different numerical approximations in the context of domain decompostion when applied to elliptic problems on rectangular domains. As we will show later in this section, the theory for the three-field formulation is derived from the Lagrangian multiplier method for implementing Dirichlet boundary conditions in weak form for elliptic problems developed by Babuska for the finite-element method. He derived optimal convergence results under the restriction that the basis functions used for the solution and the Lagrangian multipliers satisfy a certain inf-sup condition, or sometimes termed, the *LBB* condition. Imposing essential Dirichlet boundary conditions in the weak sense has found great interest in the context of wavelet discretizations (cf. [3]) and in nonconforming domain decomposition methods where one typically works with nonmatching grids on the decomposition skeleton.

In this paper, we extend the idea of the three-field technique to couple spectral-element and meshless collocation methods for solving the elliptic problem

$$Hu = \Delta u(\mathbf{x}) + g(\mathbf{x})u(\mathbf{x}) = f(\mathbf{x}), \quad \in \Omega, \tag{11}$$

with boundary conditions

$$u(\mathbf{x}) = h(\mathbf{x}) \quad \mathbf{x} \in \delta\Omega$$

where Ω is a bounded domain. The discretization of the Helmholtz operator $H = \Delta + g(\mathbf{x})$ using either spectral-elements or meshless collocation using reproducing kernels leads to the matrix with the general form

$$H_N = \mathbf{DBD}^T + \mathbf{gB} \tag{12}$$

where $\mathbf{D}$ is the discrete differential matrix for the corresponding basis and $\mathbf{B}$ is the projection matrix using spectral elements or the collocation matrix using meshless collocation, depending on the discretization. In this section we propose a method based on the three-field variational formulation which couples these approximations on a given domain Ω where a part of the domain utilizes spectral element approximation and the rest is allocated to meshless collocation using discrete reproducing kernels.

In order to introduce the method, we must first review the necessary functional spaces for the continuous version of the three-field variational formulation that will be used in the discretization as well as give a discussion of their relevant physical meaning.

3.1. *The Continuous Three-Field Formulation*

Suppose we have a subdomain of M unioned elements $\Omega^M = \cup_{i=1}^M \Omega^{e_i}$. Let $\partial\Omega^M$ denote the boundary of Ω^M and Γ denote the collection of the boundaries of Ω^{e_i} such that $\partial\Omega^{e_i} \cap \partial\Omega^M = \{0\}$, usually called the skeleton of the decomposition. We then take Γ_i to be $\partial\Omega^{e_i}$ for each i subject to the interior of Ω^M, namely $\Gamma_i \cap \partial\Omega^M = \{0\}$ for each i. Finally, we denote Ω_1 the collection of spectral elements in Ω^M and set $\Omega_2 := \Omega^M - \Omega_1$, which will be allocated to meshless reproducing kernel approximation.

Throughout the presentationn of this three-field formulation for coupling spectral-elements and meshless collocation, we take $i = 2$ and set $\Omega_1 := \Omega^{SE}$, $\Omega_2 = \Omega - \Omega^{SE}$ and $\Gamma_i := \partial\Omega_i \cap \partial\Omega$, for $i = [1, 2]$. In this case, $\Gamma = \Gamma_i$ for $i = [1, 2]$.

As with any other hybrid domain decomposition method, the functional spaces play an important role in the Hybrid SE/M method. Since we are dealing with the elliptic Helmholtz equation, the Sobolev spaces $H^1(\Omega_i)$ and $H_0^1(\Omega_i)$ on each domain Ω_i are the natural choices to handle the variational formulation which we will soon describe.

Utilizing the interfaces Γ and Γ_i leads to additional types of spaces that will be needed for domain decomposition. The first is the trace space on each Γ_i denoted as $H^{1/2}(\Gamma_i)$ which is a fractional Sobolev space which can be theoretically be obtained by the K-method interpolation as (see e.g. DeVore [9])

$$H^{1/2}(\Gamma_i) = [L^2(\Gamma_i), H^1(\Gamma_i)]_{1/2}$$

which includes functions in $H^1(\Gamma_i)$ restricted to Ω_i. We will also use the other type of trace space important in domain decomposition methods which is denoted as $H_{00}^{1/2}(\Gamma_i)$ and can be obtained by K-method interpolation as

$$H_{00}^{1/2}(\Gamma_i) = [L^2(\Gamma_i), H_0^1(\Gamma_i)]_{1/2}$$

which gives the maximal subspace of functions in $H^{1/2}(\Gamma_i)$ which belongs to $H^{1/2}(\partial\Omega_i)$ when extended by zero to the rest of $\partial\Omega_i$. Notice that this space is only used if $\partial\Omega_i \cap \Gamma_i \neq \{0\}$, otherwise $H^{1/2}(\Gamma_i)$ is used. Their corresponding norms are

$$\|\cdot\|_{\Gamma_i} := \|\cdot\|_{H^{1/2}(\Gamma_i)}$$

if $\partial\Omega_i \cap \Gamma_i \neq \{0\}$ and

$$\|\cdot\|_{\Gamma_i} := \|\cdot\|_{H^{1/2}_{00}(\Gamma_i)}$$

otherwise. We also need the dual of these trace spaces which we denote as $(H^{1/2}(\Gamma_i))' = H^{-1/2}(\Gamma_i)$. Further details of these function spaces and their properties can be found in the appendix. We now focus on the specific spaces that will be used throughout the description of the three-field variational formulation.

The first set of Sobolev spaces

$$\mathbf{V} := \Pi_{i=1}^{2} V_i, \tag{13}$$

where $V_i := V_{\Omega_i} = \{v^i \in H^1(\Omega_i) \ : \ v|_{\partial\Omega_i \cap \partial\Omega} = 0\}$ defines the space for the solution u^i of the Helmholtz equation on each subdomain Ω_i. In order for the global solution $u = (u^1, u^2)$ to be continuous across the interface Γ, while enforcing continuous flux conditions on each Γ_i

The second set of functional spaces we define are spaces of Lagrangian multipliers which provide the role of enforcing necessary flux boundary continuity over the interfaces Γ_i and are defined as $\Lambda_i := H^{-1/2}(\Gamma_i)$ for $i = [1, 2]$ which can be regarded as the dual of the trace spaces associated with the two Hilbert spaces V_i. The Lagrangian multiplier space is endowed with the standard scalar inner product $L^2(\Gamma_i)$, $\langle \Lambda_i, H^{1/2} \rangle_{\Gamma_i} = \int_{\Gamma_i} \lambda_i v^i ds$ for $v^i \in V_i$. The third function space which acts as the global continuity space for the hybrid approximation is defined on the interface Γ as restrictions of functions in $H^1_0(\Omega)$ to the interface Γ, namely

$$\Phi := \{\upsilon \in L^2(\Gamma) : \exists u \in H^1_0(\Omega), u = \upsilon \text{ on } \Gamma\}. \tag{14}$$

Global norms for the spaces $\mathbf{V}$ and $\mathbf{\Lambda}$ can be defined as broken norms over Ω_i given as

$$\|\mathbf{u}\|_{\mathbf{V}} := \Big(\sum_{i=1}^{2} \|u^i\|^2_{1,\Omega_i} \Big), \quad \|\lambda\|_{\mathbf{\Lambda}} := \Big(\sum_{i=1}^{2} \|\lambda^i\|^2_{H^{-\frac{1}{2}}}(\Gamma_i) \Big),$$

and can easily be shown to be Hilbert spaces with these induced norms. Furthermore, with the use of extension operators which we will discuss next, the interface continuity space is endowed with the norm

$$\|\varphi\|_{\Phi} := \inf_{u|_{\Gamma} = \varphi} \|u\|_{1,\Omega}$$

where the norm equivalence with $\sum_{i=1}^{2} \|\varphi\|^2_{H^{\frac{1}{2}}(\Gamma_i)}$ can be shown to hold.

With the three approximation spaces at hand, the three-field formulation of the Helmholtz problem can be written for the two subdomains utilizing the additional two interface spaces Λ_i and the global continuity space Φ. Using the dual product notation $\langle \cdot, \cdot \rangle_i = \langle H^{-1/2}(\Gamma_i), H^{1/2}(\Gamma_i) \rangle$ the following variational form is called the three-field formulation. Find $\mathbf{u} \in \mathbf{V}$, $\lambda \in \mathbf{\Lambda}$, and $\varphi \in \Phi$ such that

$$\begin{cases} i)\ \sum_{i=1}^{2}\left(a_{\Omega_i}(u^i,v^i)-\langle\lambda^i,v^i\rangle_{\Gamma_i}\right)=\sum_{i=1}^{2}(f,v^i)_{\Omega_i}, \quad \forall \mathbf{v}\in\mathbf{V}, \\ ii)\ \sum_{i=1}^{2}\langle\mu^i,u^i-\varphi\rangle_{\Gamma_i}=\mathbf{0}, \quad \forall\mu\in\mathbf{\Lambda} \\ iii)\ \sum_{i=1}^{2}\langle\lambda^i,\psi\rangle_{\Gamma_i}=\mathbf{0}, \quad \forall\psi\in\Phi \end{cases} \tag{15}$$

The bilinear operator a_{Ω_i} stems from the weak formulation of the Helmholtz equation and is defined as

$$a_{\Omega_i}(u^i,v^i)=\int_{\Omega_i}\nabla u^i\nabla v^i+gu^iv^id\Omega_i.$$

Furthermore, the inner products of the form

$$\langle H^{-1/2}(\Gamma_i),H^{1/2}(\Gamma_i)\rangle_{\Gamma_i}$$

signify the artificial boundary or interface matching conditions. To be more specific, the second equation enforces weak continuity along the interface Γ_i with the solution u^i on Ω_i with respect to the interface continuity variable φ. The third equation serves two purposes; 1) It further constraines the space of Lagrangian multipliers $\mathbf{\Lambda}$ by adding orthogonality conditions with the interface space Φ and 2) It renders the discrete formulation of the above system as a symmetric positive definite system which can then be solved for the global solution $(\mathbf{u},\lambda,\varphi)$ using a preconditioned conjugate gradient method.

In order to write (15) in a more compact form, we will use the following operator notation originally adopted in [13]

$$\begin{aligned} A:V\mapsto V', \quad (A\mathbf{u})(\mathbf{v}) &:= \sum_{i=1}^{2}a_{\Omega_i}(u^i,v^i), \\ B:V\mapsto\Lambda, \quad (B\mathbf{u})(\lambda) &:= \sum_{i=1}^{2}\langle\lambda^i,v^i\rangle_{\Gamma_i}, \\ C:\Lambda\mapsto\Phi, \quad (C\lambda)(\psi) &:= \sum_{i=1}^{2}\langle\lambda^i,\psi\rangle_{\Gamma_i}. \end{aligned} \tag{16}$$

Additionally, $F\in V'$ is given by $\langle F,v^i\rangle := \sum_{i=1}^{2}(f,v^i)_{\Omega_i}$.

Before describing the numerical formulation of the three field method for the Helmholtz equation, we first review the sufficient and necessary conditions for the existence and uniqueness of the solution to (15). For a complete analysis of the three field method for elliptic equations, the reader is referred to the thesis [13] and relevent references therein.

We first note that a key observation in the three-field formulation comes from the first two equations of (15). For a given φ on the skeleton Γ, the first two equations are local Dirichlet problems where the boundary conditions on Γ_i are imposed in the weak sense. Because of this, one can show that the local problems are well-posed for a given sufficient φ. This leads to the following lemma from [6].

LEMMA 3.1 *Let $\varphi\in\Phi$ be given. Then there exists a unique solution $(\mathbf{u},\lambda)\in\mathbf{V}\times\mathbf{\Lambda}$ of the first two*

equations of (15) with

$$\|\mathbf{u}\| + \|\lambda\| \leq C_1\big(\|F\|_{V'} + \|C\varphi\|_{\mathbf{\Lambda}'}\big). \tag{17}$$

The solution $\mathbf{u} = (u^1, u^2)$ *satisfies the variational formulation of*

$$Hu^i = f \ \ in\, \Omega_i$$

$$u^i = \varphi \ \ on\, \Gamma_i$$

i.e. $u^i \in V^i$ *solves*

$$a_{\Omega_i}(u^i, v^i) = (f, v^i)_{\Omega_i},\ \forall v^i \in H_0^1(\Omega_i),\ u^i|_{\Gamma_i} = \varphi$$

for $i = 1, 2$. *Furthermore,* $\lambda = (\lambda^i, \lambda^2)$ *is given by*

$$\langle \lambda^i, \mu^i \rangle := a_{\Omega_i}(u^i, E^i\mu^i) - (f, E^i\mu^i)_{\Omega_i},\ \forall \mu^i \in H^{\frac{1}{2}}(\Omega_i)$$

where $E^i : H^{\frac{1}{2}}(\Gamma_i) \mapsto H^1(\Omega_i)$ *is the extension operator on* Γ_i.

For a given φ on Γ, the global solution $\mathbf{u} = (u^1, u^2)$ solves in the weak sense local Helmholtz problems on each subdomain Ω_i while producing on Γ_i the lagrangian multiplier functions. The extension operator E^i on each Γ_i serves in proving existence and uniqueness of the three-field variational formulation for sufficient spaces $\mathbf{V}$, $\mathbf{\Lambda}$, and Φ. In fact, the sufficient and necessary conditions for these three function spaces can be shown by considering the local Dirichlet problems while posing the Dirichlet essential conditions in the weak sense. This goes back to the 1972 paper by Babuska [1] where he shows the sufficient and necessary conditions for the Lagrangian multiplier space $\mathbf{\Lambda}$. Reciting this result, we need for some constants $C_1, C_2 > 0$

$$\inf_{\lambda\in\mathbf{\Lambda}/\{0\}} \sup_{\mathbf{u}\in\mathbf{V}/\{0\}} \frac{\langle B\mathbf{u}, \lambda\rangle_\Gamma}{\|\mathbf{u}\|_V \, \|\lambda\|_{\mathbf{\Lambda}}} = \frac{\sum_{i=1}^2 \langle \lambda^i, u^i\rangle_i}{\|\mathbf{u}\|_V \, \|\lambda\|_{\mathbf{\Lambda}}} > C_1$$

for the B operator and

$$\inf_{\varphi\in\Phi/\{0\}} \sup_{\mu\in\mathbf{\Lambda}/\{0\}} \frac{\langle C\mu, \varphi\rangle_\Gamma}{\|\varphi\|_\Phi \, \|\mu\|_{\mathbf{\Lambda}}} = \frac{\sum_{i=1}^2 \langle \mu^i, \varphi\rangle_i}{\|\varphi\|_\Phi \, \|\mu\|_{\mathbf{\Lambda}}} > C_2$$

for the C operator. As shown in [6], if these inf-sup conditions hold, then there exists a unique solution to the three-field variational formulation (15). This leads to the following theorem.

THEOREM 3.2 *Suppose that* $f \in (V^i)'$, *for both* $\Omega_{1,2}$. *Then equation (15) possesses a unique solution* $(\mathbf{u}, \lambda, \varphi) \in (\mathbf{V} \times \mathbf{\Lambda} \times \Phi)$ *where, denoting* $w \in H^1(\Omega)$ *a weak solution to the Helmholtz equation (11) with data* $\sum_{i=1}^2 (f, v^i)_{\Omega_i}$,

$$u = (w\big|_{\Omega_1}, w\big|_{\Omega_2}), \tag{18}$$

$$\varphi = w\big|_\Gamma. \tag{19}$$

Furthermore, if $w \in H^2(\Omega_i)$, *then* $\lambda = \Big(\Big(\frac{\partial w}{\partial n_1}\Big)|_{\Gamma_1}, \Big(\frac{\partial w}{\partial n_2}\Big)|_{\Gamma_2}\Big)$.

The proof of this theorem can be found in the paper by Brezzi [6] or the dissertation of [13]

4. Discrete version of the three-field formulation

The difficulty in passing to the discrete formulation from the variational problem (15) is in choosing the appropriate discrete subspaces of $\mathbf{V}$, $\mathbf{\Lambda}$, and Φ. Arbitrarily choosing the subspaces can lead to unstable solutions of the discrete variational problem primarily due to not satisfying the discrete versions of the inf-sup conditions, so careful consideration of the spaces is necessary. In past approaches to the method, usually the discretization of the space $\mathbf{V}$ is chosen first and then $\mathbf{\Lambda}$ and Φ are chosen thereafter to satisfy the inf-sup requirements. In this section, we propose a discrete approximation to the three-fields formulation by considering the spectral-element and meshless collocation methods as the discretization tools which will then lead to the hybrid meshless/spectral-element method.

With Ω_1 defining the domain for the spectral element approximation and the regional domain Ω_2 being allocated for meshless collocation, we define the space $V_N^1 := \mathbb{P}_{N,N_e} \cap H^1(\Omega_1)$, where $\mathbb{P}_{N,N_e}$ is the space of piecewise continuous functions that map to polynomials of degree less than or equal N to the reference element Ω^e. Namely,

$$\mathbb{P}_{N,E}(\Omega_1) := \Big\{ v(\mathbf{x}^e(\mathbf{r}))|_{\Omega^e} \in \mathbb{P}_N(r) \otimes \mathbb{P}_N(s),\, e = 1, \ldots, N_e \text{ such that } \Omega^e \in \Omega_1 \Big\},$$

where $\mathbb{P}_N(r)$ is the space of all polynomials of degree less than or equal to N. To restrict this space to Ω_1, we include all Ω^{e_i} such that $\Omega_1 \cap \Omega^{e_i} \neq 0$.

The domain Ω_2 allocates a collocation approximation by considering a random (or uniform) distribution of N_M distinct collocation nodes Ω^M and on its boundary $\partial\Omega^M$ giving a set $\mathcal{X}_M^V$. We then construct the approximation space $V_{N_M}^2 := \text{span}\{\Psi_1(\mathbf{x}), \ldots, \Psi_{N_M}(\mathbf{x})\}$ as defined in section (2).

With the spaces defined on each subdomain Ω_i, the Lagrangian multiplier spaces Λ^i for the interface boundaries Γ_i can now be constructed by using the spaces V_N^1 and $V_{N_M}^2$. Since V_N^1 defines a spectral approximation of order N, we define the Lagrangian multiplier space for Γ_1 as the space of Lagrangian interpolants of order less that or equal to N and restricted to Γ_1. This is given by

$$\Lambda_N^1 = \mathbb{P}_{N,E}(\Gamma_1) := \Big\{ \lambda(\mathbf{x}^e(r))|_{\Omega^e} \in \mathbb{P}_{N-4}(r)|_{\Gamma_1},\, e = 1, \ldots, N_e \text{ such that } \Omega^e \cap \Gamma_1 \neq 0 \Big\}. \tag{20}$$

Using such a space for $H^{-\frac{1}{2}}(\Gamma_1)$, it can be shown that the discrete inf-sup condition for the interface inner product on Γ_1 is satisfied. Namely, for some constant $C_{1,N}$ dependent on the degree N of the spectral elements, we have

$$\inf_{\lambda_N^1 \in \Lambda_N^1/\{0\}} \sup_{u_N^1 \in V_N^1/\{0\}} \frac{\langle Bu_N^1, \lambda_N^1 \rangle_{\Gamma_1}}{\|u_N^1\|_{V_{N_M}^1} \|\lambda_N^1\|_{\mathbf{\Lambda_N^1}}} = \frac{\langle \lambda_N^1, u_N^1 \rangle_1}{\|u_N^1\|_{V_{N_M}^1} \|\lambda_N^1\|_{\mathbf{\Lambda_N^1}}} > C_{1,N}$$

is satisfied. This result is proved in the paper on the Mortar Spectral Element method by Ben Belgacem et al. [2] in a similar interface inner product using Lagrangian multipliers.

In order to complete the space $\mathbf{\Lambda}$ we need the additional interface space on Ω_2. On the boundary Γ_2, a second meshless collocation space for Λ_N^2 is constructed using a distribution of N_{Γ_2} nodes restricted to the interface Γ_2 producing the set X_{Γ_2}. Using the EBGRK method presented in section (2), the Lagrangian multiplier space for Γ_2 is taken to be $\Lambda_{N_M}^2 = \text{span}\{\Psi_1^\lambda(\mathbf{x}), \ldots, \Psi_{N_M}^\lambda(\mathbf{x}) : \mathbf{x} \in \Gamma_2\} \subset H^{-\frac{1}{2}}(\Gamma_2)$ where $\Psi_i^\lambda(\cdot)$ denotes the i-th discrete reproducing kernel function on X_{Γ_2}.

Lastly, in order to connect the two pairs of approximation spaces (V_N^1, Λ_N^1) and $(V_{N_M}^2, \Lambda_{N_M}^2)$ on Ω_1 and Ω_2, respectively, we build a suitable discrete subspace of Φ by taking the Lagrangian interpolants

constructed from Legendre polynomials of degree $N-2$ restricted to Γ. Namely,

$$\Phi_N := \{\varphi(\mathbf{x}^e(r))|_{\Omega^e} \in \mathbb{P}_{N-2}(r)|_\Gamma,\ e=1,\dots,N_e,\ \text{ such that }\ \Omega^e \cap \partial\Omega \neq 0\}.$$

This will ensure that the discrete inf-sup condition for Φ and Λ_N^1 on Γ_1 is satisfied. This means, for some $C_{1,N} > 0$, we have

$$\inf_{\varphi_N \in \Phi_N/\{0\}} \sup_{\lambda_N^1 \in \Lambda_N^1/\{0\}} \frac{\langle \varphi_N, \lambda_N^1 \rangle_{\Gamma_1}}{\|u_N^1\|_{V_{N_M}^1} \|\lambda_N^1\|_{\mathbf{\Lambda_N^1}}} = \frac{\langle \lambda_N^1, \varphi_N \rangle_1}{\|u_N^1\|_{V_{N_M}^1} \|\lambda_N^1\|_{\mathbf{\Lambda_N^1}}} > C_{1,N}.$$

The last issue we need to resolve in this three field formulation is complying with the strong form of the elliptic equation on Ω_2. To this end, since Ω_2 utilizes a meshless collocation technique, we define the set of test distributions on Ω_2 as $V_{\delta,N_M}^2 = \{\delta_{\mathbf{x}_i} : \mathbf{x}_i \in X_u^{N_M}\}$ where $\delta_{\mathbf{x}_i}$ is the Dirac delta function at node $\mathbf{x}_i$. The original variational formulation in (15) can now be modified as follows. Find $(u_N^1, \lambda_N^1, u_N^2, \lambda_N^2, \varphi_N) \in V_N^1 \otimes \Lambda_N^1 \otimes V_{N_M}^2 \otimes \Lambda_{N_M}^2 \otimes \Phi_N$ such that

$$\begin{cases} i)\ a_{\Omega_1}(u_N^1, \chi_N^1) - \langle \lambda_N^1, \chi_N^1 \rangle_{\Gamma_1} = (f, \chi_N^1)_{\Omega_1}, \quad \forall \chi_N^1 \in V_N^1, \\ ii)\ \langle \mu_N^1, u_N^1 - \varphi_N \rangle_{\Gamma_1} = \mathbf{0}, \quad \forall \mu_N^1 \in \Lambda_N^1, \\ iii)\ \langle \lambda_N^1, \psi_N \rangle_{\Gamma_1} = \mathbf{0}, \quad \forall \psi_N \in \Phi, \end{cases} \tag{21}$$

and

$$\begin{cases} i)\ a_{\Omega_2}(u_N^2, \chi_N^2) - \langle \lambda_N^2, \chi_N^2 \rangle_{\Gamma_2} = (f, \chi_N^2)_{\Omega_2}, \quad \forall \chi_N^2 \in V_{\delta,N_M}^2, \\ ii)\ \langle \mu_N^2, u_N^2 - \varphi_N \rangle_{\Gamma_2} = \mathbf{0}, \quad \forall \mu_N^2 \in \Lambda_{N_M}^2, \\ iii)\ \langle \lambda_N^2, \psi_N \rangle_{\Gamma_1} = \mathbf{0} \quad \forall \psi_N \in \Phi. \end{cases} \tag{22}$$

4.1. *Implementation of Three-Field Formulation*

Once the discrete approximation spaces have been chosen and numerical integration has been done, an efficient manner in solving this is to construct the Schur compliment system and then apply a conjugate gradient method as discussed in [6]. To do this, we first write (21) and (22) in algebraic form as:

$$A_i \mathbf{u}_i - B_i^T \lambda_i = \mathbf{f}_i,$$

$$-B_i \mathbf{u}_i + C_i^T \varphi = \mathbf{0},$$

$$C_i \lambda_i = \mathbf{0},$$

for $i = 1, 2$. Now applying block Gaussian elimination to the linear system, we obtain a linear system for φ as

$$\mathcal{S}\varphi = \mathbf{g}, \tag{23}$$

where $\mathcal{S} = \mathcal{S}_1 + \mathcal{S}_2$, $\mathbf{g} = \mathbf{g}_1 + \mathbf{g}_2$ and

$$\mathcal{S}_i := C_i D_i^{-1} C_i^T,\ \mathbf{g}_i := C_i D_i^{-1} B_i A_i^{-1} \mathbf{f}_i,\ D_i := B_i A_i^{-1} B_i^T,\ i = 1, 2 \tag{24}$$

The $\mathcal{S}$ matrix can be considered as the Schur compliment matrix with respect to $\mathbf{u}$ and λ of the entire system defined above. Furthermore, it was shown by Brezzi in [6] that the Schur compliment $\mathcal{S}$ is symmetric and positive definite if the matrices B_i^T and C_i have full rank. One can then apply a conjugate gradient method to the system (23) to obtain the solution of the elliptic problem on the global domain. It can be remarked that by definition of $\mathbf{g}_i$, the calculation of a conjugate gradient iteration requires the solution to the local Helmholtz equation in each subdomain Ω_i, $i = 1, 2$. Block-Jacobi preconditioning is used to solve each of these local Helmholtz problems by considering zero Neumann conditions for each local problem. This way, each local Helmholtz problem has a unique solution and in effect, the matrix A_i has an inverse which can be calculated before time-stepping.

A similar construction to the above Schur compliment technique reduces the discrete three-field formulation to solving an elliptic problem on each subdomain $\Omega_{1.2}$ in the style of [1]. To do this we write (21) and (22) in matrix form

$$\begin{bmatrix} A_1 & 0 & (B_1)^t & 0 & 0 \\ 0 & A_2 & 0 & (B_2)^t & 0 \\ B_1 & 0 & 0 & 0 & (C_1)^t \\ 0 & B_2 & 0 & 0 & (C_2)^t \\ 0 & 0 & C_1 & C_2 & 0 \end{bmatrix} \begin{bmatrix} u^1 \\ u^2 \\ \lambda_1 \\ \lambda_2 \\ \varphi \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ 0 \\ 0 \\ 0 \end{bmatrix}. \tag{25}$$

Changing the order of the unknowns and of the equations in the above system,

$$\begin{bmatrix} A_1 & (B_1)^t & 0 & 0 & 0 \\ B_1 & 0 & 0 & 0 & (C_1)^t \\ 0 & 0 & A_2 & (B_2)^t & 0 \\ 0 & 0 & B_2 & 0 & (C_2)^t \\ 0 & C_1 & 0 & C_2 & 0 \end{bmatrix} \begin{bmatrix} u^1 \\ \lambda_1 \\ u^2 \\ \lambda_2 \\ \varphi \end{bmatrix} = \begin{bmatrix} f_1 \\ 0 \\ f_2 \\ 0 \\ 0 \end{bmatrix}. \tag{26}$$

If we now set

$$\mathcal{A} = \begin{bmatrix} A_1 & (B_1)^t & 0 & 0 \\ B_1 & 0 & 0 & 0 \\ 0 & 0 & A_2 & (B_2)^t \\ 0 & 0 & B_2 & 0 \end{bmatrix}, \quad \mathbf{U} = \begin{bmatrix} u^1 \\ \lambda_1 \\ u^2 \\ \lambda_2 \end{bmatrix}, \quad \mathbf{F} = \begin{bmatrix} f_1 \\ 0 \\ f_2 \\ 0 \\ 0 \end{bmatrix}, \tag{27}$$

and

$$\mathcal{C} = [0,\ C_1,\ 0,\ C_2], \tag{28}$$

then we can write the system as

$$\begin{bmatrix} \mathcal{A} & C^t \\ \mathcal{C} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{U} \\ \varphi \end{bmatrix} = \begin{bmatrix} \mathbf{F} \\ 0 \end{bmatrix}. \tag{29}$$

One importance to remark here is that the matrix $\mathcal{A}$ is block diagonal, even in the case where the number of subdomains in the domain decomposition is greater that 2. Taking advantage of this fact, we can construct

the inverse $\mathcal{A}^{-1}$ in parallel with respect to each subdomain. Next, eliminating $\mathbf{U}$ from (29), the system becomes

$$\mathcal{C}\mathcal{A}^{-1}\mathcal{C}^t\varphi = \mathcal{C}\mathcal{A}^{-1}\mathbf{F}, \tag{30}$$

which can be solved using a preconditioned conjugate gradient method.

The last issue of the discrete three-field formulation is related to the efficient construction of the matrices C and B. As they include the integration of the basis functions for the Lagrangian multiplier spaces and the interface space and are independent of the data, they can be calculated and stored prior to time stepping as well. The matrices have the form

$$\mathbf{C}_i(j,k) = \langle \mu_{i,j}, \phi_k \rangle_\Gamma, \qquad \mu_{i,j} \in \Lambda_N^i, \quad \phi_{i,k} \in \Phi^N, \tag{31}$$

$$\mathbf{B}_i(j,k) = \langle u_{i,j}, \mu_{i,k} \rangle_{\Gamma_i}, \quad \mu_{i,j} \in \Lambda_N^i, \quad u_{i,k} \in V_N^i. \tag{32}$$

For $i = 2$, the above calculations involve integration on a spectral grid using meshless reproducing kernels. The choice of the β parameter and N_M for a given radial basis that constructs the reproducing kernel determines the stability of the entire hybrid model. A discussion of constructing an appropriate meshless approximation space is thus necessary.

5. Choice of Meshless Approximation Spaces

In regards to the choice of approximation space parameters used for the three-field variational formulation, it is evident from the review of the meshless collocation method that there are certain parameters that need to be chosen and satisfied to achieve a stable approximation. Since there are not many parameters involved in the spectral-element approximation other than the order of the Lagrangian interpolants used and the number of elements, much of the difficutly in this proposed hybrid model comes from defining the meshless collocation space and its two additional fields, namely the Lagrangian multiplier space and the interface space, λ_N^2 and Φ_N, respectively. It turns out that with $N = 10$ for the spectral element approximation and if $N_M > N^2$ where the number of nodes on the boundary of Ω_2 is at least N, then satisfying the discrete inf-sup conditions to obtain stability across the interface of the spectral element and meshless collocation approximations Γ relies on the parameter β in the meshless collocation spaces $V_{N_M}^2$ and λ_N^2 (see section (2)). Unfortunately, the β parameter for the meshless collocation space cannot be chosen arbitrarily. It is highly dependent not only on the amount of collocation nodes in the domain, but also on the radial basis function used to generate the discrete reproducing kernels. A near optimal parameter must be chosen if stable calculations are sought. For "bad" choices of N_M and/or β, the necessary discrete inf-sup conditions on the interface will not be satisfied and high oscillatory phenomona will appear on Γ and inside the domain Ω_2. Finding a near optimal parameter for a given radial basis function and N_M requires experimental choice. Fortunately, for a given Ω_2, N_M, and a radial basis, once a near optimal β is found, it can be reused for the same radial basis and N_M, regardless of the data. To numerically show dependence of β on N_M and the radial basis used, a simple numerical example is given which was reproduced from Blakely ([4]). It features the EBGRKM solution to the elliptic problem on $\Omega = [0, \pi] \times [0, \pi]$,

$$\Delta u = -2\sin(x_1)\sin(x_2) \quad x, y \in \Omega$$

$$u = 0 \quad x_1, x_2 \in \partial\Omega$$

using an empirical reproducing kernel built from the multiquadric radial basis $\phi(\|\mathbf{x}-\mathbf{x}_i\|) = \sqrt{(\|\mathbf{x}-\mathbf{x}_i\|^2 + \beta)}$. The tables (1) and (2) show the L_1 errors associated with different grid sizes on various β values. The N_M collocation nodes for V_{N_M} are randomly distributed.

Table 1. L_1 error for 10×10 and 12×12 uniform grid nodes for various β values.

10×10		12×12	
L_1 error	β	L_1 error	β
6.06394e-05	14	6.4211e-06	10
9.37156e-06	34	8.27118e-06	18
3.57931e-06	48	4.53544e-06	28
2.58906e-05	54	1.26301e-05	42
5.23094e-05	100	3.2699e-05	48
9.22903e-05	102	4.33146e-05	54

Table 2. L_1 error for 10×10, 12×12, 15×15 and 20×20 uniform grid nodes for various β values.

15×15		20×20	
L_1 error	β	L_1 error	β
4.57714e-05	6	8.1885e-07	6
1.01192e-06	10	7.41084e-06	10
4.72437e-06	18	3.71961e-05	20
9.65463e-06	32	5.83601e-05	26
2.54876e-05	42	2.74848e-05	30
4.52623e-05	62	7.28861e-05	40

5.1. *Inf-Sup Verification*

As a method for verifying the inf-sup conditions numerically, we devise a test based on the work of Brezzi et al. in [7] where arguments from linear algebra are used to construct a simple verification form la. In order to construct the test, the basis for each field, namely $\mathbf{V}_N$, $\mathbf{\Lambda}_N$ and Φ_h must be known. The goal is to verify the following conditions numerically given the spaces $\mathbf{V}_N$, $\mathbf{\Lambda}_N$ and Φ_N:

- (Control λ_h by χ_h). There exists a constant C_1 such that for any $\lambda_h \in \mathbf{\Lambda}_N$, we can find $\chi_h \in \mathbf{V}_N$

$$\|\chi_h\|_{\mathbf{V}_N} = 1 \quad \text{and} \quad \|\lambda_h\|_{\Lambda_N} \leq \frac{1}{C_1}\langle \lambda_h, \chi_h \rangle, \tag{33}$$

- (Control ϕ_h by λ_h). There exists a constant C_2 such that for any $\phi_h \in \Phi_h$, we can find $\lambda_h \in \Lambda_h$

$$\|\lambda_h\|_{\mathbf{V}_N} = 1 \quad \text{and} \quad \|\phi_h\|_{\Phi} \leq \frac{1}{C_2}\langle \phi_h, \lambda_h \rangle. \tag{34}$$

In this paper, we decribe the method for condition (34) on Ω_2 (the meshless collocation domain) and refer the reader to the paper by Brezzi et al. in [7] for the conditions on Ω_1 using finite elements, which are similar to the constructions using spectral elements.

Since condition (34) is being tested, the Lagrangian spaces on both Γ_1 and Γ_1 are needed. We begin by considering the meshless collocation side Γ_2 the space $\Lambda^2_{N_M} = \text{span}\{\Psi^{\lambda}_1(\mathbf{x}), \ldots, \Psi^{\lambda}_{N_M}(\mathbf{x}) : \ \mathbf{x} \in \mathcal{X}_{\Gamma_2}\}$ with dimension N_M. On the spectral element side of the interface Γ_1, we take $\Lambda^1_{N_S}$ as in (20) with dimension N_S. Furthermore, the spectral element continuity space on Γ_2 is taken as $\Phi_{N_q} := \{\varphi(\mathbf{x}^e(r))|_{\Omega^e} \in$

$\mathbb{P}_{N_q}(r)|_\Gamma,\ e=1,\dots,N_e,\quad$ such that $\Omega^e \cap \partial\Omega_2 \neq 0\}$. The goal is to use these spaces to determine whether or not the inf-sup conditions will be satisfied. Since the construction will use linear algebra as a tool, it is necessary to construct a mapping from the finite dimensional function spaces to $\mathbb{R}^N$. To this end,let $\{\lambda^{(1)},\lambda^{(2)},\lambda^{(2)},\dots,\lambda^{(N_t)}\}$ be a basis for $\mathbf{\Lambda}_N = \Lambda^2_{N_M} + \Lambda^1_{N_S}$ with dimension $N_t = N_M + N_S$, and $\{\phi^{(1)},\phi^{(2)},\phi^{(3)},\dots,\phi^{(N)}\}$ be a basis for Φ_N. We construct a one-to-one mapping from $\mathbf{\Lambda_N}$ to $\mathbb{R}^{N_t}$ defined by

$$\lambda \equiv (\lambda_1,\lambda_2,\dots,\lambda_{N_t}) \mapsto \sum_{i=1}^{N_t} \lambda_i \lambda^{(i)}. \tag{35}$$

In a similar manner, a one-to-one mapping from Φ_N to $\mathbb{R}^{N_q}$ is defined by

$$\phi \equiv (\phi_1,\phi_2,\dots,\phi_{N_q}) \mapsto \sum_{i=1}^{N_q} \phi_i \phi^{(i)}. \tag{36}$$

Next, three matrices are constructed out of the basis functions which will be used in constructing the inner products and norms of the discrete spaces. The matrices are constructed as

$$\begin{cases} C_{i,j} = \int_\Gamma \phi^{(i)}\lambda^{(j)} d\Gamma, & (i=1,\dots,N_q; j=1,\dots,N_t), \\ R_{i,r} = \int_\Gamma \phi^{(i)}\phi^{(r)} d\Gamma, & (i,r=1,\dots,N_q), \\ Q_{j,s} = \int_\Gamma \lambda^{(j)}\lambda^{(s)} d\Gamma, & (j,s=1,\dots,N_t). \end{cases}$$

The matrix C is the same matrix used in the discrete three field formulation in (31). The integration used for calculating the matrix Q can be done using Gauss-Lobatto-Legendre quadrature within each element on Γ. The integral in constructing R can also be done using segmented integration with Gauss-Lobatto-Legendre quadrature on the Γ_2 grid. Using these matrices and the vectors λ and ϕ, condition (34) can be written as follows. *There exisits a* $\beta_1 > 0$ *such that* $\forall \phi \in \mathbb{R}^{N_q}$, *there exists* $\lambda \in \mathbb{R}^{N_t}$ *such that*

$$(\lambda^t Q \lambda)^{1/2} = 1 \quad \text{and} \quad \lambda^t B^t \phi \geq \beta_1 (\phi^t R \phi)^{1/2} \tag{37}$$

Using the left hand side of the inequality in (37), we consider for any fixed $\phi \in \mathbb{R}^{N_q}$ the quantity

$$S(\phi) = \max_{(\lambda^t Q\lambda)^{1/2}=1} \lambda^t B^t \phi \equiv \max_{\lambda \neq 0} \frac{\lambda^t B^t \phi}{(\lambda^t Q \lambda)^{1/2}}, \tag{38}$$

to choose the optimal ϕ once λ is already chosen. To compute (38), we introduce a new variable $\mathbf{z} = Q^{1/2}\lambda$ (so $\lambda^t = \mathbf{z}^t Q^{-1/2}$) which allows us to write

$$S(\phi) = \max_{z \neq 0} \frac{\mathbf{z}^t Q^{-1/2} B^t \phi}{(\mathbf{z}^t \mathbf{z})^{1/2}}, \tag{39}$$

and so the maximum must be taken when $\mathbf{z} = Q^{-1/2} B^t \phi$ which gives

$$S(\phi) = (\phi^t B Q^{-1} B^t \phi)^{1/2}. \tag{40}$$

Now going back to the original intention of this numerical test, we want to find the biggest β_1 such that

$$S(\phi) \geq \beta_1 (\phi^t R \phi)^{-1/2}, \quad \forall \phi \in \mathbb{R}^{N_q}, \tag{41}$$

which, upon using (40) and squaring both sides of the inequality, we conclude that

$$\phi^t B Q^{-1} B^t \phi \geq \beta^2 \phi^t R \phi, \quad \forall \phi \in \mathbb{R}^{N_q}. \tag{42}$$

Removing m^t from both sides, we see that this inequality is equivalent to finding the smallest eigenvalue μ of the eigenvalue problem

$$B Q^{-1} B^t \phi = \mu R \phi \tag{43}$$

Hence, once N_q and N_t have been chosen and the cooresponding matrices C, Q and R have been calculated (Q is trivial to invert since it is diagonal), (43) can be used to determine if the inf-sup condition (37) is satisfied. For the condition to be satisfied, we want that the smallest eigenvalue is large relative to q_{Γ_2}. A good test would be to add 50 percent more nodes on Γ_2 and build a larger space $\Lambda^2_{N_M}$. If the smallest eigenvalue in (43) remains close to the previous smallest eigenvalue for the less dense node distribution on Γ_2, then the inf-sup condition (37) has been satisfied. Otherwise, the space Φ_N needs to be refined as well or the meshless approximation scheme used for $\Lambda^2_{N_M}$ is unstable.

6. Numerical Experiments

We now show some numerical examples of the convergence of three-field variational formulation applied to the Helmholtz elliptic problem on a rectangular domain where we look for an approximate solution $u_N(x, y)$ on $(x, y) \in \Omega = [-3, 3] \times [0, 4]$ such that

$$\Delta u_N + u_N = 3 \sin x \sin y \tag{44}$$

and

$$u_N(x, y) = \sin x \sin y, \quad x, y \in \delta\Omega$$

For the three-field variation solution, we split Ω into two equal area subdomains $\Omega_1 = [-3, 0] \times [0, 4]$, $\Omega_2 = [0, 3] \times [0, 4]$. On Ω_1 we construct 9 spectral elements each endowed with a Lagrangian interpolant space of order 8, and on Ω_2 we create a distribution of 900 collocation nodes and build the empirical reproducing kernel space from third order Wendland functions. Our goal in this subsection is to show stability of the hybrid meshless/spectral element formulation of the three-field method with respect to different choices of Φ_N and Λ^i_N.

To satisfy the inf-sup condition on Γ_1, we fix Λ^1_N to be the space of Lagrangian interpolants of order 10 restricted to Γ_1. As discussed in section (3), through this choice of Λ^1_N, we gain stabilty when computing the interface inner products $\langle V^1_N, \Lambda^1_N \rangle_{\Gamma_1}$. The remaining parameters one has to compute for a solution to the discrete three-field problem include the dimensions of the spectral element space Φ_N and the Lagrangian space Λ^2_N.

In the first set of numerical experiments, we take Φ_N to be the restriction of V^1_N to Γ but raise the order of the Lagrangian interpolants to 12. This will ensure that the discrete inf-sup condition is satisfied on the Γ to Γ^1 interface integral. We now have the freedom of choosing the second meshless collocation space Λ^2_N. To this end, we distribute 40 collocation nodes on Γ^1 and build the empirical reproducing kernel using the third order compactly supported Wendland radial functions with a shape parameter for the given domain

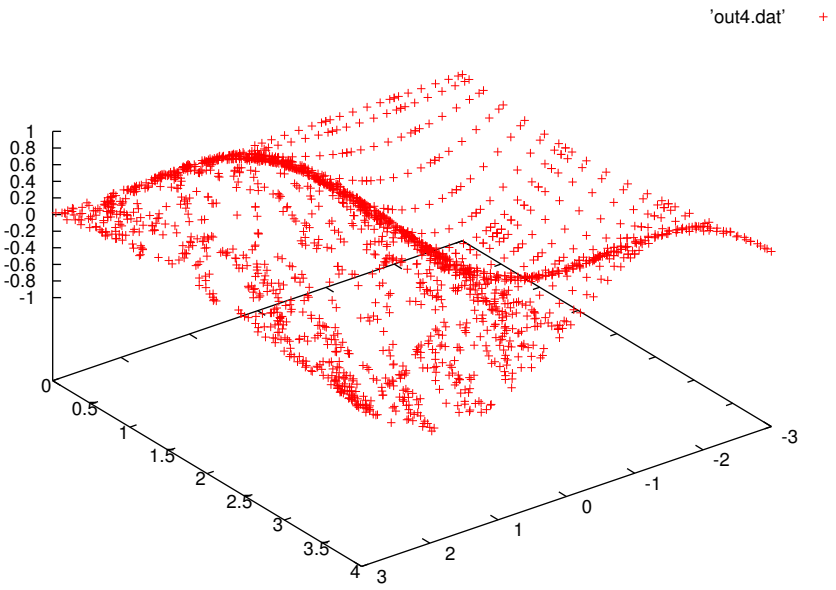

Figure 1. Approximate hybrid solution to Helmholtz problem after 100 conjugate gradient iterations

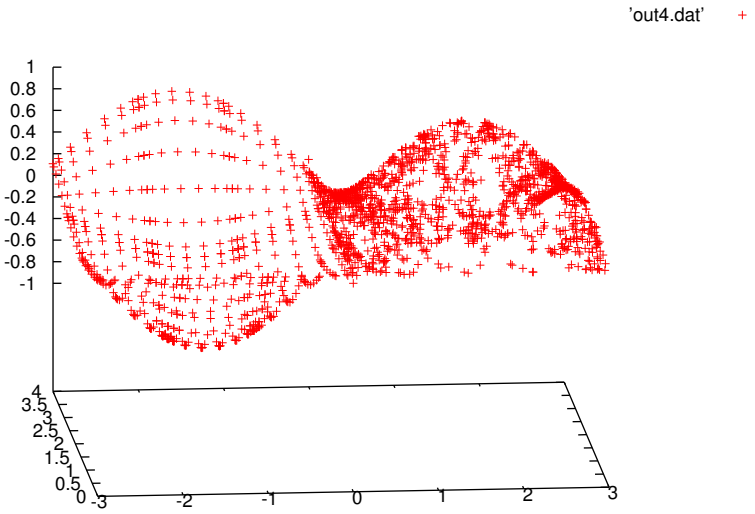

Figure 2. Approximate hybrid solution to Helmholtz problem after 100 conjugate gradient iterations

of 0.18. The preconditioner for the conjugate gradient method is simply the diagonal preconditioner of the discrete Helmholtz operator on each subdomain Ω_1 and Ω_2.

Figures (1) and (2) show the approximate hybrid solution at two different angles using the three-field method after 100 preconditioned conjugate gradient iterations with a residual threshold of 0.001.

The pointwise error of this approximate hybrid solution to the exact solution is shown in figure (3) on the top. In the same figure to the bottom, we show the effect of increasing the dimension of the

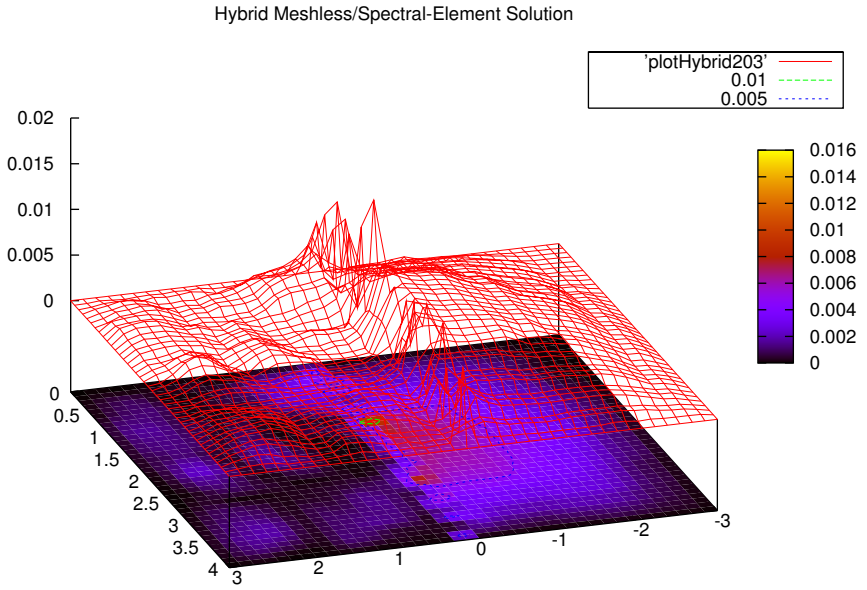

Figure 3. Pointwise error for the hybrid approximation with dimension of 400 reproducing kernels space on Ω_2.

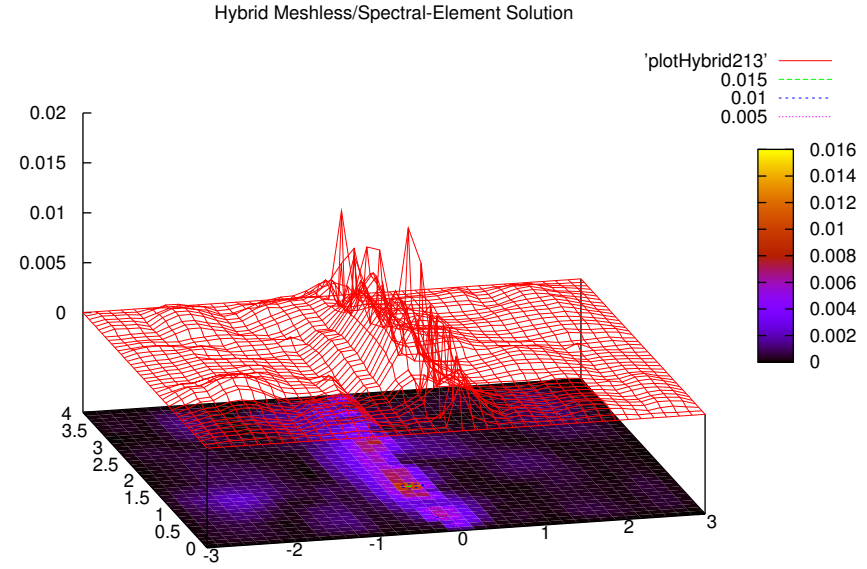

Figure 4. Pointwise error for the hybrid approximation with dimension of 900 reproducing kernels space on Ω_2.

reproducing kernel space to 900. Namely, we distribute 900 collocation points in Ω_2 and construct the empirical reproducing kernel out of Wendland functions of order 3. From the two figures, it is easy to see that the error in the approximation in both solutions at the interface is much greater than that in the two subdomains, which is ultimately due to the choice of the basis for the Λ_N^2 space.

In order to improve the convergence of the solution at the interface, we add an additional 5 collocation points on Γ_2 and add an additional node in each element on Γ for the space Φ_N. Figures (5) and (6) show

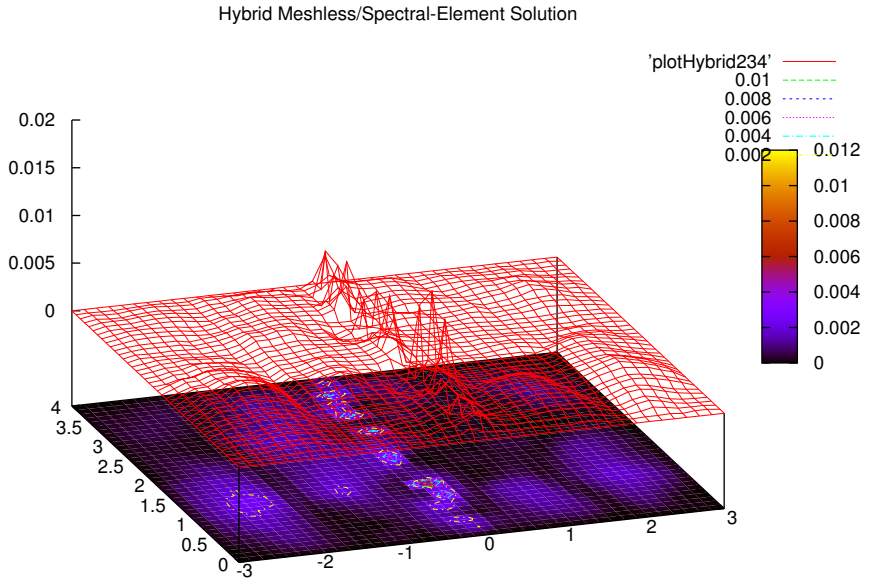

Figure 5. Pointwise error for the hybrid approximation at the interface Γ with enriched reproducing kernel space on Γ_2 and p-refinement on Φ_N. Overview of error

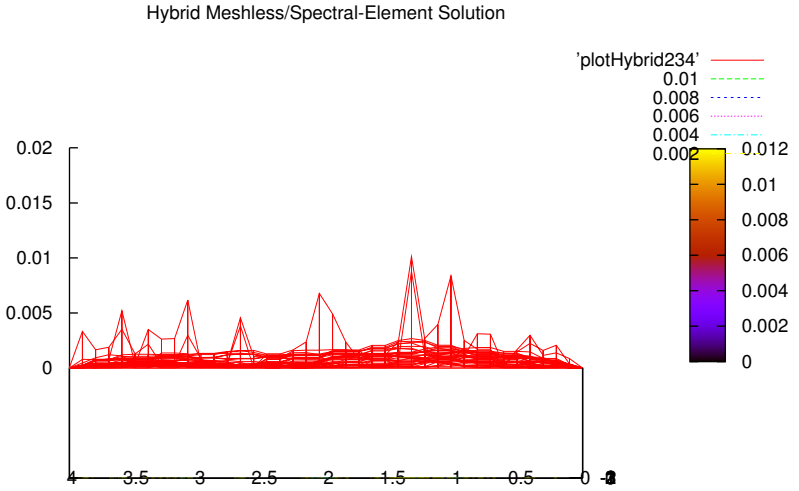

Figure 6. Pointwise error for the hybrid approximation at the interface Γ with enriched reproducing kernel space on Γ_2 and p-refinement on Φ_N. Lateral view of error

a plot of the pointwise error from above as well as from a lateral view in order to better view the error at the meshless/spectral-element interface.

7. Conclusion

In this paper, we attempted to present a hybrid meshless/spectral-element approximation scheme based on the three-field domain decomposition technique. As coupling meshless collocation and spectral elements could be of grand interest for large scale problems, we demonstrated that with careful selection of certain meshless collocation spaces, a stable hybrid approximation can be achieved for elliptic equations using the adapted three-field formulation. Moreover, a simple test for qualifying the hybrid approximation over the interface by means of the discrete inf-sup condition was also discussed.

The main difficulty of the three-field method is constructing a preconditioner to decrease the amount of iterations in the conjugate gradient solver. Many successfull methods for preconditioning the three-field method have been proposed in recent years, but are strictly geared towards finite element approaches only, namely using multigrid approaches. More effective preconditioners for the hybrid case are difficult to construct due the nature of the meshless collocation scheme but are currently being investigated.

References

[1] I. Babuska, 1973 *The finite element method with Lagrangian multipliers*, Numer. Math., 20, pp. 179–192
[2] F. Ben Belgacem, C. Bernardi, N. Chorfi, Y. Maday, 2000 *Inf-Sup conditions for the mortar spectral element discretization of the Stokes Problem* Numerishe Mathematik 85 , no. 2, 257–281
[3] S. Bertoluzza, 2003*Analysis of a stabilized three-fields domain decomposition method*, Numer. Math., 93, 611-634.
[4] C. Blakely 2006 *A New Backus-Gilbert Meshless Method for Initial-Boundary Value Problems.* International Journal on Computational Methods, Vol 3, No. 2
[5] L. P. Bos and K. Salkauskas, 1989 *Moving least-squares are Backus-Gilbert optimal*, J. Approx. Theory 59, 267-275.
[6] Brezzi, F., Marini, L. D. 1994 *A three-field domain decomposition method.* Quarteroni A. et al. [23], pp. 27-34.
[7] Brezzi, F., Marini, L. D.2005 *The three-field formulation for elasticity problems* In process
[8] M.D. Buhmann, 2003 *Radial Basis Functions: Theory and Implementations.* Cambridge University Press, Cambridge.
[9] R. DeVore *Nonlinear Approximation*, Acta Numerica, 7, (1998), 51-150
[10] G.E. Fasshauer, 2002 *Matrix-free multilevel moving least-squares methods*, in Approximation Theory, X St. Louis, MO, 271-278.
[11] G.E. Fasshauer, 2002 *Appproximate moving least-squares approximation for Time-Dependant PDEs*, in *WCCM V* July 7-12, Vienna, Austria
[12] G.E. Karniadakis, S.J. Sherwin, 1999 *Spectral/hp Element Methods for Computational Fluid Dynamics.* Oxford University Press, 404 pp.
[13] Rapin, G. 2003 *The three-field formulation for elliptic equations: stabilization and decoupling strategies.* PhD thesis, Universitaet Goettingen.
[14] R. Schaback, 2005 *Convergence of unsymmetric kernel based meshless collocation methods.* Preprint from `http://www.num.math.uni-goettingen.de/schaback/research/group.html`,
[15] R. Schaback, 1995 *Error estimates and condition numbers for radial basis function interpolation*, Adv. in Comput. Math., 3 , 251-264.
[16] H. Wendland, 2002 *Scattered Data Modelling by Radial and Related Functions*, Habilitation thesis, Univ. Gottingen.