REMARKS ON THE ACOUSTIC LIMIT FOR THE BOLTZMANN EQUATION

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ABSTRACT. We use some new nonlinear estimates found in [12] to improve the results of [6] that establish the acoustic limit for DiPerna-Lions solutions of Boltzmann equation in three ways. First, we enlarge the class of collision kernels treated to that found in [12], thereby treating all classical collision kernels to which the DiPerna-Lions theory applies. Second, we improve the scaling of the kinetic density fluctuations with Knudsen number from $O(\epsilon^m)$ for some $m > \frac{1}{2}$ to $O(\epsilon^{\frac{1}{2}})$. Third, we extend the results from periodic domains to bounded domains with impermeable boundaries, deriving the boundary condition for the acoustic system.

1. INTRODUCTION

In this note we establish the acoustic limit starting from DiPerna-Lions renormalized solutions of the Boltzmann equation considered over a bounded C^1 spatial domain $\Omega \subset \mathbb{R}^{\mathrm{D}}$. The acoustic system is the linearization about the homogeneous state of the compressible Euler system. After a suitable choice of units and Galilean frame, it governs the fluctuations in mass density $\rho(x,t)$, bulk velocity $u(x,t)$, and temperature $\theta(x,t)$ over $\Omega \times \mathbb{R}_+$ by the initial-value problem

$$
\begin{aligned}
\partial_t \rho + \nabla_x \cdot u &= 0\,, & \rho(x,0) &= \rho^{\mathrm{in}}(x)\,,\\
\partial_t u + \nabla_x(\rho + \theta) &= 0\,, & u(x,0) &= u^{\mathrm{in}}(x)\,,\\
\tfrac{\mathrm{D}}{2}\partial_t \theta + \nabla_x \cdot u &= 0\,, & \theta(x,0) &= \theta^{\mathrm{in}}(x)\,,
\end{aligned}
\tag{1.1}
$$

subject to the boundary condition

$$
\mathrm{n} \cdot u = 0\,, \quad \text{on} \quad \partial\Omega\,,
\tag{1.2}
$$

where n is the unit outward normal on $\partial\Omega$. This is one of the simplest fluid dynamical systems imaginable, being essentially the wave equation.

The acoustic system (1.1, 1.2) can be formally derived from the Boltzmann equation for kinetic densities $F(v,x,t)$ over $\mathbb{R}^{\mathrm{D}} \times \Omega \times \mathbb{R}_+$ that are close to the absolute Maxwellian

$$
M(v) = \frac{1}{(2\pi)^{\frac{\mathrm{D}}{2}}} \exp\left(-\tfrac{1}{2}|v|^2 \right).
\tag{1.3}
$$

We consider families of kinetic densities in the form $F_\epsilon(v,x,t) = M(v)G_\epsilon(v,x,t)$ where the relative kinetic densities $G_\epsilon(v,x,t)$ over $\mathbb{R}^{\mathrm{D}} \times \Omega \times \mathbb{R}_+$ are governed by the rescaled Boltzmann initial-value problem

$$
\partial_t G_\epsilon + v \cdot \nabla_x G_\epsilon = \frac{1}{\epsilon}\mathcal{Q}(G_\epsilon, G_\epsilon)\,, \qquad G_\epsilon(v,x,0) = G_\epsilon^{\mathrm{in}}(v,x)\,.
\tag{1.4}
$$

Here the Knudsen number $\epsilon > 0$ is the ratio of the mean free path to a macroscopic length scale and the collision operator $\mathcal{Q}(G_\epsilon, G_\epsilon)$ is given by

$$
\mathcal{Q}(G_\epsilon, G_\epsilon) = \iint_{\mathbb{S}^{\mathrm{D}-1}\times\mathbb{R}^{\mathrm{D}}} \left(G'_{\epsilon 1} G'_\epsilon - G_{\epsilon 1} G_\epsilon \right) b(\omega, v_1 - v)\, \mathrm{d}\omega\, M_1 \mathrm{d}v_1\,,
\tag{1.5}
$$

where the collision kernel $b(\omega, v_1 - v)$ is positive almost everywhere while $G_{\epsilon 1}$, G'_ϵ, and $G'_{\epsilon 1}$ denote $G_\epsilon(\,\cdot\,, x, t)$ evaluated at v_1, $v' = v + \omega\omega\cdot(v_1 - v)$, and $v'_1 = v - \omega\omega\cdot(v_1 - v)$ respectively.

We impose a Maxwell reflection boundary condition on $\partial\Omega$ of the form

$$\mathbf{1}_{\Sigma_+} G_\epsilon \circ \mathrm{R} = (1-\alpha)\,\mathbf{1}_{\Sigma_+}\, G_\epsilon + \alpha\,\mathbf{1}_{\Sigma_+}\sqrt{2\pi}\,\big\langle \mathbf{1}_{\Sigma_+}\, n\cdot v\, G_\epsilon\big\rangle\,. \tag{1.6}$$

Here $\alpha \in [0,1]$ is the Maxwell accommodation coefficient, $(G_\epsilon \circ \mathrm{R})(v, x, t) = G_\epsilon(\mathrm{R}(x)v, x, t)$ where $\mathrm{R}(x) = I - 2\mathrm{n}(x)\mathrm{n}(x)^T$ is the specular reflection matrix at a point $x \in \partial\Omega$, $\mathbf{1}_{\Sigma_+}$ is the indicator function of the so-called outgoing boundary set

$$\Sigma_+ = \big\{(v,x) \in \mathbb{R}^{\mathrm{D}} \times \partial\Omega \,:\, \mathrm{n}(x)\cdot v > 0\big\}\,, \tag{1.7}$$

and $\langle\,\cdot\,\rangle$ denotes the average

$$\langle \xi \rangle = \int_{\mathbb{R}^{\mathrm{D}}} \xi(v)\, M(v)\,\mathrm{d}v\,. \tag{1.8}$$

Because $\sqrt{2\pi}\,\big\langle \mathbf{1}_{\Sigma_+}\, n\cdot v\big\rangle = 1$, it is easy to see from (1.6) that on $\partial\Omega$ one has

$$\begin{aligned} \langle n\cdot v\, G_\epsilon\rangle &= \big\langle \mathbf{1}_{\Sigma_+}\, n\cdot v\,\big(G_\epsilon - G_\epsilon\circ \mathrm{R}\big)\big\rangle \\ &= \alpha\,\Big\langle \mathbf{1}_{\Sigma_+} n\cdot v\,\Big(G_\epsilon - \sqrt{2\pi}\,\big\langle \mathbf{1}_{\Sigma_+} n\cdot v\, G_\epsilon\big\rangle\Big)\Big\rangle = 0\,. \end{aligned} \tag{1.9}$$

Fluid regimes are those in which the Knudsen number ϵ is small. The acoustic system (1.1, 1.2) can be derived from (1.4, 1.6) for families of solutions $G_\epsilon(v, x, t)$ that are scaled so that

$$G_\epsilon = 1 + \delta_\epsilon g_\epsilon\,, \qquad G_\epsilon^{\mathrm{in}} = 1 + \delta_\epsilon g_\epsilon^{\mathrm{in}}\,, \tag{1.10}$$

where

$$\delta_\epsilon \to 0 \qquad \text{as} \qquad \epsilon \to 0\,, \tag{1.11}$$

and the fluctuations g_ϵ and g_ϵ^{in} converge in the sense of distributions to $g \in L^\infty(\mathrm{d}t; L^2(M\mathrm{d}v\,\mathrm{d}x))$ and $g^{\mathrm{in}} \in L^2(M\mathrm{d}v\,\mathrm{d}x)$ respectively as $\epsilon \to 0$. One finds that g has the infinitesimal Maxwellian form

$$g = \rho + v\cdot u + \big(\tfrac{1}{2}|v|^2 - \tfrac{\mathrm{D}}{2}\big)\theta\,, \tag{1.12}$$

where $(\rho, u, \theta) \in L^\infty(\mathrm{d}t; L^2(\mathrm{d}x; \mathbb{R}\times\mathbb{R}^{\mathrm{D}}\times\mathbb{R}))$ solve (1.1, 1.2) with initial data given by

$$\rho^{\mathrm{in}} = \langle g^{\mathrm{in}}\rangle\,, \qquad u^{\mathrm{in}} = \langle v\, g^{\mathrm{in}}\rangle\,, \qquad \theta^{\mathrm{in}} = \big\langle \big(\tfrac{1}{\mathrm{D}}|v|^2 - 1\big)\, g^{\mathrm{in}}\big\rangle\,. \tag{1.13}$$

The formal derivation leading to (1.1) closely follows that in [6], so its details will not be given here. The boundary condition (1.2) is obtained by noticing that (1.9) implies $\langle \mathrm{n}\cdot v\, g_\epsilon\rangle = 0$, then passing to the limit in this to get $\langle \mathrm{n}\cdot v\, g\rangle = 0$, and finally using (1.12) to obtain (1.2).

The program initiated in [1, 2, 3] seeks to justify fluid dynamical limits for Boltzmann equations in the setting of DiPerna-Lions renormalized solutions [5], which are the only temporally global, large data solutions available. The main obstruction to carrying out this program is that DiPerna-Lions solutions are not known to satisfy many properties that one formally expects for solutions of the Boltzmann equation. For example, they are not known to satisfy the formally expected local conservations laws of momentum and energy. Moreover, their regularity is poor. The justification of fluid dynamical limits in this setting is therefore highly nontrivial.

The acoustic limit was first established in this kind of setting in [4] over a periodic domain. There idea introduced there was to pass to the limit in approximate local conservations laws which are satified by DiPerna-Lions solutions. One then shows that the so-called conservation defects vanish as the Knudsen number ϵ vanishes, thereby establishing the local conservation laws in the limit. This was done in [4] using only relative entropy estimates, which restricted the result to collision kernels that are bounded and to fluctuations scaled so that

$$\delta_\epsilon \to 0 \quad \text{and} \quad \frac{\delta_\epsilon}{\epsilon} |\log(\delta_\epsilon)| \to 0 \quad \text{as} \quad \epsilon \to 0\,, \tag{1.14}$$

which is far from the formally expected optimal scaling (1.11).

In [6] the local conservation defects were removed using new dissipation rate estimates. This allowed the treatment of collision kernels that for some $C_b < \infty$ and $\beta \in [0, 1)$ satisfied

$$\int_{\mathbb{S}^{D-1}} b(\omega, v_1 - v) \, \mathrm{d}\omega \leq C_b \big(1 + |v_1 - v|^2\big)^\beta\,, \tag{1.15}$$

and of fluctuations scaled so that

$$\delta_\epsilon \to 0 \quad \text{and} \quad \frac{\delta_\epsilon}{\epsilon^{1/2}} |\log(\delta_\epsilon)|^{\beta/2} \to 0 \quad \text{as} \quad \epsilon \to 0\,. \tag{1.16}$$

The above class of collision kernels includes all classical kernels that are derived from Maxwell or hard potentials and that satisfy a weak small deflection cutoff. The scaling given by (1.16) is much less restrictive than that given by (1.16), but is far from the formally expected optimal scaling (1.11). Finally, only periodic domains are treated in [6].

Here we employ new estimates developed in [12] to improve the result of [6] in three ways. First, as in [12], we treat a broader class of collision kernels that includes those derived from soft potentials. Second, we improve the scaling of the fluctuations to $\delta_\epsilon = O(\epsilon^{1/2})$. Finally, we treat more general bounded domains and derive the boundary condition (1.2) in the limit.

We use the L^1 velocity averaging theory of Golse and Saint-Raymond [7] through the nonlinear compactness estimate of [12] to improve the scaling of the fluctuations to $\delta_\epsilon = O(\epsilon^{1/2})$. Without it we would only be able to improve the scaling to $\delta_\epsilon = o(\epsilon^{1/2})$. This is the first time the L^1 averaging theory has played any role in an acoustic limit theorem, albeit for a modest improvement in our result. We remark that averaging theory plays no role in establishing the Stokes limit with its formally expected optimal scaling of $\delta_\epsilon = o(\epsilon)$ [12].

We treat the bounded domain case in the setting of Mischler [14], who extended the DiPerna-Lions theory to bounded domains with a Maxwell reflection boundary condition. He showed that these boundary conditions are satisfied in a *renormalized* sense. This means we cannot deduce that $\langle \mathrm{n} \cdot v \, g_\epsilon \rangle = 0$ as we did in our formal argument. Rather, we apply generalizations of some boundary *a priori* estimates developed in [13] to the broader class of collision kernels in [11], and the new estimates mentioned above, to obtain $\langle \mathrm{n} \cdot v \, g \rangle = 0$ in a weak sense in the limit of $\epsilon \to 0$. We thereby derive a weak form of the boundary condition (1.2).

Finally, we remark that fully establishing the acoustic limit with its formally expected optimal scaling of the fluctuation size (1.11) is still open. This gap must be bridged before one can hope to fully establish the compressible Euler limit starting from DiPerna-Lions solutions to the Boltzmann equation. In contrast, optimal scaling can be obtained within the framework of classical solutions by using the nonlinear energy method developed by Guo. This has been done recently by the first author of this paper with Guo and Jang [9, 10].

Our paper is organized as follows. Section 2 gives its framework. Section 3 states and proves our main result modulo two steps. Section 4 removes the conservation defects. Section 5, establishes the limiting boundary condition.

2. Framework

2.1. **Formal Framework.** In this paper we use much of the same notation as in [12]. Here we just give what we need to state our theorem. For a more complete introduction to the Boltzmann equation, see [6, 12].

We let Ω be a smooth bounded domain in $\mathbb{R}^{\mathrm{D}}$, and let $\mathcal{O} = \mathbb{R}^{\mathrm{D}} \times \Omega$ be the associated phase space domain. Let $\mathrm{n}(x)$ be the outward unit normal vector at $x \in \partial\Omega$. We denote by $\mathrm{d}\sigma_x$ the Lebesgue measure on the boundary $\partial\Omega$. We define the outgoing and incoming sets, Σ_+ and Σ_-, at the boundary $\partial\Omega$ by

$$\Sigma_\pm = \left\{ (v,x) \in \mathbb{R}^{\mathrm{D}} \times \partial\Omega \,:\, \pm \mathrm{n}(x) \cdot v > 0 \right\} . \tag{2.1}$$

The absolute Maxwellian $M(v)$ given by (1.3) corresponds to the spatially homogeneous fluid state with density and temperature equal to 1 and bulk velocity equal to 0. The boundary condition (1.6) corresponds to a wall temperature of 1, so that $M(v)$ is the unique equilibrium of the fluid.

Associated with the initial data G_ϵ^{in} we have the normalization

$$\int_\Omega \langle G_\epsilon^{\mathrm{in}} \rangle \, \mathrm{d}x = 1 \, . \tag{2.2}$$

2.2. **Assumptions on the Collision Kernel.** The kernel $b(\omega, v_1 - v)$ associated with the collision operator (1.5) is positive almost everywhere. The Galilean invariance of the collisional physics implies that b has the classical form

$$b(\omega, v_1 - v) = |v_1 - v| \, \Sigma(|\omega \cdot n|, |v_1 - v|) \, , \tag{2.3}$$

where $n = (v_1 - v)/|v_1 - v|$ and Σ is the specific differential cross-section. Our five further assumptions regarding b are adopted from [12] and are technical in nature.

Our *first technical assumption* is that the collision kernel b satisfies the requirements of the DiPerna-Lions theory. That theory requires that b be locally integrable with respect to $\mathrm{d}\omega \, M_1 \mathrm{d}v_1 \, M \mathrm{d}v$, and that it moreover satisfies

$$\lim_{|v|\to\infty} \frac{1}{1+|v|^2} \int_K \bar{b}(v_1 - v) \, \mathrm{d}v_1 = 0 \, , \quad \text{for every compact } K \subset \mathbb{R}^{\mathrm{D}} \, , \tag{2.4}$$

where $\bar{b}$ is defined by

$$\bar{b}(v_1 - v) \equiv \int_{\mathbb{S}^{\mathrm{D}-1}} b(\omega, v_1 - v) \, \mathrm{d}\omega \, . \tag{2.5}$$

Galilean symmetry (2.3) implies that $\bar{b}$ is a function of $|v_1 - v|$ only.

Our *second technical assumption* regarding b is that the attenuation coefficient a, which is defined by

$$a(v) \equiv \int_{\mathbb{R}^{\mathrm{D}}} \bar{b}(v_1 - v) \, M_1 \mathrm{d}v_1 = \iint_{\mathbb{S}^{\mathrm{D}-1} \times \mathbb{R}^{\mathrm{D}}} b(\omega, v_1 - v) \, \mathrm{d}\omega \, M_1 \mathrm{d}v_1 \, , \tag{2.6}$$

is bounded below as

$$C_a \big(1 + |v|\big)^\alpha \le a(v) \, , \tag{2.7}$$

for some constants $C_a > 0$ and $\alpha \in \mathbb{R}$. Galilean symmetry (2.3) implies that a is a function of $|v|$ only.

Our *third technical assumption* regarding b is that there exists $s \in (1,\infty]$ and $C_b \in (0,\infty)$ such that

$$\left(\int_{\mathbb{R}^{\mathrm{D}}} \left| \frac{\bar{b}(v_1 - v)}{a(v_1)\, a(v)} \right|^s a(v_1)\, M_1 \mathrm{d}v_1 \right)^{\frac{1}{s}} \leq C_b \,. \tag{2.8}$$

Because this bound is uniform in v, we may take C_b to be the supremum over v of the left-hand side of (2.8).

Our *fourth technical assumption* regarding b is that the operator

$$\mathcal{K}^+ : L^2(aM\mathrm{d}v) \to L^2(aM\mathrm{d}v) \quad \text{is compact}\,, \tag{2.9}$$

where

$$\mathcal{K}^+ \tilde{g} = \frac{1}{2a} \iint_{\mathbb{S}^{\mathrm{D}-1}\times\mathbb{R}^{\mathrm{D}}} \left(\tilde{g}' + \tilde{g}_1' \right) b(\omega, v_1 - v)\, \mathrm{d}\omega\, M_1 \mathrm{d}v_1 \,. \tag{2.10}$$

We remark that $\mathcal{K}^+ : L^2(aM\mathrm{d}v) \to L^2(aM\mathrm{d}v)$ is always bounded [12].

Our *fifth technical assumption* regarding b is that for every $\delta > 0$ there exists C_δ such that $\bar{b}$ satisfies

$$\frac{\bar{b}(v_1 - v)}{1 + \delta \, \dfrac{\bar{b}(v_1 - v)}{1 + |v_1 - v|^2}} \leq C_\delta \big(1 + a(v_1)\big)\big(1 + a(v)\big) \quad \text{for every } v_1, v \in \mathbb{R}^{\mathrm{D}} \,. \tag{2.11}$$

The above assumptions are satisfied by all the classical collision kernels with a weak small deflection cutoff that derive from a repulsive intermolecular potential of the form c/r^k with $k > 2\frac{\mathrm{D}-1}{\mathrm{D}+1}$. This includes all the classical collision kernels to which the DiPerna-Lions theory applies [12]. Kernels that satisfy (1.15) clearly satisfy (2.4). If they moreover satisfy (2.7) with $\alpha = \beta$ then they also satisfy (2.8) and (2.11).

Because the kernel b satisfies (2.4), it can be normalized so that

$$\iint_{\mathbb{S}^{\mathrm{D}-1}\times\mathbb{R}^{\mathrm{D}}\times\mathbb{R}^{\mathrm{D}}} b(\omega, v_1 - v)\, \mathrm{d}\omega\, M_1 \,\mathrm{d}v_1\, M \,\mathrm{d}v = 1 \,.$$

Because $\mathrm{d}\mu = b(\omega, v_1 - v)\, \mathrm{d}\omega\, M_1 \mathrm{d}v_1\, M\mathrm{d}v$ is a positive unit measure on $\mathbb{S}^{\mathrm{D}-1} \times \mathbb{R}^{\mathrm{D}} \times \mathbb{R}^{\mathrm{D}}$, we denote by $\langle\!\langle \Xi \rangle\!\rangle$ the average over this measure of any integrable function $\Xi = \Xi(\omega, v_1, v)$

$$\langle\!\langle \Xi \rangle\!\rangle = \iiint_{\mathbb{S}^{\mathrm{D}-1}\times\mathbb{R}^{\mathrm{D}}\times\mathbb{R}^{\mathrm{D}}} \Xi(\omega, v_1, v)\, \mathrm{d}\mu \,. \tag{2.12}$$

2.3. **DiPerna-Lions-Mischler Theory.** As in [4, 6, 12], we will work in the framework of DiPerna-Lions solutions to the scaled Boltzmann equation on the phase space $\mathbb{R}^{\mathrm{D}} \times \Omega$

$$\begin{aligned} \partial_t G_\epsilon + v \cdot \nabla_x G_\epsilon &= \frac{1}{\epsilon} \mathcal{Q}(G_\epsilon, G_\epsilon) \quad \text{on} \quad \mathcal{O} \times \mathbb{R}_+ \,, \\ G_\epsilon(v, x, 0) &= G_\epsilon^{\mathrm{in}}(v, x) \quad \text{on} \quad \mathcal{O} \,, \end{aligned} \tag{2.13}$$

with the Maxwell reflection boundary condition (1.6) which can be expressed as

$$\gamma_- G_\epsilon = (1 - \alpha) L(\gamma_+ G_\epsilon) + \alpha \langle \gamma_+ G_\epsilon \rangle_{\partial\Omega} \quad \text{on} \quad \Sigma_- \times \mathbb{R}_- \,, \tag{2.14}$$

where $\gamma_\pm G = \mathbf{1}_{\Sigma_\pm} \gamma G$ are the traces of G on the outgoing and incoming sets $\Sigma_\pm$. Here the local reflection operator L is given by

$$L\phi(v\,, x) = \phi(\mathrm{R}(x) v\,, x)\,, \tag{2.15}$$

where $\mathrm{R}(x)v = v - 2v\cdot\mathrm{n}(x)\mathrm{n}(x)$ is the specular reflection of v. The diffusion reflection operator is defined as

$$\langle\phi\rangle_{\partial\Omega} = \sqrt{2\pi}\int\limits_{v\cdot\mathrm{n}(x)>0}\phi(v\,,x)\mathrm{n}(x)\cdot v\,M\mathrm{d}v\,. \tag{2.16}$$

DiPerna-Lions theory requires that both the equation and boundary conditions in (2.13) should be understood in the renormalized sense, see (3.10) and (3.14). These solutions were initially constructed by DiPerna and Lions [5] over the whole space $\mathbb{R}^{\mathrm{D}}$ for any initial data satisfying natural physical bounds. For bounded domain case, Mischler [14] recently developed a general theory to treat Maxwell boundary conditions (1.6).

The DiPerna-Lions theory does not yield solutions that are known to solve the Boltzmann equation in the usual weak solutions. Rather, it gives the existence of a global weal solution to a class of formally equivalent initial value problems that are obtained by multiplying (2.13) by $\Gamma'(G_\epsilon)$, where Γ' is the derivative of an admissible function Γ:

$$(\partial_t + v\cdot\nabla_x)\Gamma(G_\epsilon) = \frac{1}{\epsilon}\Gamma'(G_\epsilon)\mathcal{Q}(G_\epsilon\,,G_\epsilon)\quad\text{on}\quad \mathcal{O}\times\mathbb{R}^+\,. \tag{2.17}$$

A function $\Gamma : [0\,,\infty) \to \mathbb{R}$ is called admissible if it is continuously differentiable and for some constant $C_\Gamma < \infty$ its derivative satisfies

$$|\Gamma'(z)|\sqrt{1+z} \le C_\Gamma\,. \tag{2.18}$$

In the case of a bounded domain, one also has to give a sense to the boundary condition, and this can be included in the weak formulation. In [14], Mischler proved the following Trace Theorem:

Theorem 2.1. (Trace Theorem [14]) *For every* fixed $\epsilon > 0$*,* $T > 0$*, let* $G_\epsilon \in L^1(\mathcal{O}\times[0,T])$ *and* Γ *an admissible function so that* $\frac{1}{\epsilon}\Gamma'(G_\epsilon)\mathcal{Q}(G_\epsilon, G_\epsilon) \in L^1(\mathcal{O}\times[0,T])$ *and the equation* (2.17) *in the sense of distribution. Then for every* $t \in [0\,,T]$*, there exists* $\gamma G_\epsilon(\cdot\,,\cdot\,,t) \in C([0,T])\,;L^1(\mathcal{O}))$ *and* $\gamma G_\epsilon \in L^1(\Sigma\times[0,T])$*, satisfying the following Green formula*

$$\begin{aligned}
&\int\limits_\Omega \langle\Gamma(G_\epsilon(t_2))Y\rangle\,\mathrm{d}x - \int\limits_\Omega\langle\Gamma(G_\epsilon(t_1))Y\rangle\,\mathrm{d}x\\
&-\int_{t_1}^{t_2}\int\limits_\Omega\langle\Gamma(G_\epsilon)v\cdot\nabla_x Y\rangle\,\mathrm{d}x\,\mathrm{d}t + \int_{t_1}^{t_2}\int\limits_{\partial\Omega}\langle\Gamma(\gamma G_\epsilon)Y(v\cdot\mathrm{n}(x))\rangle\,\mathrm{d}\sigma_x\,\mathrm{d}t\\
&= \frac{1}{\epsilon}\int_{t_1}^{t_2}\int\limits_\Omega\langle\Gamma'(G_\epsilon)\mathcal{Q}(G_\epsilon\,,G_\epsilon)Y\rangle\,\mathrm{d}x\,\mathrm{d}t\,.
\end{aligned} \tag{2.19}$$

for every $Y \in C^2\cap L^\infty(\mathbb{R}^{\mathrm{D}}\times\bar\Omega)$ *and every* $[t_1\,,t_2] \subset [0\,,\infty]$*. Moreover, the trace function is renormalized as*

$$\gamma\Gamma(G_\epsilon) = \Gamma(\gamma G_\epsilon)\,. \tag{2.20}$$

The Maxwell boundary condition (1.6) can be understood in the following sense:

$$\Gamma(\gamma_- G_\epsilon) = \Gamma\left((1-\alpha)L(\gamma_+G_\epsilon) + \alpha\langle\gamma_+G_\epsilon\rangle_{\partial\Omega}\right)\quad\text{on}\quad\Sigma_-\times\mathbb{R}^+\,, \tag{2.21}$$

where the equality holds in the sense of distribution.

Theorem 2.2. (Renormalzied solutions [5, 14]) *Let b satisfy the conditions listed in section 2.2. Given any initial data G_ϵ^{in} satisfying*

$$\iint_{\mathcal{O}} G_\epsilon^{\rm in}(1+|v|^2+|\log G_\epsilon^{\rm in}|)\, M{\rm d}v\,{\rm d}x < +\infty\,, \tag{2.22}$$

there exists at least one $G_\epsilon \geq 0$ in $C([0\,,\infty);\, w\text{-}L^1(\,M{\rm d}v\,{\rm d}x))$ such that (2.19) *and* (2.21) *hold for all admissible functions Γ. Moreover, G_ϵ satisfies the following global entropy inequality for all $t>0$:*

$$H(G_\epsilon(t)) + \int_0^t \left[\frac{1}{\epsilon}R(G_\epsilon(s)) + \frac{\alpha}{\sqrt{2\pi}}\mathcal{E}(\gamma_+ G_\epsilon)\right]{\rm d}s \leq H(G_\epsilon^{\rm in})\,, \tag{2.23}$$

where the relative entropy functional is given by

$$H(G) = \int_\Omega \langle G\log(G) - G + 1\rangle\,{\rm d}x\,, \tag{2.24}$$

the entropy dissipation rate functional is given by

$$R(G) = \frac{1}{4}\int_{\mathbb{R}^{\rm D}} \left\langle\!\!\left\langle \log\left(\frac{G_1'G'}{G_1G}\right)\left(G_1'G' - G_1G\right)\right\rangle\!\!\right\rangle {\rm d}x\,, \tag{2.25}$$

and the so-called Darrozès-Guiraud information is given by

$$\mathcal{E}(\gamma_+ G_\epsilon) = \int_{\partial\Omega} \left[\langle h(\delta_\epsilon\gamma_+ g_\epsilon)\rangle_{\partial\Omega} - h(\delta_\epsilon\langle\gamma_+ g_\epsilon\rangle_{\partial\Omega})\right]{\rm d}\sigma_x\,. \tag{2.26}$$

3. Main Result

3.1. **Main Theorem.** We will consider families G_ϵ of DiPerna-Lions renormalized solutions to (2.13) such that $G_\epsilon^{\rm in} \geq 0$ satisfies the entropy bound

$$H(G_\epsilon^{\rm in}) \leq C^{\rm in}\delta_\epsilon^2 \tag{3.1}$$

for some $C^{\rm in} < \infty$ and $\delta_\epsilon > 0$ that satisfies the scaling $\delta_\epsilon \to 0 \quad \text{as} \quad \epsilon \to 0\,.$

The entropy functional H provides a natural measure of the proximity of G to the equilibrium $G = 1$. We define the families $g_\epsilon^{\rm in}$ and g_ϵ of fluctuations about $G = 1$ by the relations

$$G_\epsilon^{\rm in} = 1 + \delta_\epsilon g_\epsilon^{\rm in}\,, \qquad G_\epsilon = 1 + \delta_\epsilon g_\epsilon\,. \tag{3.2}$$

One easily sees that H asymptotically behaves like half the square of the L^2-norm of these fluctuations as $\epsilon \to 0$. Hence, (2.23) is consistent with these fluctuations being of order 1. Just as the relative entropy H controls the fluctuations g_ϵ, the dissipation rate R given by (2.25) controls the scaled collision integrals defined by

$$q_\epsilon = \frac{1}{\sqrt{\epsilon}\delta_\epsilon}\left(G_{\epsilon 1}'G_\epsilon' - G_{\epsilon 1}G_\epsilon\right). \tag{3.3}$$

Here we only state the weak acoustic limit theorem because the corresponding strong limit theorem is the same as in [6] and its proof based on the weak limit theorem and relative entropy convergence is also the same.

Theorem 3.1. (Weak Acoustic Limit Theorem) *Let b be a collision kernel that satisfies the assumptions in Section 2.2. Let $G_\epsilon^{\rm in}$ be a family in the entropy class $E(M{\rm d}v\,{\rm d}x) = \{G_\epsilon^{\rm in} \geq$

$0 : H(G^{\rm in}) < \infty\}$ that satisfies the normalization (2.2) *and the entropy bound* (3.1) *for some $C^{\rm in} < \infty$ and $\delta_\epsilon > 0$ satisfies the scaling*

$$\delta_\epsilon = O\big(\sqrt{\epsilon}\big)\,. \tag{3.4}$$

Assume, moreover, that for some $(\rho^{\rm in}, u^{\rm in}, \theta^{\rm in}) \in L^2(\mathrm{d}x; \mathbb{R}\times\mathbb{R}^{\rm D}\times\mathbb{R})$ the family of fluctuations $g_\epsilon^{\rm in}$ satisfies

$$(\rho^{\rm in}, u^{\rm in}, \theta^{\rm in}) = \lim_{\epsilon\to 0}\Big(\langle g_\epsilon^{\rm in}\rangle\,,\,\langle v\, g_\epsilon^{\rm in}\rangle\,,\,\big\langle\big(\tfrac{1}{\rm D}|v|^2 - 1\big)\, g_\epsilon^{\rm in}\big\rangle\Big) \tag{3.5}$$

in the sense of distributions.

Let G_ϵ be any family of DiPerna-Lions renormalized solutions to the Boltzmann equation (2.13) *that have $G_\epsilon^{\rm in}$ as initial values.*

Then, as $\epsilon \to 0$, the family of fluctuations g_ϵ satisfies

$$g_\epsilon \to \rho + v\cdot u + (\tfrac12|v|^2 - \tfrac{\rm D}{2})\theta \quad w\text{-}L^1_{loc}(\mathrm{d}t; w\text{-}L^1(\sigma M \mathrm{d}v\,\mathrm{d}x))\,, \tag{3.6}$$

where $(\rho, u, \theta) \in C([0,\infty); L^2(\mathrm{d}x; \mathbb{R}\times\mathbb{R}^{\rm D}\times\mathbb{R}))$ is the unique solution to the acoustic system (1.1) *with the boundary condition* (1.2) *and with initial data $(\rho^{\rm in}, u^{\rm in}, \theta^{\rm in})$ obtain from* (3.5)*. In addition, one has that*

$$\Big(\langle g_\epsilon\rangle\,,\,\langle v\, g_\epsilon\rangle\,,\,\big\langle\big(\tfrac{1}{\rm D}|v|^2 - 1\big)\; g_\epsilon\big\rangle\Big) \to (\rho, u, \theta) \quad \text{in } C([0,\infty); w\text{-}L^1(\mathrm{d}x; \mathbb{R}\times\mathbb{R}^{\rm D}\times\mathbb{R}))\,. \tag{3.7}$$

and that ρ satisfies

$$\int_\Omega \rho\,\mathrm{d}x = 0\,. \tag{3.8}$$

This result improves upon the acoustic limit result in [6] in three ways. First, its assumption on the collision kernel b is the same as [12], so it covers a broad class of cut-off kernels and especially soft potentials. Second, the scaling assumption in [12] requires $\delta_\epsilon = \epsilon^m$, $m > \frac12$. Here we can treat the borderline case $m = \frac12$, whereas formal derivation of acoustic system only requires $\delta_\epsilon \to 0$ as $\epsilon \to 0$, i.e. $m > 0$. This more restrictive requirement arises from the way in which we remove the local conservation laws defects of the DiPerna-Lions solutions. Third, we derive the weak form of the boundary condition $\mathrm{n}\cdot u = 0$. It is the first time such a boundary condition for the acoustic system is derived from the Boltzmann equation with the Maxwell boundary condition.

3.2. **Proof of the Main Theorem.** In order to derive the fluid equations with boundary conditions, we need to pass to the limit in approximate local conservation laws built from the renormalized Boltzmann equation (2.13). We choose the renormalization used in [12] — namely,

$$\Gamma(Z) = \frac{Z-1}{1+(Z-1)^2}\,. \tag{3.9}$$

After dividing by δ_ϵ, equation (2.13) becomes

$$\partial_t \tilde g_\epsilon + v\cdot\nabla_x \tilde g_\epsilon = \frac{1}{\sqrt{\epsilon}}\,\Gamma'(G_\epsilon)\iint_{\mathbb{S}^{\rm D-1}\times\mathbb{R}^{\rm D}} q_\epsilon\, b(\omega\,, v_1 - v)\,\mathrm{d}\omega\, M_1 \mathrm{d}v_1\,, \tag{3.10}$$

where $\tilde g_\epsilon = \Gamma(G_\epsilon)/\delta_\epsilon$. By introducing $N_\epsilon = 1 + \delta_\epsilon^2 g_\epsilon^2$, we can write

$$\tilde g_\epsilon = \frac{g_\epsilon}{N_\epsilon}\,, \qquad \Gamma'(G_\epsilon) = \frac{2}{N_\epsilon^2} - \frac{1}{N_\epsilon}\,. \tag{3.11}$$

When moment of the renormalized Boltzmann equation (3.10) is formally taken with respect to any $\zeta \in \operatorname{span}\{1, v_1, \cdots, v_{\mathrm{D}}, |v|^2\}$, one obtains

$$\partial_t\langle\zeta\,\tilde{g}_\epsilon\rangle + \nabla_x\cdot\langle v\,\zeta\,\tilde{g}_\epsilon\rangle = \frac{1}{\sqrt{\epsilon}}\langle\!\langle \zeta\,\Gamma'(G_\epsilon)\,q_\epsilon\rangle\!\rangle\,. \tag{3.12}$$

This fails to be a local conservation law because the so-called *conservation defect* on the right-hand side is generally nonzero. The idea of the proof is to show that as $\epsilon \to 0$ this defect vanishes, while the left-hand side converges to the left-hand side of the local conservation law corresponding to ζ.

It can be shown that every DiPerna-Lions solution satisfies (3.12) in the sense that for every $\chi \in C^1(\Omega)$ and every $[t_1, t_2] \subset [0, \infty)$ it satisfies

$$\begin{aligned}\int_\Omega \chi\,\langle\zeta\,\tilde{g}_\epsilon(t_2)\rangle\,\mathrm{d}x - \int_\Omega \chi\,\langle\zeta\,\tilde{g}_\epsilon(t_1)\rangle\,\mathrm{d}x + \int_{t_1}^{t_2}\int_{\partial\Omega}\chi\,\mathrm{n}\cdot\langle v\,\zeta\,\tilde{g}_\epsilon\rangle\,\mathrm{d}\sigma_x\,\mathrm{d}t \\ - \int_{t_1}^{t_2}\int_\Omega \nabla_x\chi\cdot\langle v\,\zeta\,\tilde{g}_\epsilon\rangle\,\mathrm{d}x\,\mathrm{d}t = \int_{t_1}^{t_2}\int_\Omega \chi\,\frac{1}{\sqrt{\epsilon}}\langle\!\langle\zeta\,\Gamma'(G_\epsilon)\,q_\epsilon\rangle\!\rangle\,\mathrm{d}x\,\mathrm{d}t\,.\end{aligned} \tag{3.13}$$

Moreover, from (2.20) and (2.21), the boundary condition can be understood in the following renormalized sense:

$$\gamma_-\tilde{g}_\epsilon = (1-\alpha)L\gamma_+\tilde{g}_\epsilon + \alpha\langle\gamma_+\tilde{g}_\epsilon\rangle_{\partial\Omega} \quad \text{on} \quad \Sigma_- \times \mathbb{R}^+\,, \tag{3.14}$$

where the equality holds in the sense of distribution.

The Main Theorem will be proved in two steps: the interior equations and the boundary condition. The acoustic system (1.1) interior is justified by showing that the limit of (3.13) (**with the test function χ supported in the domain** Ω) as $\epsilon \to 0$ is the weak form of the acoustic system. The convergence of the density terms, the flux terms are exactly the same as [6, 12], so we skip the proof here. The only difference is the removal of the conservation defect which is proved in [6] under the restriction $\delta_\epsilon = \epsilon^m$, $m > \frac{1}{2}$. In the rest of this note, we remark that for the borderline case $m = \frac{1}{2}$, the local conservation law defects also vanish as $\epsilon \to 0$.

The second step is to justify the boundary condition (1.2). We do not have enough control to pass to the limit in nonlinear boundary terms, i.e. the boundary terms in (3.13) for conservation of momentum and energy. However, the boundary condition can be justified by only taking limit $\epsilon \to 0$ in (3.13) for conservation of mass. The details will be given in the last section.

4. Removal of the Conservation Defects

The conservation defects have the form

$$\frac{1}{\sqrt{\epsilon}}\langle\!\langle\zeta\,\Gamma'(G_\epsilon)\,q_\epsilon\rangle\!\rangle = \frac{1}{\sqrt{\epsilon}}\left\langle\!\!\left\langle \zeta\left(\frac{2}{N_\epsilon^2} - \frac{1}{N_\epsilon}\right)q_\epsilon\right\rangle\!\!\right\rangle\,. \tag{4.1}$$

In order to establish momentum and energy conservation laws, we must show that these defects vanish as $\epsilon \to 0$. This is done with the following proposition.

Proposition 4.1. *For $n = 1$ and $n = 2$, and for every $\zeta \in$ span$\{1, v_1, \cdots, v_{\mathrm{D}}, |v|^2\}$ one has*

$$\frac{1}{\sqrt{\epsilon}}\left\langle\!\!\left\langle\zeta\,\frac{q_\epsilon}{N_\epsilon^n}\right\rangle\!\!\right\rangle \to 0 \quad \text{in } w\text{-}L^1_{loc}(\mathrm{d}t; w\text{-}L^1(\mathrm{d}x)) \text{ as } \epsilon \to 0\,. \tag{4.2}$$

Proof. Similar to the proof of Proposition 10.1 in [12], for $n = 1$, we obtain the decomposition

$$\frac{1}{\sqrt{\epsilon}}\left\langle\!\!\left\langle \zeta \, \frac{q_\epsilon}{N_\epsilon} \right\rangle\!\!\right\rangle = \frac{\delta_\epsilon^2}{\sqrt{\epsilon}}\left\langle\!\!\left\langle \zeta \, \frac{g_{\epsilon 1}^2 q_\epsilon}{N_{\epsilon 1} N_\epsilon} \right\rangle\!\!\right\rangle + \left\langle\!\!\left\langle \zeta \, \frac{\delta_\epsilon^2 (g_{\epsilon 1} + g_\epsilon) q_\epsilon^2}{N'_{\epsilon 1} N'_\epsilon N_{\epsilon 1} N_\epsilon} \right\rangle\!\!\right\rangle - \frac{\delta_\epsilon^2}{\sqrt{\epsilon}}\left\langle\!\!\left\langle \zeta' \, \frac{g'_{\epsilon 1} g'_\epsilon \, q_\epsilon}{N'_{\epsilon 1} N'_\epsilon N_{\epsilon 1} N_\epsilon} \, J_\epsilon \right\rangle\!\!\right\rangle , \tag{4.3}$$

where J_ϵ is given by

$$J_\epsilon = 2 + \delta_\epsilon \big(g'_{\epsilon 1} + g'_\epsilon + g_{\epsilon 1} + g_\epsilon \big) - \delta_\epsilon^2 \big(g'_{\epsilon 1} g'_\epsilon - g_{\epsilon 1} g_\epsilon \big) \, . \tag{4.4}$$

We now dominate the integrands of the three terms on the right-hand side of (4.3). Because for every $\zeta \in \operatorname{span}\{1\,, v_1\,, \cdots, v_{\mathrm{D}}\,, |v|^2\}$ there exists a constant $C < \infty$ such that $|\zeta| \leq C\sigma$ where $\sigma \equiv 1 + |v|^2$, the integrand of the first term is dominated by

$$\frac{\delta_\epsilon^2}{\sqrt{\epsilon}} \, \sigma \, \frac{g_{\epsilon 1}^2 |q_\epsilon|}{N_{\epsilon 1} N_\epsilon} \, . \tag{4.5}$$

Because $\frac{\delta_\epsilon |g_{\epsilon 1} + g_\epsilon|}{\sqrt{N'_{\epsilon 1} N'_\epsilon N_{\epsilon 1} N_\epsilon}} \leq 2$, the integrand of the second term is dominated by

$$\sigma \, \frac{\delta_\epsilon q_\epsilon^2}{\sqrt{N'_{\epsilon 1} N'_\epsilon N_{\epsilon 1} N_\epsilon}} \, . \tag{4.6}$$

Finally, because $\frac{|J_\epsilon|}{\sqrt{N'_{\epsilon 1} N'_\epsilon N_{\epsilon 1} N_\epsilon}} \leq 8$, the integrannd of the third term is dominated by

$$\frac{\delta_\epsilon^2}{\sqrt{\epsilon}} \, \sigma' \, \frac{|g'_{\epsilon 1} g'_\epsilon| |q_\epsilon|}{\sqrt{N'_{\epsilon 1} N'_\epsilon N_{\epsilon 1} N_\epsilon}} \, . \tag{4.7}$$

Hence, the result (4.2) for the case $n = 1$ will follow once we establish that the terms (4.5), (4.6), and (4.7) vanish as $\epsilon \to 0$.

The term (4.6) can be treated easily. By the inequality $n'_{\epsilon 1} n'_\epsilon n_{\epsilon 1} n_\epsilon \leq 2\sqrt{N'_{\epsilon 1} N'_\epsilon N_{\epsilon 1} N_\epsilon}$, where $n_\epsilon = 1 + \frac{\delta_\epsilon}{3} g_\epsilon$, from estimate

$$\sigma \, \frac{q_\epsilon^2}{n'_{\epsilon 1} n'_\epsilon n_{\epsilon 1} n_\epsilon} = O(|\log(\sqrt{\epsilon}\delta_\epsilon)|) \quad \text{in} \quad L^1_{loc}(\mathrm{d}t; L^1(\mathrm{d}\mu \mathrm{d}x)) \quad \text{as} \quad \epsilon \to 0 \, , \tag{4.8}$$

which is proved in Lemma 9.4 of [6], it follows that

$$\sigma \frac{\delta_\epsilon q_\epsilon^2}{\sqrt{N'_{\epsilon 1} N'_\epsilon N_{\epsilon 1} N_\epsilon}} = O(\delta_\epsilon |\log(\sqrt{\epsilon}\delta_\epsilon)|) \to 0 \quad \text{in} \quad L^1_{loc}(\mathrm{d}t; L^1(\mathrm{d}\mu \mathrm{d}x)) \quad \text{as} \quad \epsilon \to 0 \, . \tag{4.9}$$

Terms (4.5) and (4.7) require more work, which is stated in the following lemma.

Lemma 4.1.

$$\frac{\delta_\epsilon^2}{\sqrt{\epsilon}} \, \sigma \, \frac{g_{\epsilon 1}^2 |q_\epsilon|}{N_{\epsilon 1} N_\epsilon} \to 0 \quad \textit{in} \quad L^1_{loc}(\mathrm{d}t; L^1(\mathrm{d}\mu \mathrm{d}x)) \quad \textit{as} \quad \epsilon \to 0 \, , \tag{4.10}$$

$$\frac{\delta_\epsilon^2}{\sqrt{\epsilon}} \, \sigma' \, \frac{|g'_{\epsilon 1} g'_\epsilon| |q_\epsilon|}{\sqrt{N'_{\epsilon 1} N'_\epsilon N_{\epsilon 1} N_\epsilon}} \to 0 \quad \textit{in} \quad L^1_{loc}(\mathrm{d}t; L^1(\mathrm{d}\mu \mathrm{d}x)) \quad \textit{as} \quad \epsilon \to 0 \, . \tag{4.11}$$

The key to proving Lemma 4.1 is the following compactness result which also played an essential role in [12].

$$\frac{g_\epsilon^2}{\sqrt{N_\epsilon}} \quad \text{is relatively compact in } \; w\text{-}L^1_{loc}(\mathrm{d}t; w\text{-}L^1(aM\mathrm{d}v\mathrm{d}x))\,. \tag{4.12}$$

The method to prove Lemma 4.1 is the same as [6], added an important observation which is inspired from [12]. In [6], the entropy dissipation rate control is employed to estimate (4.5) and (4.7), coupled with the nonlinear estimate

$$\sigma\,\frac{g_\epsilon^2}{\sqrt{N_\epsilon}} = O(|\log(\delta_\epsilon)|) \quad \text{in } \; L^\infty(\mathrm{d}t; L^1(M\mathrm{d}v\mathrm{d}x)) \;\text{ as }\; \epsilon \to 0\,. \tag{4.13}$$

In [12], rather than the estimate (4.13), the weak compactness (4.12) was proved and used to established Navier-Stokes limit. It is observed here that this nonlinear weak compactness (4.12) can be employed to prove the conservation defect theorem in the linear acoustic limit when the scaling is $\delta_\epsilon = \sqrt{\epsilon}$, thus improve the result in [6] to the boaderline case $m = \frac{1}{2}$.

The entropy inequality yields that

$$\frac{1}{\epsilon\delta_\epsilon^2}\int_0^\infty\!\!\int \left\langle\!\!\left\langle \frac{1}{4} r\left(\frac{\sqrt{\epsilon}\delta_\epsilon q_\epsilon}{G_{\epsilon 1}G_\epsilon}\right) G_{\epsilon 1}G_\epsilon \right\rangle\!\!\right\rangle \mathrm{d}x\mathrm{d}t \le C^{\mathrm{in}}\,, \tag{4.14}$$

where the function r is defined over $z > -1$ by

$$r(z) = z\log(1+z)\,. \tag{4.15}$$

The function r is strictly convex over $z > -1$. The proofs of (4.10) and (4.11) are each based on a delicate use of the classical Young inequality satisfied by r and its Legendre dual r^*, namely, the inequality

$$pz \le r^*(p) + r(z) \quad \text{for every } \; p \in \mathbb{R} \;\text{ and }\; z > -1\,. \tag{4.16}$$

Upon choosing

$$p = \frac{\sqrt{\epsilon}\delta_\epsilon y}{\alpha} \quad \text{and} \quad z = \frac{\sqrt{\epsilon}\delta_\epsilon |q_\epsilon|}{G_{\epsilon 1}G_\epsilon}\,, \tag{4.17}$$

and noticing that $r(|z|) \le r(z)$ for every $z > -1$, for every positive α and y one obtains

$$y|q_\epsilon| \le \frac{\alpha}{\epsilon\delta_\epsilon^2}\, r^*\left(\frac{\sqrt{\epsilon}\delta_\epsilon y}{\alpha}\right) G_{\epsilon 1}G_\epsilon + \frac{\alpha}{\epsilon\delta_\epsilon^2}\, r\left(\frac{\sqrt{\epsilon}\delta_\epsilon q_\epsilon}{G_{\epsilon 1}G_\epsilon}\right) G_{\epsilon 1}G_\epsilon\,. \tag{4.18}$$

This inequality is the starting point for the proofs of Lemma 4.1. These proofs also use the facts, recalled from [3], that r^* is superquadratic in the sense

$$r^*(\lambda p) \le \lambda^2 r^*(p)\,, \quad \text{for every } \; p > 0 \;\text{ and }\; \lambda \in [0,1]\,, \tag{4.19}$$

and that r^* has the exponential asymptotics $r^*(p)\;\exp(p)$ as $p \to \infty$.

For the proof of (4.10) we use the inequality (4.18) with $y = \frac{\sigma}{4s^*}\frac{\delta_\epsilon^2}{\sqrt{\epsilon}}\frac{g_{\epsilon 1}^2}{N_{\epsilon 1}N_\epsilon}$, where $s^* \in [0,\infty)$ as in Lemma 10.1 in [12]. We then apply the superquadratic property (4.19) with $\lambda = \frac{\delta_\epsilon^3 g_{\epsilon 1}^2}{\alpha N_{\epsilon 1}N_\epsilon}$ and $p = \frac{\sigma}{4s^*}$, where we note that $\lambda \le 1$ whenever $\delta_\epsilon \le \alpha$. This leads to

$$\frac{\sigma}{4s^*}\frac{\delta_\epsilon^2}{\sqrt{\epsilon}}\frac{g_{\epsilon 1}^2|q_\epsilon|}{N_{\epsilon 1}N_\epsilon} \le \frac{1}{\alpha}\frac{\delta_\epsilon^4}{\epsilon}\frac{g_{\epsilon 1}^4}{N_{\epsilon 1}N_\epsilon} r^*\left(\frac{\sigma}{4s^*}\right) G_{\epsilon 1}G_\epsilon + \frac{\alpha}{\epsilon\delta_\epsilon^2} r\left(\frac{\sqrt{\epsilon}\delta_\epsilon q_\epsilon}{G_{\epsilon 1}G_\epsilon}\right) G_{\epsilon 1}G_\epsilon\,. \tag{4.20}$$

Because $G_{\epsilon 1}G_\epsilon \leq 2\sqrt{N_{\epsilon 1}N_\epsilon}$ while $N_\epsilon \geq 1$, the first term on the right-hand side above is bounded by

$$\frac{2}{\alpha}\frac{\delta_\epsilon^2}{\epsilon}\frac{\delta_\epsilon^2 g_{\epsilon 1}^2}{N_{\epsilon 1}}\frac{g_{\epsilon 1}^2}{\sqrt{N_{\epsilon 1}}}r^*\left(\frac{\sigma}{4s^*}\right). \tag{4.21}$$

Note that $\delta_\epsilon^2 = \epsilon$, so the first factor above is bounded by $2/\alpha$ and tends to zero almost everywhere as $\epsilon \to 0$. The rest of proof is exactly the same as in the proof of Lemma 10.2 in [12].

For the proof of (4.11) we use the inequality (4.18) with $y = \frac{\sigma}{4s^*}\frac{\delta_\epsilon^2}{\sqrt{\epsilon}}\frac{|g'_{\epsilon 1}g'_\epsilon|}{\sqrt{N'_{\epsilon 1}N'_\epsilon N_{\epsilon 1}N_\epsilon}}$. Then apply the superquadratic property (4.19) and (4.18), we have

$$\frac{\sigma'}{4s^*}\frac{\delta_\epsilon^2}{\sqrt{\epsilon}}\frac{|g'_{\epsilon 1}g'_\epsilon||q_\epsilon|}{\sqrt{N'_{\epsilon 1}N'_\epsilon N_{\epsilon 1}N_\epsilon}} \leq \frac{1}{\alpha}\frac{\delta_\epsilon^4}{\epsilon}\frac{g'^2_{\epsilon 1}g'^2_\epsilon}{N'_{\epsilon 1}N'_\epsilon N_{\epsilon 1}N_\epsilon}r^*\left(\frac{\sigma}{4s^*}\right)G_{\epsilon 1}G_\epsilon + \frac{\alpha}{\epsilon\delta_\epsilon^2}r\left(\frac{\sqrt{\epsilon}\delta_\epsilon q_\epsilon}{G_{\epsilon 1}G_\epsilon}\right)G_{\epsilon 1}G_\epsilon . \tag{4.22}$$

the first term on the right-hand side above is bounded by

$$\frac{2}{\alpha}\frac{\delta_\epsilon^2}{\epsilon}\frac{\delta_\epsilon^2 g'^2_{\epsilon 1}}{N'_{\epsilon 1}}\frac{g'^2_{\epsilon 1}}{\sqrt{N'_{\epsilon 1}}}r^*\left(\frac{\sigma}{4s^*}\right). \tag{4.23}$$

Again we use the scaling assumption $\delta_\epsilon = \sqrt{\epsilon}$, the first factor above is bounded by $2/\alpha$ and tends to zero almost everywhere as $\epsilon \to 0$. The rest of proof is exactly the same as in the proof of Lemma 10.3 in [12]. We thereby complete the proof of Proposition 4.1.

□

5. Limiting Boundary Condition

In this section, we derive the weak form of the limiting boundary condition $u \cdot \mathrm{n} = 0$. First we state some the *a priori* estimate on γg_ϵ from the boundary term in the entropy inequality (2.23). The proof is the same as the Lemma 6.1 in [13] and Lemma 6 in [11] for more general collision kernels. So we just state the lemma without giving proof.

Lemma 5.1. *Define* $\gamma_\epsilon = \gamma_+ g_\epsilon - \langle \gamma_+ g_\epsilon \rangle_{\partial\Omega}$ *and*

$$\gamma_\epsilon^{(1)} = \gamma_\epsilon \mathbf{1}_{\gamma_+ G_\epsilon \leq 2\langle G_\epsilon\rangle_{\partial\Omega} \leq 4\gamma_+ G_\epsilon}, \quad \gamma_\epsilon^{(2)} = \gamma_\epsilon - \gamma_\epsilon^{(1)}. \tag{5.1}$$

Then each of these is bounded as follows:

$$\sqrt{\alpha}\frac{\gamma_\epsilon^{(1)}}{[1+\epsilon\gamma_+ g_\epsilon^2]^{1/2}} \quad in \quad L^2_{loc}(\mathrm{d}t; L^2(M|v\cdot \mathrm{n}(x)|\mathrm{d}v\mathrm{d}\sigma_x)), \tag{5.2}$$

$$\sqrt{\alpha}\frac{\gamma_\epsilon^{(1)}}{[1+\epsilon\gamma_+ \hat{g}_\epsilon^2]^{1/2}} \quad in \quad L^2_{loc}(\mathrm{d}t; L^2(M|v\cdot \mathrm{n}(x)|\mathrm{d}v\mathrm{d}\sigma_x)), \tag{5.3}$$

$$\frac{\alpha}{\sqrt{\epsilon}}\gamma_\epsilon^{(2)} \quad in \quad L^1_{loc}(\mathrm{d}t; L^1(M|v\cdot \mathrm{n}(x)|\mathrm{d}v\mathrm{d}\sigma_x)), \tag{5.4}$$

where

$$\begin{aligned} \gamma_+\hat{g}_\epsilon &= (1-\alpha)\gamma_+ g_\epsilon + \alpha\langle \gamma_+ g_\epsilon\rangle_{\partial\Omega}, \\ &= \gamma_+ g_\epsilon - \alpha\gamma_\epsilon . \end{aligned} \tag{5.5}$$

It is proved in [5] (generalized to bounded domain with Maxwell reflection condition in [14]) that DiPerna-Lions solutions to the Boltzmann equation satisfy a weak form of the local conservation law of mass

$$\partial_t \langle g_\epsilon \rangle + \nabla_x \cdot \langle v\, g_\epsilon \rangle = 0\,. \tag{5.6}$$

Lemma 5.2. *Let G_ϵ be a family of renormalized solutions to the Boltzmann equation* (2.13)*, g_ϵ is the associated fluctuations defined as $G_\epsilon = 1 + \sqrt{\epsilon} g_\epsilon$. Let $\mathrm{n}(x)$ be a vector field that belongs to $W^{1,\infty}(\Omega;\mathbb{R}^{\mathrm{D}})$ and coincides with the outward unit normal vector at the boundary $\partial\Omega$. Then*

$$\mathrm{n}\cdot \langle v\gamma g_\epsilon \rangle \quad \textit{is relatively compact in } L^1_{loc}(\mathrm{d}t; W^{-1,1}(\mathrm{d}\sigma_x))\,. \tag{5.7}$$

Proof. The proof relies on the weak formulation of the local conservation law of mass (5.6) and the relative compactness of $\langle g_\epsilon \rangle$ and $\langle v\, g_\epsilon \rangle$. Let $\varphi(x,t)$ be a test function belonging to $W^{1,\infty}(\overline{\Omega}\times\mathbb{R}_+)$ and compactly supported in time t. Then

$$\begin{aligned} &\int_0^T \int_{\partial\Omega} \varphi(t,x)\, \mathrm{n}(x)\cdot \langle v\, \gamma g_\epsilon \rangle \,\mathrm{d}\sigma_x \,\mathrm{d}t \\ &= \int_0^T \int_\Omega \langle g_\epsilon \rangle \partial_t \varphi \,\mathrm{d}x\,\mathrm{d}t + \int_0^T \int_\Omega \langle v\, g_\epsilon \rangle \cdot \nabla_x \varphi \,\mathrm{d}x\,\mathrm{d}t\,. \end{aligned} \tag{5.8}$$

The Weak Acoustic Limit Theorem 3.1 asserts that $\langle g_\epsilon \rangle$ and $\langle v\, g_\epsilon \rangle$ are relatively compact in $C([0,\infty)\,; \text{w-}L^1(\mathrm{d}x))$. It then follows from (5.8) that $\mathrm{n}\cdot \langle v\,\gamma g_\epsilon \rangle$ is relatively compact in $\text{w-}L^1_{loc}(\mathrm{d}t; \text{w-}L^1(\mathrm{d}\sigma_x))$. □

The next Lemma claims that the limit of any convergent subsequence of $\mathrm{n}\cdot \langle v\,\gamma g_\epsilon \rangle$ is zero.

Lemma 5.3. *Let G_ϵ be a family of renormalized solutions to the Boltzmann equation* (2.13) *satisfying the Maxwell boundary condition. Let g_ϵ be the associated fluctuations defined by $G_\epsilon = 1 + \sqrt{\epsilon} g_\epsilon$. Then*

$$\lim_{\epsilon_n\to 0} \int_0^T \int_{\partial\Omega} \mathrm{n}(x)\cdot \langle v\,\gamma g_{\epsilon_n} \rangle\, \varphi(x,t)\,\mathrm{d}\sigma_x\,\mathrm{d}t = 0\,, \tag{5.9}$$

for any sequence $\epsilon_n \to 0$ such that $\mathrm{n}\cdot\langle v\,\gamma g_{\epsilon_n}\rangle$ converges, and for any test function $\varphi \in W^{1,\infty}(\overline{\Omega}\times\mathbb{R}_+)$ that is compactly supported in time.

Proof. We need only to prove (5.9) for $\tilde{g}_\epsilon$, because we can take φ in (5.8) as the test function in the moment equation (3.12) for $\zeta = 1$, then substracted from (5.8):

$$\begin{aligned} &\int_0^T \int_{\partial\Omega} \mathrm{n}(x)\cdot \langle v\,(\gamma g_\epsilon - \gamma\tilde{g}_\epsilon)\rangle\, \varphi(x,t)\,\mathrm{d}\sigma_x \mathrm{d}t \\ &= \int_0^T \int_\Omega \langle g_\epsilon - \tilde{g}_\epsilon \rangle \partial_t\varphi\,\mathrm{d}x\,\mathrm{d}t + \int_0^T \int_\Omega \langle v\,(g_\epsilon - \tilde{g}_\epsilon)\rangle\cdot\nabla_x\varphi\,\mathrm{d}x\,\mathrm{d}t + \int_0^T \int_\Omega \varphi\,\frac{1}{\sqrt{\epsilon}}\left\langle\!\left\langle \Gamma'(G_\epsilon)\, q_\epsilon \right\rangle\!\right\rangle \mathrm{d}x\,\mathrm{d}t\,. \end{aligned} \tag{5.10}$$

It is proved in [12] that $g_\epsilon - \tilde{g}_\epsilon \to 0$ in $L^1_{loc}(\mathrm{d}t; L^1(M\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t))$ thus the first two terms on the right-hand side of (5.10) vanish as $\epsilon \to 0$. The third term on the right-hand side of (5.10) also vanishes $\epsilon\to 0$ by Proposition 4.1.

From (3.14), the renormalized form of the Maxwell boundary condition reads

$$\gamma_-\tilde{g}_\epsilon = (1-\alpha)\frac{L\gamma_+ g_\epsilon}{1+\epsilon^2(L\gamma_+\hat{g}_\epsilon)^2} + \alpha\frac{\langle\gamma_+ g_\epsilon\rangle_{\partial\Omega}}{1+\epsilon^2(L\gamma_+\hat{g}_\epsilon)^2}\,, \tag{5.11}$$

Applying (5.11), we have

$$
\begin{aligned}
&\int_0^T \int_{\partial\Omega} \langle v\gamma g_\epsilon\rangle \cdot \mathrm{n}(x)\varphi(t,x)\,\mathrm{d}\sigma_x \mathrm{d}t \\
&= \alpha \int_0^T \int_{\partial\Omega}\int_{v\cdot\mathrm{n}>0} \frac{\gamma_+ g_\epsilon - \langle\gamma_+ g_\epsilon\rangle_{\partial\Omega}}{1+\epsilon\gamma_+\hat{g}_\epsilon^2}(v\cdot\mathrm{n})M\,\mathrm{d}v\varphi\,\mathrm{d}\sigma_x\,\mathrm{d}t \\
&\quad + \int_0^T \int_{\partial\Omega}\int_{v\cdot\mathrm{n}>0} \gamma_+ g_\epsilon \left(\frac{1}{1+\epsilon\gamma_+ g_\epsilon^2} - \frac{1}{1+\epsilon\gamma_+\hat{g}_\epsilon^2}\right)(v\cdot\mathrm{n})M\,\mathrm{d}v\varphi\,\mathrm{d}\sigma_x\,\mathrm{d}t \\
&= \alpha \int_0^T \int_{\partial\Omega}\int_{v\cdot\mathrm{n}>0} \frac{\gamma_\epsilon(1-\epsilon\gamma_+ g_\epsilon\gamma_+\hat{g}_\epsilon)}{(1+\epsilon\gamma_+ g_\epsilon^2)(1+\epsilon\gamma_+\hat{g}_\epsilon^2)}(v\cdot\mathrm{n})M\,\mathrm{d}v\varphi\,\mathrm{d}\sigma_x\,\mathrm{d}t \\
&= \alpha \int_0^T \int_{\partial\Omega}\int_{v\cdot\mathrm{n}>0} \frac{\gamma_\epsilon^{(2)}(1-\epsilon\gamma_+ g_\epsilon\gamma_+\hat{g}_\epsilon)}{(1+\epsilon\gamma_+ g_\epsilon^2)(1+\epsilon\gamma_+\hat{g}_\epsilon^2)}(v\cdot\mathrm{n})M\,\mathrm{d}v\varphi\,\mathrm{d}\sigma_x\,\mathrm{d}t \\
&\quad + \alpha \int_0^T \int_{\partial\Omega}\int_{v\cdot\mathrm{n}>0} \gamma_\epsilon^{(1)}\frac{\epsilon\gamma_+ g_\epsilon^2 + \epsilon\gamma_+\hat{g}_\epsilon^2 + \epsilon^2\gamma_+ g_\epsilon^2\gamma_+\hat{g}_\epsilon^2}{(1+\epsilon\gamma_+ g_\epsilon^2)(1+\epsilon\gamma_+\hat{g}_\epsilon^2)}(v\cdot\mathrm{n})M\,\mathrm{d}v\varphi\,\mathrm{d}\sigma_x\,\mathrm{d}t
\end{aligned}
\tag{5.12}
$$

Using the bound (5.4) for $\gamma_\epsilon^{(2)}$, it is easy to estimate that the first term on the right-hand side of (5.12) is bounded by $C\sqrt{\epsilon}$. For the $\gamma_\epsilon^{(1)}$ part, from the L^2 bounds (5.2) and (5.3),

$$
\sqrt{\alpha}\frac{\gamma_\epsilon^{(1)}}{\sqrt{1+\epsilon\gamma_+ g_\epsilon^2}} \quad \text{and} \quad \sqrt{\alpha}\frac{\gamma_\epsilon^{(1)}}{\sqrt{1+\epsilon\gamma_+\hat{g}_\epsilon^2}}
\tag{5.13}
$$

are relatively compact in w-$L^1_{loc}(\mathrm{d}t;\text{w-}L^1(|v\cdot\mathrm{n}|M\mathrm{d}v\,\mathrm{d}\sigma_x))$. Use the fact that

$$
\sqrt{\alpha}\frac{\sqrt{\epsilon}\gamma_+ g_\epsilon\sqrt{\epsilon}\gamma_+\hat{g}_\epsilon}{\sqrt{1+\epsilon\gamma_+ g_\epsilon^2}(1+\epsilon\gamma_+\hat{g}_\epsilon^2)}\,, \quad \sqrt{\alpha}\frac{\sqrt{\epsilon}\gamma_+ g_\epsilon\sqrt{\epsilon}\gamma_+\hat{g}_\epsilon}{\sqrt{1+\epsilon\gamma_+\hat{g}_\epsilon^2}(1+\epsilon\gamma_+ g_\epsilon^2)}
\tag{5.14}
$$

are bounded in L^∞ and goes to 0 a.e. Then by the Product Limit Theorem of [3], the product of (5.13) and (5.14) shows the second term on the right-hand side of (5.12) goes to 0 in $L^1_{loc}(\mathrm{d}t;L^1(|v\cdot\mathrm{n}|M\mathrm{d}v\,\mathrm{d}\sigma_x))$ as $\epsilon\to 0$. Thus we finish the proof of the lemma. □

Remark: If γg_ϵ satisfies the Maxwell reflection boundary condition in the usual sense, i.e if

$$
\gamma_- g_\epsilon = (1-\alpha)L(\gamma_+ g_\epsilon) + \alpha\langle\gamma_+ g_\epsilon\rangle_{\partial\Omega} \quad \text{on} \quad \Sigma_-\,,
\tag{5.15}
$$

then it is easy to show then for every $\epsilon > 0$, the normal component of the momentum on the boundary vanishes, i.e.

$$
\mathrm{n}\cdot\langle v\gamma g_\epsilon\rangle = 0 \quad \text{on} \quad \partial\Omega\,.
\tag{5.16}
$$

However, DiPerna-Lions solutions are not known to satisfy (5.16). It seems to us that to prove (5.16) is as difficult as establishing the local conservation law of momentum for DiPerna-Lions solutions. So, Lemma 5.3 is in the same spirit that the local conservation laws of momentum and energy are not known for DiPerna-Lions solutions, but are satisfied in the limit as $\epsilon\to 0$ proved first in [4]. It is an interesting open question that if DiPerna-Lions solution satisfy (5.16) in the usual sense for *every* $\epsilon > 0$.

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