Multistring Solutions of the Self-Graviting Massive $W-$Boson

Dongho Chae
Department of Mathematics
Sungkyunkwan University
Suwon 440-746, Korea
e-mail: chae@skku.edu

Abstract

We consider a semilinear elliptic system which include the model system of the $W-$strings in the cosmology as a special case. We prove existence of multi-string solutions and obtain precise asymptotic decay estimates near infinity for the solutions. As a special case of this result we solve an open problem in [4]

1 Introduction

Let $\lambda_1, \lambda_2, \lambda_2, \lambda_4 > 0$ be given. We consider the following system for (u, η) in $\mathbb{R}^2$.

$$\Delta u = -\lambda_1 e^\eta - \lambda_2 e^u + 4\pi \sum_{j=1}^N \delta(z - z_j), \quad (1.1)$$

$$\Delta \eta = -\lambda_3 e^\eta - \lambda_4 e^u \quad (1.2)$$

equipped with the boundary condition

$$\int_{\mathbb{R}^2} e^u dx + \int_{\mathbb{R}^2} e^\eta dx < \infty, \quad (1.3)$$

where we denoted $z = x_1 + ix_2 \in \mathbb{C} = \mathbb{R}^2$. The system (1.1)-(1.2) reduces to the Bogomol'nyi type of equation modelling the cosmic strings with matter field given by the massive $W-$boson of the electroweak theory if we choose the coefficients as,

$$\lambda_1 = 2m_W^2, \quad \lambda_2 = 4e^2, \lambda_3 = \frac{16\pi G m_W^4}{e^2}, \quad \lambda_4 = 32\pi G m_W^2, \tag{1.4}$$

where m_W is the mass of the $W-$boson, e is the charge of the electron, and G is the gravitational constant([1],[4]). The points $\{z_1, \cdots, z_N\}$ corresponds to the location on the $(x_1, x_2)-$plane of parallel (along the x_3-axis) strings. See [4] for the derivation of this system from the corresponding Einsten-Weinberg-Salam theory as well as interesting physical backgrounds of the model. In [4] the construction of radially symmetric solution(in the case $z_1 = \cdots = z_N$) is discussed by further reduction the system into a single equation, and solving the ordinary differential equation. When the locations of strings are different to each other, however, we cannot assume the radial symmetry of the solutions, and no existence theory is available. In particular, the author of [4] left the construction of solution in this case as an open problem. One of our main purpose in this paper is to solve this problem. Actually, we solve the existence problem for more general coefficient cases as in (1.1)-(1.2). The following is our main theorem.

Theorem 1.1 *Let $N \in \mathbb{N} \cup \{0\}$, and $\mathcal{Z} = \{z_j\}_{j=1}^N$ be given in $\mathbb{R}^2$ allowing multiplicities. Then, there exists a constant $\varepsilon_1 > 0$ such that for any $\varepsilon \in (0, \varepsilon_1)$ and any $c_0 >$ there exists a family of solutions to (1.1)-(1.3), (u, η). Moreover, the solutions we constructed have the following representations:*

$$\begin{aligned} u(z) &= \ln \rho^I_{\varepsilon, a^*_\varepsilon}(z) + \varepsilon^2 w_1(\varepsilon|z|) + \varepsilon^2 v^*_{1,\varepsilon}(\varepsilon z), && (1.5)\\ \eta(z) &= \ln \rho^{II}_{\varepsilon, a^*_\varepsilon}(z) + \varepsilon^2 w_2(\varepsilon|z|) + \varepsilon^2 v^*_{2,\varepsilon}(\varepsilon z), && (1.6)\end{aligned}$$

where the functions $\rho^I_{\varepsilon,a}(z), \rho^{II}_{\varepsilon,a}(z)$ are defined by

$$\rho^I_{\varepsilon,a}(z) = \frac{8\varepsilon^{2N+2}|f(z)|^2}{\lambda_2 \left(1 + \varepsilon^{2N+2}|F(z) + \frac{a}{\varepsilon^{N+1}}|^2\right)^2}, \tag{1.7}$$

and

$$\rho^{II}_{\varepsilon,a}(z) = \frac{c_0 \varepsilon^4}{\left(1 + \varepsilon^{2N+2}|F(z) + \frac{a}{\varepsilon^{N+1}}|^2\right)^{\frac{2\lambda_4}{\lambda_2}}} \tag{1.8}$$

with

$$f(z) = (N+1)\prod_{j=1}^N (z - z_j), \quad F(z) = \int_0^z f(\xi) d\xi \tag{1.9}$$

for $k = 1, 2$, $\varepsilon > 0$ and $a = a_1 + ia_2 \in \mathbb{C}$. The smooth radial functions, w_1, w_2 in (1.5) and (1.6) respectively satisfy the asymptotic formula,

$$w_1(|z|) = -C_1 \ln|z| + O(1), \qquad w_2(|z|) = -C_2 \ln|z| + O(1) \tag{1.10}$$

as $|z|\to\infty$, where

$$C_1 = \frac{c_0\lambda_1\lambda_2\lambda_4}{2(N+1)(\lambda_2+\lambda_4)(\lambda_2+2\lambda_4)}, \tag{1.11}$$

$$C_2 = \begin{cases} \dfrac{C_1\lambda_4}{\lambda_2}\left[= \dfrac{c_0\lambda_1\lambda_4^2}{2(N+1)(\lambda_2+\lambda_4)(\lambda_2+2\lambda_4)}\right] & \text{if } \lambda_1\lambda_4-\lambda_2\lambda_3=0, \\ \dfrac{C_1\lambda_4}{\lambda_2} - \dfrac{(\lambda_1\lambda_4-\lambda_2\lambda_3)c_0}{2(N+1)\lambda_2}B\left(\dfrac{1}{N+1}, \dfrac{2\lambda_4}{\lambda_2}-\dfrac{1}{N+1}\right) & \text{if } N+1>\dfrac{\lambda_2}{2\lambda_4} \end{cases} \tag{1.12}$$

with the beta function(Euler's integral of the first kind) defined by

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt. \quad \forall x,y>0$$

(See [3].) The function $v_{1,\varepsilon}^*, v_{2,\varepsilon}^*$ in (1.5) and (1.6) respectively satisfy

$$\sup_{z\in\mathbb{R}^2} \frac{|v_{1,\varepsilon}^*(\varepsilon z)| + |v_{2,\varepsilon}^*(\varepsilon z)|}{\ln(e+|z|)} \le o(1) \qquad \text{as } \varepsilon\to 0. \tag{1.13}$$

Remark 1.1. In the physical model of the cosmic strings of $W-$boson, (1.4), we note that the first case of (1.12) holds($\lambda_1\lambda_4-\lambda_2\lambda_3=0$), and we have $C_2>0$ as well as $C_1>0$. Thus, we have extra(additional) contributions from the second terms of to the decays of u and η in (1.5) and (1.6) respectively.

2 Proof of Theorem 1.1

We note that for any $\varepsilon>0$ and $a\in\mathbb{C}$, $\ln\rho_{\varepsilon,a}^I(z)$, is a solution of the Liouville equation.

$$\Delta\ln\rho_{\varepsilon,a}^I(z) = -\lambda_2\rho_{\varepsilon,a}^I(z) + 4\pi\sum_{j=1}^N \delta(z-z_{1,j}). \tag{2.1}$$

We consider the following equation for $\rho_{a,\varepsilon}^{II}(z)$

$$\Delta\ln\rho_{a,\varepsilon}^{II}(z) = -\lambda_4\rho_{a,\varepsilon}^I(z). \tag{2.2}$$

From (2.1) we have

$$\Delta\left[\ln\rho_{a,\varepsilon}^I(z) - \sum_{j=1}^N \ln|z-z_j|^2\right] = -\lambda_2\rho_{a,\varepsilon}^I(z). \tag{2.3}$$

Combining (2.2) with (2.3), we obtain

$$\Delta\left\{\lambda_4\left[\ln\rho_{a,\varepsilon}^I(z) - \sum_{j=1}^N \ln|z-z_j|^2\right] - \lambda_2\ln\rho_{a,\varepsilon}^{II}(z)\right\} = 0,$$

from which we derive

$$\ln \rho^{II}_{a,\varepsilon}(z) = \frac{\lambda_4}{\lambda_2}\left[\ln \rho^{I}_{a,\varepsilon}(z) - \sum_{j=1}^{N} \ln |z - z_j|^2\right] + h(z),$$

where $h(z)$ is a harmonic function. Choosing $h(z)$ as the constant,

$$h(z) \equiv \frac{\lambda_4}{\lambda_2} \ln\left(\varepsilon^{\frac{4\lambda_2}{\lambda_4} - 2N - 2} \lambda_2^{\frac{\lambda_2}{\lambda_4}} [8(N+1)^2]^{-1} c_0^{\frac{\lambda_2}{\lambda_4}}\right),$$

we get the form of $\rho^{II}_{a,\varepsilon}(z)$ given in (1.8). We set

$$g^{I}_{\varepsilon,a}(z) = \frac{1}{\varepsilon^2}\rho^{I}_{\varepsilon,a}\left(\frac{z}{\varepsilon}\right), \quad g^{II}_{\varepsilon,a}(z) = \frac{1}{\varepsilon^4}\rho^{II}_{\varepsilon,a}\left(\frac{z}{\varepsilon}\right),$$

and define $\rho_1(r)$ and $\rho_2(r)$ by

$$\rho_1(r) = \frac{8(N+1)^2 r^{2N}}{\lambda_2(1 + r^{2N+2})^2} = \lim_{\varepsilon \to 0} g^{I}_{\varepsilon,0}(z),$$

and

$$\rho_2(r) = \frac{c_0}{(1 + r^{2N+2})^{\frac{2\lambda_4}{\lambda_2}}} = \lim_{\varepsilon \to 0} g^{II}_{\varepsilon,0}(z)$$

respectively. We transform $(u, \eta) \mapsto (v_1, v_2)$ by the formula

$$u(z) = \ln \rho^{I}_{\varepsilon,a}(z) + \varepsilon^2 w_1(\varepsilon|z|) + \varepsilon^2 v_1(\varepsilon z), \tag{2.4}$$

$$\eta(z) = \ln \rho^{II}_{\varepsilon,b}(z) + \varepsilon^2 w_2(\varepsilon|z|) + \varepsilon^2 v_2(\varepsilon z), \tag{2.5}$$

where w_1 and w_2 are the radial functions to be determined below. Then, using (2.1), the system can be written as the functional equation, $P(v_1, v_2, a, \varepsilon) = (0, 0)$, where

$$P_1(v_1, v_2, a, \varepsilon) = \Delta v_1 + \lambda_1 g^{II}_{a,\varepsilon}(z) e^{\varepsilon^2(w_2+v_2)} + \lambda_2 \frac{g^{I}_{\varepsilon,a}(z)}{\varepsilon^2}(e^{\varepsilon^2(w_1+v_1)} - 1) + \Delta w_1, \tag{2.6}$$

and

$$P_2(v_1, v_2, a, \varepsilon) = \Delta v_2 + \lambda_3 g^{II}_{\varepsilon,a}(z) e^{\varepsilon^2(w_2+v_2)} + \lambda_4 \frac{g^{I}_{\varepsilon,a}(z)}{\varepsilon^2}(e^{\varepsilon^2(w_1+v_1)} - 1) + \Delta w_2. \tag{2.7}$$

Now we introduce the functions spaces introduced in [2]. For $\alpha > 0$ the Banach spaces X_α and Y_α are defined as

$$X_\alpha = \{u \in L^2_{loc}(\mathbb{R}^2) \mid \int_{\mathbb{R}^2} (1 + |x|^{2+\alpha})|u(x)|^2 dx < \infty\}$$

equipped with the norm $\|u\|^2_{X_\alpha} = \int_{\mathbb{R}^2}(1+|x|^{2+\alpha})|u(x)|^2dx$, and

$$Y_\alpha = \{u \in W^{2,2}_{loc}(\mathbb{R}^2) \mid \|\Delta u\|^2_{X_\alpha} + \left\| \frac{u(x)}{1+|x|^{1+\frac{\alpha}{2}}} \right\|^2_{L^2(\mathbb{R}^2)} < \infty\}$$

equipped with the norm $\|u\|^2_{Y_\alpha} = \|\Delta u\|^2_{X_\alpha} + \left\| \frac{u(x)}{1+|x|^{1+\frac{\alpha}{2}}} \right\|^2_{L^2(\mathbb{R}^2)}$. We recall the following propositions proved in [2].

Proposition 2.1 *Let Y_α be the function space introduced above. Then we have the followings.*

(i) If $v \in Y_\alpha$ is a harmonic function, then $v \equiv$ constant.

(ii) There exists a constant $C > 0$ such that for all $v \in Y_\alpha$

$$|v(x)| \le C\|v\|_{Y_\alpha} \ln(e + |x|), \qquad \forall x \in \mathbb{R}^2.$$

Proposition 2.2 *Let $\alpha \in (0, \frac{1}{2})$, and let us set*

$$L = \Delta + \rho : Y_\alpha \to X_\alpha. \tag{2.8}$$

where

$$\rho(z) = \rho(|z|) = \frac{8(N+1)^2|z|^{2N}}{(1+|z|^{2N+2})^2}.$$

We have

$$KerL = Span\{\varphi_+, \varphi_-, \varphi_0\}, \tag{2.9}$$

where we denoted

$$\varphi_+(r,\theta) = \frac{r^{N+1}\cos(N+1)\theta}{1+r^{2N+2}}, \quad \varphi_-(r,\theta) = \frac{r^{N+1}\sin(N+1)\theta}{1+r^{2N+2}}, \tag{2.10}$$

and

$$\varphi_0 = \frac{1-r^{2N+2}}{1+r^{2N+2}}. \tag{2.11}$$

Moreover, we have

$$ImL = \{f \in X_\alpha | \int_{\mathbb{R}^2} f\varphi_\pm = 0\}. \tag{2.12}$$

Hereafter, we fix $\alpha = \frac{1}{4}$, and set $X_{\frac{1}{4}} = X$ and $Y_{\frac{1}{4}} = Y$.
Using Proposition 2.1 (ii), one can check easily that for $\varepsilon > 0$ P is a well defined continuous mapping from B_{ε_0} into X^2, where we set $B_{\varepsilon_0} = \{\|v_1\|^2_Y + \|v_2\|^2_Y + |a|^2 < \varepsilon_0\}$, for sufficiently small ε_0. In order to extend continuously P to $\varepsilon = 0$ the radial functions $w_1(r), w_2(r)$ should satisfy

$$\Delta w_1 + \lambda_2\rho_1 w_1 + \lambda_1\rho_2 = 0 \tag{2.13}$$

$$\Delta w_2 + \lambda_4\rho_1 w_1 + \lambda_3\rho_2 = 0 \tag{2.14}$$

For the existence and asymptotic properties of w_1 and w_2 we have the following lemma, which is a part of Theorem 1.1.

Lemma 2.1 *There exist radial solutions $w_1(|z|), w_2(|z|)$ of (2.13)-(2.14) belonging to Y, which satisfy the asymptotic formula in (1.10),(1.11),(1.12).*

Proof: Let us set $f(r) = \rho_1(r)$. Then, it is found in [2] that the ordinary differential equation(with respect to r), $\Delta w_1 + C_1\rho_1 w_1 = f(r)$ has a solution $w_1(r) \in Y$ given by

$$w_1(r) = \varphi_0(r)\left\{\int_0^r \frac{\phi_f(s)-\phi_f(1)}{(1-s)^2}ds + \frac{\phi_f(1)r}{1-r}\right\} \tag{2.15}$$

with

$$\phi_f(r) := \left(\frac{1+r^{2N+2}}{1-r^{2N+2}}\right)^2 \frac{(1-r)^2}{r}\int_0^r \varphi_0(t)tf(t)dt,$$

where $\phi_f(1)$ and $w_1(1)$ are defined as limits of $\phi_f(r)$ and $w_1(r)$ as $r \to 1$. From the formula (2.15) we find that

$$w_1(r) = \varphi_0(r)\int_2^r \left(\frac{1+s^{2N+2}}{1-s^{2N+2}}\right)^2 \frac{I(s)}{s}ds + (\text{bounded function of } r)$$

as $r \to \infty$, where

$$I(s) = \lambda_1 \int_0^s \varphi_0(t)t\rho_2(t)dt.$$

Since $\varphi_0(r) \to -1$ as $r \to \infty$, the first part of (1.10) follows if we show

$$I = I(\infty) = \lambda_1\int_0^\infty \varphi_0(r)r\rho_2(r)dr = C_1.$$

Changing variable $r^{2N+2} = t$, we evaluate

$$\begin{aligned}
I &= \lambda_1\int_0^\infty \varphi_0(r)\rho_2(r)rdr \\
&= c_0\lambda_1\int_0^\infty \left[\frac{r^{2N}}{(1+r^{2N_2+2})^{3+\frac{2\lambda_4}{\lambda_2}}} - \frac{r^{4N+2}}{(1+r^{2N_2+2})^{3+\frac{2\lambda_4}{\lambda_2}}}\right]rdr \\
&= \frac{c_0\lambda_1}{2(N+1)}\left[\int_0^\infty \frac{1}{(1+t)^{3+\frac{2\lambda_4}{\lambda_2}}}dt - \int_0^\infty \frac{t}{(1+t)^{3+\frac{2\lambda_4}{\lambda_2}}}dt\right] \\
&= \frac{c_0\lambda_1}{2(N+1)}\left[\frac{1}{2+\frac{2\lambda_4}{\lambda_2}} - \frac{1}{\left(2+\frac{2\lambda_4}{\lambda_2}\right)\left(1+\frac{2\lambda_4}{\lambda_2}\right)}\right] \\
&= \frac{c_0\lambda_1\lambda_2\lambda_4}{2(N+1)(\lambda_2+\lambda_4)(\lambda_2+2\lambda_4)} = C_1.
\end{aligned} \tag{2.16}$$

In order to obtain C_2 we find from (2.13) and (2.14) that

$$\Delta(\lambda_4 w_1 - \lambda_2 w_2) = (-\lambda_1\lambda_4 + \lambda_2\lambda_3)\rho_2,$$

from which we have

$$
\begin{aligned}
w_2(z) &= \frac{\lambda_4}{\lambda_2} w_1(z) + \frac{\lambda_1\lambda_4 - \lambda_2\lambda_3}{2\pi\lambda_2} \int_{\mathbb{R}^2} \ln(|z-y|)\rho_2(|y|)dy \\
&= -\frac{\lambda_4 C_1}{\lambda_2} \ln|z| + \frac{\lambda_1\lambda_4 - \lambda_2\lambda_3}{2\pi\lambda_2} \left[\int_{\mathbb{R}^2} \rho_2(|y|)dy\right] \ln|z| + O(1)
\end{aligned}
\tag{2.17}
$$

as $|z| \to \infty$. In the case $\lambda_1\lambda_4 - \lambda_2\lambda_3 = 0$, we have $C_2 = \frac{\lambda_4 C_1}{\lambda_2}$. In the case $\frac{2\lambda_4}{\lambda_2} > \frac{1}{N+1}$, we compute the integral as follows.

$$
\begin{aligned}
\int_{\mathbb{R}^2} \rho_2(|y|)dy &= 2\pi c_0 \int_0^\infty \frac{r}{(1+r^{2N+2})^{\frac{2\lambda_4}{\lambda_2}}} dr \\
&= \frac{\pi c_0}{N+1} \int_0^\infty \frac{t^{-\frac{N}{N+1}}}{(1+t)^{\frac{2\lambda_4}{\lambda_2}}} dt \quad (r^{2N+2} = t) \\
&= \frac{\pi c_0}{N+1} B\left(\frac{1}{N+1}, \frac{2\lambda_4}{\lambda_2} - \frac{1}{N+1}\right),
\end{aligned}
\tag{2.18}
$$

where we used the formula(See pp. 322[3]) for the beta function

$$
\int_0^\infty \frac{x^{\mu-1}}{(1+x)^\nu} dx = B(\mu, \nu-\mu), \qquad \text{where } \nu > \mu.
$$

Substituting (2.18) into (2.17), we have $w_2(z) = -C_2 \ln|z| + O(1)$ as $|z| \to \infty$, where C_2 is given by (1.12). This completes the proof of Lemma 2.1 $\Box$

Now we compute the linearized operator of P.

By direct computation we have

$$
\lim_{\varepsilon\to 0} \left.\frac{\partial g^I_{a,\varepsilon}(z)}{\partial a_1}\right|_{a=0} = -4\rho_1\varphi_+, \quad \lim_{\varepsilon\to 0} \left.\frac{\partial g^I_{a,\varepsilon}(z)}{\partial a_2}\right|_{a=0} = -4\rho_1\varphi_-,
$$

$$
\lim_{\varepsilon\to 0} \left.\frac{\partial g^{II}_{a,\varepsilon}(z)}{\partial a_1}\right|_{a=0} = -4\rho_2\varphi_+, \quad \lim_{\varepsilon\to 0} \left.\frac{\partial g^{II}_{a,\varepsilon}(z)}{\partial a_2}\right|_{a=0} = -4\rho_2\varphi_-.
$$

Let us set $P'_{u,\eta,a}(0,0,0,0) = \mathcal{A}$. Then, using the above preliminary computations, we obtain

$$
\mathcal{A}_1[\nu_1,\nu_2,\alpha] = \Delta\nu_1 + \lambda_2\rho_1\nu_1 - 4(\lambda_2 w_1\rho_1 + \lambda_1\rho_2)(\varphi_+\alpha_1 + \varphi_-\alpha_2),
$$

and

$$
\mathcal{A}_2[\nu_1,\nu_2,\alpha] = \Delta\nu_2 + \lambda_4\rho_1\nu_1 - 4(\lambda_4 w_1\rho_1 + \lambda_3\rho_2)(\varphi_+\alpha_1 + \varphi_-\alpha_2).
$$

We establish the following lemma for the operator $\mathcal{A}$.

Lemma 2.2 *The operator $\mathcal{A} : Y^2 \times \mathbb{C} \times \mathbb{R}_+$ defined above is onto. Moreover, kernel of $\mathcal{A}$ is given by*

$$Ker\mathcal{A} = Span\{(0,1); (\varphi_\pm, \frac{\lambda_4}{\lambda_2}\varphi_\pm), (\varphi_0, \frac{\lambda_4}{\lambda_2}\varphi_0)\} \times \{(0,0)\}.$$

Thus, if we decompose $Y^2 \times \mathbb{C} = U \oplus Ker\mathcal{A}$, where we set $U = (Ker\mathcal{A})^\perp$, then $\mathcal{A}$ is an isomorphism from U onto X^2.

In order to prove the above lemma we need to establish the following.

Proposition 2.3

$$I_\pm := \int_{\mathbb{R}^2} (\lambda_2 w_1 \rho_1 + \lambda_1 \rho_2)\varphi_\pm dx \neq 0. \tag{2.19}$$

Proof: In order to transform the integrals we use the formula

$$L\left[\frac{1}{16(1+r^{2N+2})^2}\right] = \frac{(N+1)^2 r^{4N+2}}{(1+r^{2N+2})^4}, \qquad \forall N \in \mathbb{Z}_+$$

which can be verified by an elementary computation. Using this, we have the following

$$\begin{aligned}
\int_{\mathbb{R}^2} (\lambda_2 w_1\rho_1 + \lambda_1\rho_2)\varphi_\pm^2 dx &= \int_0^{2\pi}\int_0^\infty (\lambda_2 w_1\rho_1 + \lambda_1\rho_2)\frac{r^{2N+2}}{(1+r^{2N+2})^2}\left\{\begin{array}{c}\cos^2(N+1)\theta \\ \sin^2(N+1)\theta\end{array}\right\} r dr d\theta \\
&= \pi\int_0^\infty \left[\frac{8(N+1)^2 r^{2N}}{(1+r^{2N+2})^2} w_1 + \lambda_1\rho_2\right]\frac{r^{2N+2}}{(1+r^{2N+2})^2} r dr \\
&= \pi\int_0^\infty \left[\frac{1}{2}L\left\{\frac{1}{(1+r^{2N+2})^2}\right\} w_1 + \frac{\lambda_1\rho_2 r^{2N+2}}{(1+r^{2N+2})^2}\right] r dr \\
&= \pi\int_0^\infty \left[\frac{1}{2}Lw_1\cdot\frac{1}{(1+r^{2N+2})^2} + \frac{\lambda_1\rho_2 r^{2N+2}}{(1+r^{2N+2})^2}\right] r dr \\
&= \pi\lambda_1 c_0\int_0^\infty \left[-\frac{\rho_2}{2(1+r^{2N+2})^2} + \frac{\rho_2 r^{2N+2}}{(1+r^{2N+2})^2}\right] r dr \\
&= \frac{\pi\lambda_1 c_0}{2}\int_0^\infty \frac{r^{2N+2}-1}{(1+r^{2N+2})^{2+\frac{2\lambda_4}{\lambda_2}}} r dr = \frac{\pi\lambda_1 c_0}{4}\int_0^\infty \frac{t^{N+1}-1}{(1+t^{N+1})^{2+\frac{2\lambda_4}{\lambda_2}}} dt \quad (r^2 = t) \\
&= \frac{\pi\lambda_1 c_0}{4}\left[\int_0^1 \frac{t^{N+1}-1}{(1+t^{N+1})^{2+\frac{2\lambda_4}{\lambda_2}}} dt + \int_1^\infty \frac{t^{N+1}-1}{(1+t^{N+1})^{2+\frac{2\lambda_4}{\lambda_2}}} dt\right]
\end{aligned}$$

(Changing variable $t \to 1/t$ in the second integral,)

$$\begin{aligned}
&= \frac{\pi\lambda_1 c_0}{4}\left[\int_0^1 \frac{t^{N+1}-1}{(1+t^{N+1})^{2+\frac{2\lambda_4}{\lambda_2}}} dt + \int_0^1 \frac{(1-t^{N+1})t^{\frac{2\lambda_4}{\lambda_2}}}{(1+t^{N+1})^{2+\frac{2\lambda_4}{\lambda_2}}} dt\right] \\
&= \frac{\pi\lambda_1 c_0}{4}\int_0^\infty \frac{(t^{N+1}-1)(1-t^{\frac{2\lambda_4}{\lambda_2}})}{(1+t^{N+1})^{2+\frac{2\lambda_4}{\lambda_2}}} dt < 0.
\end{aligned}$$

This completes the proof of the proposition.□

We are now ready to prove Lemma 2.2.

Proof of Lemma 2.2: Given $(f_1, f_2) \in X^2$, we want first to show that there exists $(\nu_1, \nu_2) \in Y^2$, $\alpha_1, \alpha_2 \in \mathbb{R}$ such that

$$\mathcal{A}(\nu_1, \nu_2, \alpha_1, \alpha_2) = (f_1, f_2),$$

which can be rewritten as

$$\Delta\nu_1 + \lambda_2\rho_1\nu_1 - 4(\lambda_2 w_1\rho_1 + \lambda_1\rho_2)(\varphi_+\alpha_1 + \varphi_-\alpha_2) = f_1, \tag{2.20}$$

and

$$\Delta\nu_2 + \lambda_4\rho_1\nu_1 - 4(\lambda_4 w_1\rho_1 + \lambda_3\rho_2)(\varphi_+\alpha_1 + \varphi_-\alpha_2) = f_2. \tag{2.21}$$

Let us set

$$\alpha_1 = \frac{1}{4I_+}\int_{\mathbb{R}^2} f_1\varphi_+ dx, \qquad \alpha_2 = \frac{1}{4I_-}\int_{\mathbb{R}^2} f_2\varphi_- dx, \tag{2.22}$$

where $I_\pm \neq 0$ is defined in (2.19). We introduce $\tilde{f}$ by

$$\tilde{f}_1 = f_1 - \alpha_1\varphi_+ - \alpha_2\varphi_-. \tag{2.23}$$

Using the fact

$$\int_0^{2\pi} \varphi_+\varphi_- d\theta = 0, \tag{2.24}$$

we find easily

$$\int_{\mathbb{R}^2} \tilde{f}_1\varphi_\pm dx = 0. \tag{2.25}$$

Hence, by (2.12) there exists $\nu_1 \in Y$ such that $\Delta\nu_1 + \lambda_2\rho_1\nu_1 = \tilde{f}_1$. Thus we have found $(\nu_1, \alpha_1, \alpha_2) \in Y \times \mathbb{R}^2$ satisfying (2.20). Given such $(\nu_1, \alpha_1, \alpha_2)$, the function

$$\nu_2(z) = \frac{1}{2\pi}\int_{\mathbb{R}^2} \ln(|z-y|)g(y)dy + c_1, \tag{2.26}$$

where

$$g = f_2 - \lambda_4\rho_1\nu_1 + 4(\lambda_4 w_1\rho_1 + \lambda_3\rho_2)(\varphi_+\alpha_1 + \varphi_-\alpha_2),$$

and c_1 is any constant, satisfies (2.21), and belongs to Y. We have just finished the proof that $\mathcal{A} : Y^2 \times \mathbb{R}^2 \to X^2$ is onto.
We now show that the restricted operator(denoted by the same symbol),

$\mathcal{A} : (KerL \oplus Span\{1\})^{\perp} \times \mathbb{R}^2 \to X^2$ is one to one. Given $(\nu_1, \nu_2, \alpha_1, \alpha_2) \in (KerL \oplus Span\{1\})^{\perp} \times \mathbb{R}^2$, let us consider the equation, $\mathcal{A}(\nu_1, \nu_2, \alpha_1, \alpha_2) = (0, 0)$, which corresponds to

$$\Delta \nu_1 + \lambda_2 \rho_1 \nu_1 - 4(\lambda_2 w_1 \rho_1 + \lambda_1 \rho_2)(\varphi_+ \alpha_1 + \varphi_- \alpha_2) = 0, \tag{2.27}$$

and

$$\Delta \nu_2 + \lambda_4 \rho_1 \nu_1 - 4(\lambda_4 w_1 \rho_1 + \lambda_3 \rho_2)(\varphi_+ \alpha_1 + \varphi_- \alpha_2) = 0. \tag{2.28}$$

Taking $L^2(\mathbb{R}^2)$ inner product of (2.27) with $\varphi_{\pm}$, and using (2.19), we find $\alpha_1 = \alpha_2 = 0$. Thus, (2.27) implies $\nu_1 \in KerL$. This, combined with the hypothesis $\nu_1 \in (KerL)^{\perp}$ leads to $\nu_1 = 0$. Now, (2.28) is reduced to $\Delta \nu_2 = 0$. Since $\nu_2 \in Y$, Proposition 2.1 implies $\nu_2 =$ constant. Since $\nu_2 \in (Span\{1\})^{\perp}$ by hypothesis, we have $\nu_2 = 0$. This completes the proof of the lemma. $\square$

We are now ready to prove our main theorem.
Proof of Theorem 1.1: Let us set

$$U = (\mathrm{Ker} L \oplus \mathrm{Span}\{1\})^{\perp} \times R^2.$$

Then, Lemma 2.2 shows that $P'_{(v_1,v_2,\alpha)}(0,0,0,0) : U \to X^2$ is an isomorphism. Then, the standard implicit function theorem(See e.g. [5]), applied to the functional $P : U \times (-\varepsilon_0, \varepsilon_0) \to X^2$, implies that there exists a constant $\varepsilon_1 \in (0, \varepsilon_0)$ and a continuous function $\varepsilon \mapsto \psi^*_\varepsilon := (v^*_{1,\varepsilon}, v^*_{2,\varepsilon}, a^*_\varepsilon)$ from $(0, \varepsilon_1)$ into a neighborhood of 0 in U such that

$$P(v^*_{1,\varepsilon}, v^*_{2,\varepsilon}, a^*_\varepsilon) = (0,0), \quad \text{for all } \varepsilon \in (0, \varepsilon_1).$$

This completes the proof of Theorem 1.1. The representation of solutions u_1, u_2, and the explicit form of $\rho^I_{\varepsilon, a^*_\varepsilon}(z)$, $\rho^{II}_{\varepsilon, a^*_\varepsilon}(z)$, , together with the asymptotic behaviors of w_1, w_2 described in Lemma 2.1, and the fact that $v^*_{1,\varepsilon}, v^*_{2,\varepsilon} \in Y$, combined with Proposition 2.1, implies that the solutions satisfy the boundary condition in (1.3). Now, from Proposition 2.1 we obtain that for each $j = 1, 2$,

$$|v^*_{j,\varepsilon}(x)| \le C \|v^*_{j,\varepsilon}\|_Y (\ln^+ |x| + 1) \le C \|\psi_\varepsilon\|_U (\ln^+ |x| + 1). \tag{2.29}$$

This implies then

$$|v^*_{j,\varepsilon}(\varepsilon x)| \le C \|\psi_\varepsilon\|_U (\ln^+ |\varepsilon x| + 1) \le C \|\psi_\varepsilon\|_U (\ln^+ |x| + 1).cxxc$$

From the continuity of the function $\varepsilon \mapsto \psi_\varepsilon$ from $(0, \varepsilon_0)$ into U and the fact $\psi^*_0 = 0$ we have

$$\|\psi_\varepsilon\|_U \to 0 \qquad \text{as } \varepsilon \to 0. \tag{2.30}$$

The proof of (1.13) follows from (2.29) combined with (2.30). This completes the proof of Theorem 1.1$\square$

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