On the spectral dynamics of the deformation tensor and new a priori estimates for the 3D Euler equations

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Abstract

In this paper we study the dynamics of eigenvalues of the deformation tensor for solutions of the 3D incompressible Euler equations. Using the evolution equation of the L^2 norm of spectra, we deduce new a priori estimates of the L^2 norm of vorticity. As an immediate corollary of the estimate we obtain a new sufficient condition of L^2 norm control of vorticity. We also obtain decay in time estimates of the ratios of the eigenvalues. In the remarks we discuss what these estimates suggest in the study of searching initial data leading to a possible finite time singularities. We find that the dynamical behaviors of L^2 norm of vorticity are controlled completely by the second largest eigenvalue of the deformation tensor.

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1 Introduction

We are concerned with the following Euler equations for the homogeneous incompressible fluid flows in $\Omega \subset \mathbb{R}^3$.

$$\frac{\partial v}{\partial t} + (v \cdot \nabla) v \ = -\nabla p, \tag{1.1}$$

$$\mathrm{div}\ \ v = 0, \tag{1.2}$$

$$v(x,0) = v_0(x), \tag{1.3}$$

where $v = (v_1, v_2, v_3)$, $v_j = v_j(x,t)$, $j = 1,2,3$, is the velocity of the flow, $p = p(x,t)$ is the scalar pressure, and v_0 is the given initial velocity, satisfying div $v_0 = 0$. For simplicity of presentation we assume $\Omega = \mathbb{T}^3$, the 3D periodic box. Most of the results in this paper, however, are valid also in the whole of $\mathbb{R}^3$, or bounded domain with smooth boundary, at least after obvious modifications. Given $m \in \mathbb{N} \cup \{0\}$, let $H^m(\mathbb{T}^3)$ be the standard Sobolev space on $\mathbb{T}^3$,

$$H^m(\mathbb{T}^3) = \{f \in L^2(\mathbb{T}^3) \,|\, \|f\|^2_{H^m} = \sum_{|\alpha| \leq m} \int_{\mathbb{T}^3} |D^\alpha f(x)|^2 dx < \infty\},$$

where $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ with $|\alpha| = \alpha_1 + \alpha_2 + \alpha_3$ is the usual multi-index notation. We introduce the space of solenoidal vector fields,

$$\mathbb{H}^m_\sigma = \{u \in [H^m(\mathbb{T}^3)]^3 \,|\, \mathrm{div}\, u = 0\}.$$

Then, for $v_0 \in \mathbb{H}^m_\sigma$ with $m > 5/2$, the local in time unique existence of solution to (1.1)-(1.3), which belongs to $C([0,T]; \mathbb{H}^m_\sigma)$ for some $T = T(\|v_0\|_{H^m})$, was established in [17, 22]. This was later extended to the local existence in various other function spaces by many authors([18, 8, 9, 24, 25, 3, 4, 5, 6]). The question of finite time blow-up/global regularity of such locally constructed solution is one of the most outstanding open problems in the mathematical fluid mechanics. For physical meaning and other significance of this problem as well as many instructive examples of solutions we refer [10, 21]. For a mathematical or numerical test of the actual finite time blow-up of a given solution, it is important to have a good blow-up criterion. In this direction there is a celebrated result by Beale-Kato-Majda([2]), now called the BKM criterion. This criterion is later refined in [19, 6, 9], using refined versions of logarithmic Sobolev inequalities. As for another approach to the blow-up criterion, there is a pioneering work on the geometric type of blow-up criterion due to Constantin-Fefferman-Majda([13])(see also [10]), which was initiated in [12], and the idea of which was used in the recent works in [7, 15]. For different type of geometric approach to the blow-up problem(considering vortex tube initial data) we refer [14]. We also mention a recent interesting result in [1] for rotating flows, where they discovered that rotation has some sense of regularization effect. In this paper we study the regularity/blow-up problem, using the spectral dynamics of the deformation tensor for the solution of the Euler equations. Previous spectral approaches to the singularity problems in the nonlinear partial differential equations are studied in [20], however their study is not for the real Euler equations, but for its model equations, avoiding the difficulty of the nonlocality caused by the Riesz transform appearing in the equations when the pressure is eliminated. Moreover, their spectrum is for the matrices of the velocity gradient, not for the deformation tensor, which is the symmetric part of the velocity gradient. In the next section we derive an evolution equation, in the L^2 sense, of the eigenvalues of the deformation tensor. From this equation we derive new a priori estimates for the L^2 norm of vorticity. The inequality itself already tells us interesting dynamical mechanism of compression and stretching of infinitesimal fluid volume elements leading to possible blow-up. The inequality immediately leads to very simple and elegant sufficient condition of L^2 norm control of vorticity of the 3D Euler equations. In the section 3 we consider special classes of initial data. For such initial data we can have better estimates exponential growth/decay of the L^2 norm of vorticity. We also deduce decay estimates in time of a ratio of eigenvalues of the deformation tensor.

2 Dynamics of eigenvalues of the deformation tensor

We use the following notations for matrix components.

$$V_{ij} = \frac{\partial v_j}{\partial x_i}, \quad S_{ij} = \frac{V_{ij} + V_{ji}}{2}, \quad A_{ij} = \frac{V_{ij} - V_{ji}}{2},$$

where $i, j = 1, 2, 3$. Then, obviously we have $V_{ij} = S_{ij} + A_{ij}$. For a given 3D velocity field $v(x,t)$, describing fluid motions, S_{ij} is called the deformation tensor, while A_{ij} is related to the vorticity field $\omega = \text{curl } v$ by

$$A_{ij} = \frac{1}{2} \sum_{k=1}^{3} \varepsilon_{ijk} \omega_k,$$

where ε_{ijk}, the skewsymmetric tensor with the normalization $\varepsilon_{123} = 1$. For incompressible fluid we have $Tr(S) = \sum_{i=1}^{3} S_{ii} = \operatorname{div} v = 0$. We now state the theorem on the evolutions of the eigenvalues of the deformation tensor associated with the solution of the Euler system (1.1)-(1.3).

Theorem 2.1 *Let $\lambda_1(x,t), \lambda_2(x,t), \lambda_3(x,t)$ be the eigenvalues of the deformation tensor $S = (S_{ij})_{i,j=1}^3$ associated to the classical solution $v(x,t)$ of*

(1.1)-(1.3). Then, the following equation holds.

$$\frac{d}{dt}\int_{\mathbb{T}^3}(\lambda_1^2+\lambda_2^2+\lambda_3^2)dx = -4\int_{\mathbb{T}^3}\lambda_1\lambda_2\lambda_3 dx. \tag{2.1}$$

Proof. We take L^2 inner product (1.1) with Δu, and integrate by part to derive

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\nabla v\|_{L^2}^2 &= \int_{\mathbb{T}^3}(v\cdot\nabla)v\cdot\Delta v dx = -\sum_{i,j,k=1}^{3}\int_{\mathbb{T}^3}\frac{\partial v_j}{\partial x_k}\frac{\partial v_k}{\partial x_i}\frac{\partial v_j}{\partial x_i}dx\\
&= -\sum_{i,j,k=1}^{3}\int_{\mathbb{T}^3}S_{kj}V_{ik}V_{ij}dx = -\sum_{i,j,k=1}^{3}\int_{\mathbb{T}^3}S_{kj}(S_{ik}+A_{ik})(S_{ij}+A_{ij})dx\\
&= -\sum_{i,j,k=1}^{3}\int_{\mathbb{T}^3}(S_{kj}A_{ik}A_{ij}+S_{kj}S_{ik}S_{ij})dx\\
&= -\frac{1}{4}\sum_{i,j,k=1}^{3}\int_{\mathbb{T}^3}S_{kj}\left[\sum_{m=1}^{3}\delta_{kj}\omega_m\omega_m-\omega_j\omega_k\right]dx-\sum_{i,j,k=1}^{3}\int_{\mathbb{T}^3}S_{kj}S_{ik}S_{ij}dx\\
&= \frac{1}{4}\sum_{j,k=1}^{3}\int_{\mathbb{T}^3}S_{jk}\omega_j\omega_k dx-\sum_{i,j,k=1}^{3}\int_{\mathbb{T}^3}S_{kj}S_{ik}S_{ij}dx.
\end{aligned}\tag{2.2}$$

Next, we consider the vorticity equation for the 3D Euler equations,

$$\frac{\partial\omega}{\partial t}+(v\cdot\nabla)\omega=(\omega\cdot\nabla)v, \tag{2.3}$$

which is obtained from (1.1) by taking curl$(\cdot)$ operation. Taking L^2 inner product (2.3) with ω, we obtain, after integration by part,

$$\frac{1}{2}\frac{d}{dt}\|\omega\|_{L^2}^2=\int_{\mathbb{T}^3}(\omega\cdot\nabla)v\cdot\omega dx=\sum_{j,k=1}^{3}\int_{\mathbb{T}^3}S_{jk}\omega_j\omega_k dx. \tag{2.4}$$

Since we have the equality,

$$\int_{\mathbb{T}^3}|\nabla v|^2dx=\int_{\mathbb{T}^3}|\omega|^2dx, \tag{2.5}$$

from (2.2) and (2.4) we obtain

$$\frac{1}{4}\sum_{j,k=1}^{3}\int_{\mathbb{T}^3}S_{jk}\omega_j\omega_k dx-\sum_{i,j,k=1}^{3}\int_{\mathbb{T}^3}S_{kj}S_{ik}S_{ij}dx=\sum_{j,k=1}^{3}\int_{\mathbb{T}^3}S_{jk}\omega_j\omega_k dx,$$

Hence,

$$\sum_{j,k=1}^{3}\int_{\mathbb{T}^3}S_{jk}\omega_j\omega_k dx=-\frac{4}{3}\sum_{i,j,k=1}^{3}\int_{\mathbb{T}^3}S_{kj}S_{ik}S_{ij}dx. \tag{2.6}$$

We also have the following pointwise equality,

$$\begin{aligned}|\nabla v|^2 &= \sum_{j,k=1}^{3} V_{jk}V_{jk} = \sum_{j,k=1}^{3}(S_{jk}+A_{jk})(S_{jk}+A_{jk}) \\ &= \sum_{j,k=1}^{3}\left(S_{jk}S_{jk}+\frac{1}{4}\sum_{m,n}^{3}\varepsilon_{jkm}\varepsilon_{jkn}\omega_m\omega_n\right) \\ &= \sum_{j,k=1}^{3} S_{jk}S_{jk}+\frac{1}{2}|\omega|^2. \end{aligned}\tag{2.7}$$

Integrating (2.7) over $\mathbb{T}^3$, and using (2.5), we obtain

$$\int_{\mathbb{T}^3}|\omega|^2dx = 2\sum_{j,k=1}^{3}\int_{\mathbb{T}^3} S_{jk}S_{jk}dx = 2\int_{\mathbb{T}^3}(\lambda_1^2+\lambda_2^2+\lambda_3^2)dx. \tag{2.8}$$

We also observe,

$$\sum_{i,j,k=1}^{3} S_{kj}S_{ik}S_{ij} = \lambda_1^3+\lambda_2^3+\lambda_3^3 = 3\lambda_1\lambda_2\lambda_3, \tag{2.9}$$

which follows from the following algebra, using $\lambda_1+\lambda_2+\lambda_3=0$,

$$\begin{aligned}0 &= (\lambda_1+\lambda_2+\lambda_3)^3 \\ &= \lambda_1^3+\lambda_2^3+\lambda_3^3+3\lambda_1^2(\lambda_2+\lambda_3)+3\lambda_2^2(\lambda_1+\lambda_3)+3\lambda_3(\lambda_1+\lambda_2)+6\lambda_1\lambda_2\lambda_3 \\ &= \lambda_1^3+\lambda_2^3+\lambda_3^3-3(\lambda_1^3+\lambda_2^3+\lambda_3^3)+6\lambda_1\lambda_2\lambda_3.\end{aligned}$$

Substituting (2.8) and (2.6), combined with (2.9), into (2.4), we have (2.1). □

The following is a new a priori estimate for the L^2 norm of vorticity for the 3D incompressible Euler equations.

Theorem 2.2 *Let $v(t) \in C([0,T);\mathbb{H}^m_\sigma)$, $m > 5/2$ be the local classical solution of the 3D Euler equations with initial data $v_0 \in \mathbb{H}^m_\sigma$. Let $\lambda_1(x,t) \geq \lambda_2(x,t) \geq \lambda_3(x,t)$ are the eigenvalues of the deformation tensor $S_{ij}(v) = \frac{1}{2}(\frac{\partial v_j}{\partial x_i}+\frac{\partial v_i}{\partial x_j})$. We denote $\lambda_2^+(x,t) = \max\{\lambda_2(x,t),0\}$, and $\lambda_2^-(x,t) = \min\{\lambda_2(x,t),0\}$. Then, the following (a priori) estimates hold.*

$$\begin{aligned}\|\omega_0\|_{L^2}\exp\left[\int_0^t\left(\frac{1}{2}\inf_{x\in\mathbb{T}^3}\lambda_2^+(x,t)-\sup_{x\in\mathbb{T}^3}|\lambda_2^-(x,t)|\right)dt\right] &\leq \|\omega(t)\|_{L^2} \\ \leq \|\omega_0\|_{L^2}\exp\left[\int_0^t\left(\sup_{x\in\mathbb{T}^3}\lambda_2^+(x,t)-\frac{1}{2}\inf_{x\in\mathbb{T}^3}|\lambda_2^-(x,t)|\right)dt\right]&\end{aligned}\tag{2.10}$$

for all $t \in (0,T)$.

Remark 2.1 The above estimate says, for example, that if we have the following comparability conditions,

$$\sup_{x\in\mathbb{T}^3} \lambda_2^+(x,t) \simeq \inf_{x\in\mathbb{T}^3} |\lambda_2^-(x,t)| \simeq g(t)$$

for some time interval $[0,T]$, then

$$\|\omega(t)\|_{L^2} \lesssim O\left(\exp\left[C\int_0^t g(s)ds\right]\right) \qquad \forall t\in[0,T]$$

for some constant C.

Remark 2.2 We note that $\lambda_2^+(x,t) > 0$ implies we have stretching of infinitesimal fluid volume in two directions and compression in the other one direction(planar stretching) at (x,t), while $|\lambda_2^-(x,t)| > 0$ implies stretching in one direction and compressions in two directions(linear stretching). The above estimate says that the dominance competition between planar stretching and linear stretching is an important mechanism controlling the growth/decay in time of the L^2 norm of vorticity.

Proof of Theorem 2.2 Since $\lambda_1+\lambda_2+\lambda_3=0$, and $\lambda_1\geq\lambda_2\geq\lambda_3$, we have $\lambda_1\geq 0, \lambda_3\leq 0$, and

$$|\lambda_2| \leq \min\{\lambda_1, |\lambda_3|\}. \tag{2.11}$$

We first observe that from (2.8),

$$\begin{aligned}
\int_{\mathbb{T}^3} |\omega|^2 dx &= 2\int_{\mathbb{T}^3}(\lambda_1^2+\lambda_2^2+\lambda_3^2)dx \\
&= 4\int_{\mathbb{T}^3}(\lambda_1^2+\lambda_1\lambda_2+\lambda_2^2)dx \quad (\lambda_3=-\lambda_1-\lambda_2) \\
&= 4\int_{\mathbb{T}^3}(\lambda_2^2+\lambda_2\lambda_3+\lambda_3^2)dx \quad (\lambda_1=-\lambda_2-\lambda_3)
\end{aligned} \tag{2.12}$$

We estimate the 'vortex stretching term' as

$$\begin{aligned}
-4\int_{\mathbb{T}^3}\lambda_1\lambda_2\lambda_3 dx &= -4\int_{\mathbb{T}^3}\lambda_2^+\lambda_1\lambda_3 dx - 4\int_{\mathbb{T}^3}\lambda_2^-\lambda_1\lambda_3 dx \\
&= 4\int_{\mathbb{T}^3}\lambda_2^+\lambda_1(\lambda_1+\lambda_2)dx - 4\int_{\mathbb{T}^3}|\lambda_2^-|(\lambda_2+\lambda_3)\lambda_3 dx \\
&= 4\int_{\mathbb{T}^3}\lambda_2^+(\lambda_1^2+\lambda_1\lambda_2)dx - 2\int_{\mathbb{T}^3}|\lambda_2^-|(2\lambda_2\lambda_3+2\lambda_3^2)dx \\
&\leq 4\sup_{x\in\mathbb{T}^3}\lambda_2^+(x,t)\int_{\mathbb{T}^3}(\lambda_1^2+\lambda_1\lambda_2+\lambda_2^2)dx \\
&\qquad -2\inf_{x\in\mathbb{T}^3}|\lambda_2^-(x,t)|\int_{\mathbb{T}^3}(\lambda_2^2+\lambda_2\lambda_3+\lambda_3^2)dx
\end{aligned}$$

$$
\begin{aligned}
&= \; 4\sup_{x\in\mathbb{T}^3}\lambda_2^+(x,t)\int_{\mathbb{T}^3}(\lambda_1^2+\lambda_1\lambda_2+\lambda_2^2)dx \\
&\qquad -2\inf_{x\in\mathbb{T}^3}|\lambda_2^-(x,t)|\int_{\mathbb{T}^3}(\lambda_1^2+\lambda_1\lambda_2+\lambda_2^2)dx, \qquad (2.13)
\end{aligned}
$$

where we used (2.11) and (2.12). This, combined with (2.1) and (2.12), yields

$$
\frac{d}{dt}\int_{\mathbb{T}^3}|\omega(x,t)|^2dx \le \left[2\sup_{x\in\mathbb{T}^3}\lambda_2^+(x,t) - \inf_{x\in\mathbb{T}^3}|\lambda_2^-(x,t)|\right]\int_{\mathbb{T}^3}|\omega(x,t)|^2dx. \qquad (2.14)
$$

Applying the Gronwall lemma, we have the second inequality of (2.10). In order to prove the first inequality of (2.10) we estimate from below starting from equality part of (2.13)

$$
\begin{aligned}
-4\int_{\mathbb{T}^3}\lambda_1\lambda_2\lambda_3dx &= 4\int_{\mathbb{T}^3}\lambda_2^+\lambda_1(\lambda_1+\lambda_2)dx - 4\int_{\mathbb{T}^3}|\lambda_2^-|(\lambda_2+\lambda_3)\lambda_3dx \\
&= \; 2\int_{\mathbb{T}^3}\lambda_2^+(2\lambda_1^2+2\lambda_1\lambda_2)dx - 4\int_{\mathbb{T}^3}|\lambda_2^-|(\lambda_2\lambda_3+\lambda_3^2)dx \\
&\ge \; 2\int_{\mathbb{T}^3}\lambda_2^+(\lambda_1^2+\lambda_1\lambda_2+\lambda_2^2)dx - 4\int_{\mathbb{T}^3}|\lambda_2^-|(\lambda_2^2+\lambda_2\lambda_3+\lambda_3^2)dx \\
&\ge \; 2\inf_{x\in\mathbb{T}^3}\lambda_2^+(x,t)\int_{\mathbb{T}^3}(\lambda_1^2+\lambda_1\lambda_2+\lambda_2^2)dx \\
&\qquad -4\sup_{x\in\mathbb{T}^3}|\lambda_2^-(x,t)|\int_{\mathbb{T}^3}(\lambda_2^2+\lambda_2\lambda_3+\lambda_3^2)dx, \qquad (2.15)
\end{aligned}
$$

where we used (2.11) again. Similarly to the above, combining this with (2.1) and (2.12), yields

$$
\frac{d}{dt}\int_{\mathbb{T}^3}|\omega(x,t)|^2dx \ge \left[\inf_{x\in\mathbb{T}^3}\lambda_2^+(x,t) - 2\sup_{x\in\mathbb{T}^3}|\lambda_2^-(x,t)|\right]\int_{\mathbb{T}^3}|\omega(x,t)|^2dx, \qquad (2.16)
$$

and, applying the Gronwall lemma we finish the first inequality of (2.10). $\square$

Corollary 2.1 *Let $v_0 \in \mathbb{H}^m_\sigma$ be given, and $\lambda_1(x,t), \lambda_2(x,t), \lambda_3(x,t)$ are as in Theorem 2.2. Suppose*

$$
\limsup_{t\to T_*}\|\omega(t)\|_{L^2} = \infty. \qquad (2.17)
$$

Then, necessarily

$$
\int_0^{T_*}\|\lambda_2^+(t)\|_{L^\infty}dt = \infty. \qquad (2.18)
$$

Proof. We just observe that from (2.10), we have immediately

$$
\|\omega(t)\|_{L^2} \le \|\omega_0\|_{L^2}\exp\left(\int_0^t\|\lambda_2^+(s)\|_{L^\infty}ds\right).
$$

This implies the corollary. □

Remark 2.3 The above corollary says that if singularity happens in the L^2 norm of vorticity, then it should be caused by the uncontrollable intensification of planar stretching.

Remark 2.4 In the 3D Navier-Stokes equations the L^2 norm singularity of vorticity is equivalent to the that of any high norms in Sobolev space(see e.g. [23, 11]). Hence, the above corollary says that the regularity/singularity of the 3D Navier-Stokes equations are controlled by the integral, $\int_0^t \|\lambda_2^+(s)\|_{L^\infty} ds$.

Remark 2.5 In [16] the author also investigated another sufficient condition for the singularity of L^2 norm of vorticity of the 3D Euler equations, using simultaneously the eigenvector and eigenvalues of the deformation tensor and the hessian of the pressure. Our condition is completely different from it, and has direct physical interpretation.

3 Applications for some classes of initial data

In order to state our theorem in this section we introduce some definitions. Given a differentiable vector field $f = (f_1, f_2, f_3)$ on $\mathbb{T}^3$, we denote by the scalar field $\lambda_i(f)$, i=1,2,3, the eigenvalues of the deformation tensor associated with f. Below we always assume the ordering, $\lambda_1(f) \geq \lambda_2(f) \geq \lambda_3(f)$. We also fix $m > 5/2$ below. We recall that if $f \in \mathbb{H}^m_\sigma$, then $\lambda_1(f) + \lambda_2(f) + \lambda_3(f) = 0$, which is another representation of div $f = 0$.

Let us begin with introduction of admissible classes $\mathcal{A}_\pm$ defined by

$$\mathcal{A}_+ = \{ f \in \mathbb{H}^m_\sigma(\mathbb{T}^3) \,|\, \inf_{x \in \mathbb{T}^3} \lambda_2(f)(x) > 0 \},$$

and

$$\mathcal{A}_- = \{ f \in \mathbb{H}^m_\sigma(\mathbb{T}^3) \,|\, \sup_{x \in \mathbb{T}^3} \lambda_2(f)(x) < 0 \}.$$

Physically $\mathcal{A}_+$ consists of solenoidal vector fields with planar stretching(see Remark 2.2) everywhere, while $\mathcal{A}_-$ consists of everywhere linear stretching vector fields. Although they do not represent real physical flows, they might be useful in the study of searching initial data leading to finite time singularity for the 3D Euler equations.

Given $v_0 \in \mathbb{H}^m_\sigma$, let $T_*(v_0)$ be the maximal time of unique existence of solution in $\mathbb{H}^m_\sigma$ for the system (1.1)-(1.3). Let $S_t : \mathbb{H}^m_\sigma \to \mathbb{H}^m_\sigma$ be the solution operator, mapping from initial data to the solution $v(t)$. Given $f \in \mathcal{A}_+$, we define the first zero touching time of $\lambda_2(f)$ as

$$T(f) = \inf\{ t \in (0, T_*(v_0)) \,|\, \exists x \in \mathbb{T}^3 \text{ such that } \lambda_2(S_t f)(x) < 0 \}.$$

Similarly for $f \in \mathcal{A}_-$, we define

$$T(f) = \inf\{t \in (0, T_*(v_0)) \,|\, \exists x \in \mathbb{T}^3 \text{ such that } \lambda_2(S_t f)(x) > 0\}.$$

The following theorem is actually an immediate corollary of Theorem 2.2, combined with the above definition of $\mathcal{A}_\pm$ and $T(f)$. We just observe that for $v_0 \in \mathcal{A}_+$(resp. $\mathcal{A}_-$) we have $\lambda_2^- = 0, \lambda_2^+ = \lambda_2$(resp. $\lambda_2^+ = 0, \lambda_2^- = \lambda_2$) on $\mathbb{T}^3 \times (0, T(v_0))$.

Theorem 3.1 *Let $v_0 \in \mathcal{A}_\pm$ be given. We set $\lambda_1(x,t) \geq \lambda_2(x,t) \geq \lambda_3(x,t)$ as the eigenvalues of the deformation tensor associated with $v(x,t) = (S_t v_0)(x)$ defined $t \in (0, T(v_0))$. Then, for all $t \in (0, T(v_0))$ we have the following estimates:*
(i) If $v_0 \in \mathcal{A}_+$, then

$$\exp\left(\frac{1}{2}\int_0^t \inf_{x\in\mathbb{T}^3} |\lambda_2(x,s)| ds\right) \leq \frac{\|\omega(t)\|_{L^2}}{\|\omega_0\|_{L^2}} \leq \exp\left(\int_0^t \sup_{x\in\mathbb{T}^3} |\lambda_2(x,s)| ds\right). \tag{3.1}$$

(ii) If $v_0 \in \mathcal{A}_-$, then

$$\exp\left(-\int_0^t \sup_{x\in\mathbb{T}^3} |\lambda_2(x,s)| ds\right) \leq \frac{\|\omega(t)\|_{L^2}}{\|\omega_0\|_{L^2}} \leq \exp\left(-\frac{1}{2}\int_0^t \inf_{x\in\mathbb{T}^3} |\lambda_2(x,s)| ds\right). \tag{3.2}$$

Remark 3.1 If we have the comparability conditions,

$$\inf_{x\in\mathbb{T}^3} |\lambda_2(x,t)| \quad \simeq \quad \sup_{x\in\mathbb{T}^3} |\lambda_2(x,t)| \simeq g(t) \quad \forall t \in (0, T(v_0)),$$

which is the case for sufficiently small box $\mathbb{T}^3$, then we have

$$\frac{\|\omega(t)\|_{L^2}}{\|\omega_0\|_{L^2}} \simeq \begin{cases} \exp\left(\int_0^t g(s) ds\right) & \text{if} \quad v_0 \in \mathcal{A}_+ \\ \exp\left(-\int_0^t g(s) ds\right) & \text{if} \quad v_0 \in \mathcal{A}_- \end{cases}$$

for $t \in (0, T(v_0))$. In particular, if we could find $v_0 \in \mathcal{A}_+$ such that

$$\inf_{x\in\mathbb{T}^3} |\lambda_2(x,t)| \gtrsim O\left(\frac{1}{t_* - t}\right) \tag{3.3}$$

for time interval near t_*, then such data would lead to singularity at t_*.

Below we have some decay in time estimates for some ratio of eigenvalues.

Theorem 3.2 *Let $v_0 \in \mathcal{A}_\pm$ be given, and we set $\lambda_1(x,t) \geq \lambda_2(x,t) \geq \lambda_3(x,t)$ as in Theorem 3.1. We define*

$$\varepsilon(x,t) = \frac{|\lambda_2(x,t)|}{\lambda(x,t)} \quad \forall (x,t) \in \mathbb{T}^3 \times (0, T(v_0)), \tag{3.4}$$

where we set

$$\lambda(x,t) = \begin{cases} \lambda_1(x,t) & \text{if} \quad v_0 \in \mathcal{A}_+ \\ -\lambda_3(x,t) & \text{if} \quad v_0 \in \mathcal{A}_-. \end{cases}$$

Then, there exists a constant $C = C(v_0, |\Omega|)$, with $|\Omega|$ denoting the volume of the box $\Omega = \mathbb{T}^3$, such that

$$\inf_{(x,s)\in\mathbb{T}^3\times(0,t)} \varepsilon(x,s) < \frac{C}{\sqrt{t}} \quad \forall t \in (0, T(v_0)). \tag{3.5}$$

Remark 3.2 Regarding the problem of searching a finite time blowing up solution, again, the proof of the above theorem, in particular, the estimate (3.10) below, combined with Remark 2.3, suggests the following, :
Given $\delta > 0$, let us suppose we could find $v_0 \in \mathcal{A}_+$ such that for the associated solution $v(x,t) = (S_t v_0)(x)$ the estimate

$$\inf_{(x,s)\in\mathbb{T}^3\times(0,t)} \varepsilon(x,s) \gtrsim O\left(\frac{1}{t^{\frac{1}{2}+\delta}}\right), \tag{3.6}$$

holds true, for sufficiently large time t. Then such v_0 will lead to the finite time singularity. In order to check the behavior (3.6) for a given solution we need a sharper and/or localized version of the equation (2.1) for the dynamics of eigenvalues of the deformation tensor.

Proof of Theorem 3.2 We divide the proof the into two separate cases.

(i) The case $v_0 \in \mathcal{A}_+$:

We parameterize the eigenvalues of the deformation tensor of the solution $v(x,t)$ of (1.1)-(1.3) by

$\lambda_1(x,t) = \lambda(x,t) > 0$, $\lambda_2 = \varepsilon(x,t)\lambda(x,t) > 0$, $\lambda_3(x,t) = -(1+\varepsilon(x,t))\lambda(x,t) < 0$.

for all $(x,t) \in \mathbb{T}^3 \times (0,T)$. We observe that

$$0 < \varepsilon(x,t) \le 1 \quad \forall (x,t) \in \mathbb{T}^3 \times [0, T(v_0)). \tag{3.7}$$

The equation (2.1) can be written as

$$\frac{d}{dt}\int_{\mathbb{T}^3} \lambda^2(\varepsilon^2+\varepsilon+1)dx = 2\int_{\mathbb{T}^3} \lambda^3(\varepsilon^2+\varepsilon)dx \quad \forall t \in (0, T(v_0)). \tag{3.8}$$

From the estimate

$$\begin{aligned} \int_{\mathbb{T}^3} \lambda^2(\varepsilon^2+\varepsilon+1)dx &= \int_{\mathbb{T}^3} \lambda^2(\varepsilon^2+\varepsilon)^{\frac{2}{3}} \frac{(\varepsilon^2+\varepsilon+1)}{(\varepsilon^2+\varepsilon)^{\frac{2}{3}}}dx \\ &\le \left[\int_{\mathbb{T}^3} \lambda^3(\varepsilon^2+\varepsilon)dx\right]^{\frac{2}{3}} \left[\int_{\mathbb{T}^3} \frac{(\varepsilon^2+\varepsilon+1)^3}{(\varepsilon^2+\varepsilon)^2}dx\right]^{\frac{1}{3}} \\ &\le \frac{3}{\sqrt[3]{4}}\left[\int_{\mathbb{T}^3} \frac{1}{\varepsilon^4}dx\right]^{\frac{1}{3}} \left[\int_{\mathbb{T}^3} \lambda^3(\varepsilon^2+\varepsilon)dx\right]^{\frac{2}{3}}, \end{aligned}$$

where we used (3.7), we have

$$\int_{\mathbb{T}^3} \lambda^3(\varepsilon^2+\varepsilon)dx \geq \frac{2}{\sqrt{27}}\left[\int_{\mathbb{T}^3}\frac{1}{\varepsilon^4}dx\right]^{-\frac{1}{2}}\left[\int_{\mathbb{T}^3}\lambda^2(\varepsilon^2+\varepsilon+1)dx\right]^{\frac{3}{2}}dx,$$

which, combined with (3.8), yields

$$\frac{d}{dt}\int_{\mathbb{T}^3} \lambda^2(\varepsilon^2+\varepsilon+1)dx \geq \frac{4}{\sqrt{27}}\left[\int_{\mathbb{T}^3}\frac{1}{\varepsilon^4}dx\right]^{-\frac{1}{2}}\left[\int_{\mathbb{T}^3}\lambda^2(\varepsilon^2+\varepsilon+1)dx\right]^{\frac{3}{2}}. \quad (3.9)$$

Setting

$$y(t) = \left[\int_{\mathbb{T}^3}\lambda^2(\varepsilon^2+\varepsilon+1)dx\right]^{\frac{1}{2}},$$

we have

$$\frac{dy}{dt} \geq \frac{2}{\sqrt{27}}\left[\int_{\mathbb{T}^3}\frac{1}{\varepsilon^4}dx\right]^{-\frac{1}{2}} y^2.$$

Solving the differential inequality, we have

$$y(t) \geq \frac{y_0}{1-\frac{2y_0}{\sqrt{27}}\int_0^t\left[\int_{\mathbb{T}^3}\frac{1}{\varepsilon^4}dx\right]^{-\frac{1}{2}}ds}.$$

Since $y^2(t) = \frac{1}{2}\|\omega(t)\|_{L^2}^2$, we have just derived

$$\|\omega(t)\|_{L^2} \geq \frac{\sqrt{2}\|\omega_0\|_{L^2}}{\sqrt{2}-\frac{2\|\omega_0\|_{L^2}}{\sqrt{27}}\int_0^t\left[\int_{\mathbb{T}^3}\frac{1}{\varepsilon^4}dx\right]^{-\frac{1}{2}}ds} \quad \forall t\in[0,T(v_0)).$$

Since the denominator should be positive for all $t\in[0,T(v_0)]$, we obtain that

$$\frac{2\|\omega_0\|_{L^2}}{\sqrt{27}}\int_0^t\left[\int_{\mathbb{T}^3}\frac{1}{\varepsilon^4}dx\right]^{-\frac{1}{2}}ds < \sqrt{2}.$$

Estimating from below the left hand side, we are lead to the inequality

$$t\inf_{(x,s)\in\mathbb{T}^3\times(0,t)}\varepsilon^2(x,s) \leq |\Omega|^{\frac{1}{2}}\int_0^t\left[\int_{\mathbb{T}^3}\frac{1}{\varepsilon^4}dx\right]^{-\frac{1}{2}}ds < \frac{\sqrt{27}|\Omega|^{\frac{1}{2}}}{\sqrt{2}\|\omega_0\|_{L^2}}, \quad (3.10)$$

which implies (3.5) for the case $v_0\in\mathcal{A}_+$.

(ii) The case $v_0\in\mathcal{A}_-$:

In this case parameterize the eigenvalues as

$$\lambda_1(x,t) = (1+\varepsilon(x,t))\lambda(x,t) > 0, \quad \lambda_2 = -\varepsilon(x,t)\lambda(x,t) > 0$$

$$\lambda_3(x,t) = -\lambda(x,t) > 0,$$
where as previously we have $0 < \varepsilon(x,t) \leq 1$ for all $(x,t) \in \mathbb{T}^3 \times (0, T(v_0))$. The equation (2.1) can now be written as
$$\frac{d}{dt}\int_{\mathbb{T}^3} \lambda^2(\varepsilon^2+\varepsilon+1)dx = -2\int_{\mathbb{T}^3} \lambda^3(\varepsilon^2+\varepsilon)dx. \tag{3.11}$$
Similarly to the above, we obtain
$$\frac{d}{dt}\int_{\mathbb{T}^3} \lambda^2(\varepsilon^2+\varepsilon+1)dx \leq -\frac{2}{\sqrt{27}}\left[\int_{\mathbb{T}^3}\frac{1}{\varepsilon^4}dx\right]^{-\frac{1}{2}}\left[\int_{\mathbb{T}^3} \lambda^2(\varepsilon^2+\varepsilon+1)dx\right]^{\frac{3}{2}}. \tag{3.12}$$
Hence, by similar procedure to the previous case, we have
$$\|\omega(t)\|_{L^2} \leq \frac{\sqrt{2}\|\omega_0\|_{L^2}}{\sqrt{2} + \frac{2\|\omega_0\|_{L^2}}{\sqrt{27}}\int_0^t \left[\int_{\mathbb{T}^3}\frac{1}{\varepsilon^4}dx\right]^{-\frac{1}{2}} ds}. \tag{3.13}$$
Now we recall the helicity conservation(see e.g. [21]),
$$H(t) = \int_{\mathbb{T}^3} v(x,t)\cdot\omega(x,t)dx = \int_{\mathbb{T}^3} v_0(x)\cdot\omega_0(x)dx = H_0,$$
which implies
$$H_0 \leq \|v(t)\|_{L^2}\|\omega(t)\|_{L^2} = \|v_0\|_{L^2}\|\omega(t)\|_{L^2} = \sqrt{2E_0}\|\omega(t)\|_{L^2}, \tag{3.14}$$
where we used the energy conservation
$$E(t) = \frac{1}{2}\|v(t)\|_{L^2}^2 = \frac{1}{2}\|v_0\|_{L^2}^2 = E_0.$$
Combining (3.13) with (3.14), we have
$$\frac{H_0}{\sqrt{2E_0}} \leq \frac{\sqrt{2}\|\omega_0\|_{L^2}}{\sqrt{2} + \frac{2\|\omega_0\|_{L^2}}{\sqrt{27}}\int_0^t \left[\int_{\mathbb{T}^3}\frac{1}{\varepsilon^4}dx\right]^{-\frac{1}{2}} ds},$$
from which we derive
$$\int_0^t \left[\int_{\mathbb{T}^3}\frac{1}{\varepsilon^4}dx\right]^{-\frac{1}{2}} ds \leq \sqrt{27}\left(\frac{\sqrt{E_0}}{H_0} - \frac{1}{\sqrt{2}\|\omega_0\|_{L^2}}\right). \tag{3.15}$$
Estimating from below the left hand side of (3.15), we deduce
$$t \inf_{(x,s)\in\mathbb{T}^3\times[0,s]} \varepsilon^2(x,s) \leq \sqrt{27}|\Omega|^{\frac{1}{2}}\left(\frac{\sqrt{E_0}}{H_0} - \frac{1}{\sqrt{2}\|\omega_0\|_{L^2}}\right) \tag{3.16}$$
for all $t \in (0, T(v_0))$. This finishes the proof of (3.5) for $v_0 \in \mathcal{A}_-$. $\square$

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References

[1] A. Babin, A. Mahalov and B. Nicolaenko, *3D Navier-Stokes and Euler equations with Initial Data Characterized by Uniformly Large Vorticity,* Indiana University Mathematics Journal, **50** (1), (2001), pp. 1-35.

[2] J.T. Beale, T. Kato and A. Majda, *Remarks on the Breakdown of Smooth Solutions for the 3-D Euler Equations,* Comm. Math. Phys., **94**, (1984), pp. 61-66.

[3] D. Chae, *On the Well-Posedness of the Euler Equations in the Besov and the Triebel-Lizorkin Spaces,* Proceedings of the Conference, "Tosio Kato's Method and Principle for Evolution Equations in Mathematical Physics", June 27-29, (2001).

[4] D. Chae, *On the Well-Posedness of the Euler Equations in the Triebel-Lizorkin Spaces,* Comm. Pure Appl. Math., **55**, (2002), pp. 654-678.

[5] D. Chae, *On the Euler Equations in the Critical Triebel-Lizorkin Spaces*, Arch. Rational Mech. Anal., **170**, no. 3, (2003), pp.185-210

[6] D. Chae, *Local Existence and Blow-up Criterion for the Euler Equations in the Besov Spaces*, Asymptotic Analysis, **38**, no. 3-4, (2004), pp. 339-358.

[7] D. Chae, *Remarks on the blow-up criterion of the 3D Euler equations*, Nonlinearity, **18**, (2005), pp. 1021-1029.

[8] J. -Y. Chemin, *Régularité des trajectoires des particules d'un fluide incompressible remplissant l'espace,* J. Math. Pures Appl., **71**(5) , (1992), pp. 407-417.

[9] J. -Y. Chemin, *Perfect incompressible fluids,* Clarendon Press, Oxford, (1998).

[10] P. Constantin, *Geometric Statistics in Turbulence,* SIAM Rev., **36**, (1994), pp. 73-98.

[11] P. Constantin and C. Foias, *Navier-Stokes equations,* Univ. Chicago Press, Chicago, (1988).

[12] P. Constantin, C. Fefferman, *Direction of vorticity and the problem of global regularity for the Navier-Stokes equations,* Indiana Univ. Math. J., **42**, (1993), pp. 775-789.

[13] P. Constantin, C. Fefferman and A. Majda, *Geometric Constraints on Potential Singularity Solutions for the 3D Euler Equations,* Comm. Partial Diff. Eqns., **21**, (1996), pp.559-571.

[14] D. Cordoba and C. Fefferman, *On the collapse of tubes carried by 3D incompressible flows,* Comm. Math. Phys., **222** (2), (2001), pp. 293-298.

[15] J. Deng, T. Y. Hou and X. Yu, *Geometric and Nonblowup of 3D Incompressible Euler Flow,* Comm. P.D.E, **30**, (2005), pp. 225-243.

[16] X. He, *A sufficient condition for a finite-time L_2 singularity of the 3d Euler Equations,* to appear in Math. Proc. Camb. Phil. Soc. (2005).

[17] T. Kato, *Nonstationary flows of viscous and ideal fluids in* $\mathbb{R}^3$, J. Functional Anal. **9**, (1972), pp. 296-305.

[18] T. Kato and G. Ponce, *Commutator estimates and the Euler and Navier-Stokes equations,* Comm. Pure Appl. Math., **41**, (1988), pp. 891-907.

[19] H. Kozono and Y. Taniuchi, *Limiting case of the Sobolev inequality in BMO, with applications to the Euler equations,* Comm. Math. Phys., **214**, (2000), pp. 191-200.

[20] H. Liu and E. Tadmor, *Spectral Dyanamics of the Velocity Gradient Field in Restricted Flows,* Comm. Math. Phys., **228**, (2002), pp. 435-466.

[21] A. Majda and A. Bertozzi *Vorticity and Incompressible Flow,* Cambridge Univ. Press,(2002).

[22] R. Temam, *On the Euler equations of incompressible flows,* J. Functional Anal., **20**, (1975), pp. 32-43.

[23] R. Temam, *Navier-Stokes equations,* 2nd ed., North-Holland, Amsterdam, (1986).

[24] M. Vishik, *Hydrodynamics in Besov spaces,* Arch. Rational Mech. Anal, **145**, (1998), pp. 197-214.

[25] M. Vishik, *Incompressible flows of an ideal fluid with vorticity in borderline spaces of Besov type,* Ann. Scient. Éc. Norm. Sup., 4^e série, t. 32, (1999), pp. 769-812.