Remarks on the blow-up criterion of the 3D Euler equations

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Abstract

In this note we prove that the finite time blow-up of classical solutions of the 3-D homogeneous incompressible Euler equations is controlled by the Besov space, $\dot{B}^0_{\infty,1}$, norm of the *two components* of the vorticity. For the axisymmetric flows with swirl we deduce that the blow-up of solution is controlled by the same Besov space norm of the *angular component* of the vorticity. For the proof of these results we use the Beale-Kato-Majda criterion, and the special structure of the vortex stretching term in the vorticity formulation of the Euler equation.

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1 Introduction

We are concerned on the Euler equations for the homogeneous incompressible fluid flows in $\mathbb{R}^3 \times (0, \infty)$.

$$\frac{\partial v}{\partial t} + (v \cdot \nabla)v = -\nabla p, \tag{1.1}$$

$$\mathrm{div}\ \ v = 0, \tag{1.2}$$

$$v(x, 0) = v_0(x), \quad x \in \mathbb{R}^3 \tag{1.3}$$

where $v = (v^1, v^2, v^3)$, $v^j = v^j(x, t)$, $j = 1, 2, 3$ is the velocity of the fluid flows, $p = p(x, t)$ is the scalar pressure, and v_0 is the given initial velocity satisfying div $v_0 = 0$. Taking curl of (1.1), we obtain the following vorticity formulation for the vorticity field ω =curl v.

$$\frac{\partial \omega}{\partial t} + (v \cdot \nabla)\omega = \omega \cdot \nabla v, \tag{1.4}$$

$$\text{div }\ v = 0, \quad \text{curl }\ v = \omega, \tag{1.5}$$

$$\omega(x,0) = \omega_0(x), \quad x \in \mathbb{R}^3 \tag{1.6}$$

The local in time solution of the Euler equations in the Sobolev space $H^m(\mathbb{R}^n)$ for $m > n/2 + 1$, $n = 2,3$ was obtained by Kato in [13], and there are several other local well-posedness results after that, using various function spaces([14, 15, 8, 2, 3]). The most outstanding open problem for the Euler equations is whether or not there exists any smooth initial data, say $v_0 \in C_0^\infty(\mathbb{R}^3)$, which evolves in finite time into a blowing up solution(breakdown of the initial data regularity). In this direction there is a celebrated criterion of the blow-up due to Beale, Kato and Majda(called the BKM criterion)[1], which states for $m > \frac{5}{2}$

$$\limsup_{t \nearrow T_*} \|v(t)\|_{H^m} = \infty \quad \text{if and only if} \quad \int_0^{T_*} \|\omega(t)\|_{L^\infty} dt = \infty. \tag{1.7}$$

(See [16, 2, 3, 17] for the refinements of this result by replacing the L^∞ norm of the vorticity by weaker norms close to the L^∞ norm.) In this note we are concerned on refining the BKM criterion by reducing the number of components of the vorticity vector field. For the statement of our main results we introduce a particular Besov space, $\dot{B}^0_{\infty,1}$. Given $f \in \mathcal{S}$, the Schwartz class of rapidly deceasing functions. Its Fourier transform $\hat{f}$ is defined by

$$\mathcal{F}(f) = \hat{f}(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x) dx.$$

We consider $\varphi \in \mathcal{S}$ satisfying the following three conditions:

(i) $\operatorname{Supp}\hat{\varphi} \subset \{\xi \in \mathbb{R}^n \,|\, \frac{1}{2} \le |\xi| \le 2\}$,

(ii) $\hat{\varphi}(\xi) \ge C > 0$ if $\frac{2}{3} < |\xi| < \frac{3}{2}$,

(iii) $\sum_{j\in\mathbb{Z}} \hat{\varphi}_j(\xi) = 1$, where $\hat{\varphi}_j = \hat{\varphi}(2^{-j}\xi)$.

Construction of such sequence of functions $\{\varphi_j\}_{j\in\mathbb{Z}}$ is well-known(See e.g. [21]). Note that $\hat{\varphi}_j$ is supported on the annulus of radius about 2^j.
Then, $\dot{B}^0_{\infty,1}$ is defined by

$$f \in \dot{B}^0_{\infty,1} \Longleftrightarrow \|f\|_{\dot{B}^0_{\infty,1}} = \sum_{j\in\mathbb{Z}} \|\varphi_j * f\|_{L^\infty} < \infty,$$

where $*$ is the standard notation for convolution, $(f*g)(x) = \int_{\mathbb{R}^n} f(x-y)g(y)dy$. Note that the condition (iii)(partition of unity) above implies immediately that $\dot{B}^0_{\infty,1} \hookrightarrow L^\infty$. The space $\dot{B}^0_{\infty,1}$ can be embedded into the class of continuous bounded functions, thus having slightly better regularity than L^∞, but containing as a subspace the Hölder space C^γ, for any $\gamma > 0$. One distinct feature of $\dot{B}^0_{\infty,1}$, compared with L^∞ is that the singular integral operators of the Calderon-Zygmund type map $\dot{B}^0_{\infty,1}$ into itself boundedly, the property which L^∞ does not have. We now state our main theorems.

Theorem 1.1 *Let $m > 5/2$. Suppose $v \in C([0,T_1);H^m(\mathbb{R}^3))$ is the local classical solution of (1.1)-(1.3) for some $T_1 > 0$, corresponding to the initial data $v_0 \in H^m(\mathbb{R}^3)$, and $\omega =$ curl v is its vorticity. We decompose $\omega = \tilde{\omega} + \omega^3 e_3$, where $\tilde{\omega} = \omega^1 e_1 + \omega^2 e_2$, and $\{e_1, e_2, e_3\}$ is the canonical basis of $\mathbb{R}^3$. Then,*

$$\limsup_{t \nearrow T} \|v(t)\|_{H^m} = \infty \quad \textit{if and only if} \quad \int_0^T \|\tilde{\omega}(t)\|^2_{\dot{B}^0_{\infty,1}} dt = \infty. \tag{1.8}$$

Remark 1.1. Actually $\tilde{\omega}$ could be the projected component of ω onto any plane in $\mathbb{R}^3$. For the solution $v = (v^1, v^2, 0)$ of the Euler equations on the $x_1 - x_2$ plane, the vorticity is $\omega = \omega^3 e_3$ with $\omega_3 = \partial_{x_1} v^2 - \partial_{x_2} v^1$, and $\tilde{\omega} \equiv 0$. Hence, as a trivial application of the above theorem we obtain the global in time existence of classical solutions for the 2-D Euler equations.

Remark 1.2. For the 3-D Navier-Stokes equations it is possible to control the regularity also by $\tilde{\omega}$, but using the same scale invariant norm as for the whole components of the vorticity field, ω as obtained in [4].

Next, we consider the axisymmetric solution of the Euler equations, which means velocity field $v(r, x_3, t)$, solving the Euler equations, and having the representation

$$v(r, x_3, t) = v^r(r, x_3, t) e_r + v^\theta(r, x_3, t) e_\theta + v^3(r, x_3, t) e_3$$

in the cylindrical coordinate system, where

$$e_r = (\frac{x_1}{r}, \frac{x_2}{r}, 0), \quad e_\theta = (-\frac{x_2}{r}, \frac{x_1}{r}, 0), \quad e_3 = (0,0,1), \quad r = \sqrt{x_1^2 + x_2^2}.$$

In this case also the question of finite time blow-up of solution is widely open(See [11],[5],[6] for previous studies in such case). The vorticity $\omega =$ curl v is computed as

$$\omega = \omega^r e_r + \omega^\theta e_\theta + \omega^3 e_3,$$

where

$$\omega^r = -\partial_{x_3} v^\theta, \quad \omega^\theta = \partial_{x_3} v^r - \partial_r v^3, \quad \omega^3 = \frac{1}{r} \partial_r (r v^\theta).$$

We denote

$$\tilde{v} = v^r e_r + v^3 e_3, \qquad \tilde{\omega} = \omega^r e_r + \omega^3 e_3.$$

Hence, $\omega = \tilde{\omega} + \vec{\omega}_\theta$, where $\vec{\omega}_\theta = \omega^\theta e_\theta$. The Euler equations for the axisymmetric solution are

$$\frac{\partial v^r}{\partial t} + (\tilde{v} \cdot \tilde{\nabla}) v^r = -\frac{\partial p}{\partial r}, \tag{1.9}$$

$$\frac{\partial v^\theta}{\partial t} + (\tilde{v} \cdot \tilde{\nabla}) v^\theta = -\frac{v^r v^\theta}{r}, \tag{1.10}$$

$$\frac{\partial v^3}{\partial t} + (\tilde{v} \cdot \tilde{\nabla}) v^3 = -\frac{\partial p}{\partial x_3}, \tag{1.11}$$

$$\text{div } \tilde{v} = 0, \tag{1.12}$$

$$v(r, x_3, 0) = v_0(r, x_3), \tag{1.13}$$

where $\tilde{\nabla} = e_r \frac{\partial}{\partial r} + e_3 \frac{\partial}{\partial x_3}$. In the axisymmetry the Euler equations in the vorticity formulation becomes

$$\frac{\partial \omega^r}{\partial t} + (\tilde{v} \cdot \tilde{\nabla})\omega^r (\tilde{\omega} \cdot \tilde{\nabla}) v^r \tag{1.14}$$

$$\frac{\partial \omega^3}{\partial t} + (\tilde{v} \cdot \tilde{\nabla})\omega^3 (\tilde{\omega} \cdot \tilde{\nabla}) v^3 \tag{1.15}$$

$$\left[\frac{\partial}{\partial t} + \tilde{v} \cdot \tilde{\nabla}\right] \left(\frac{\omega^\theta}{r}\right) = (\tilde{\omega} \cdot \tilde{\nabla}) \left(\frac{v^\theta}{r}\right) \tag{1.16}$$

$$\text{div } \tilde{v} = 0, \quad \text{curl } \tilde{v} = \vec{\omega}^\theta. \tag{1.17}$$

We now state our main theorem for the axisymmetric solutions of the Euler equations.

Theorem 1.2 *Let v be the local classical axisymmetric solution of the 3-D Euler equations considered in Theorem 1.1, corresponding to an axisymmetric initial data $v_0 \in H^m(\mathbb{R}^3)$. As in the above we decompose $\omega = \tilde{\omega} + \vec{\omega}_\theta$, where $\tilde{\omega} = \omega^r e_r + \omega^3 e_3$ and $\vec{\omega}_\theta = \omega^\theta e_\theta$. Then,*

$$\limsup_{t \nearrow T} \|v(t)\|_{H^m} = \infty \quad \textit{if and only if} \quad \int_0^T \|\vec{\omega}_\theta(t)\|_{\dot{B}^0_{\infty,1}} dt = \infty. \tag{1.18}$$

Remark 1.3. We note that for the axisymmetric 3-D Navier-Stokes equations with swirl it is possible to control the regularity also by $\vec{\omega}_\theta$ without strengthening its norm as obtained in [4].

Remark 1.4. We compare this result with that of [6], where we proved that the regularity/singularity is controlled by the integral $\int_0^T \|\vec{\omega}_\theta(t)\|_{L^\infty}(1+\log^+ \|\vec{\omega}_\theta(t)\|_{C^\gamma})dt$. We note that this integral contains C^γ norm of $\vec{\omega}_\theta$, higher than $\dot{B}^0_{\infty,1}$ norm.

2 Proof of the Main Results

Multiplying (1.5) by e_3, we obtain

$$\frac{\partial \omega^3}{\partial t} + (v \cdot \nabla)\omega^3 = (\omega \cdot \nabla) v \cdot e_3. \tag{2.1}$$

Given a vector field $v(x,t)$, we consider the particle trajectory mapping $X(\alpha, t)$ defined by the system of ordinary differential equation,

$$\frac{\partial X(\alpha, t)}{\partial t} = v(X(\alpha, t), t), \quad X(\alpha, 0) = \alpha \in \mathbb{R}^3.$$

Integrating (2.1) along $X(\alpha, t)$, we have

$$\omega^3(X(\alpha,t),t) = \omega_0^3(\alpha) + \int_0^t [(\omega\cdot\nabla)v\cdot e_3](X(\alpha,s),s)ds.$$

Hence, taking supremum over $\alpha \in \mathbb{R}^3$ yields

$$\|\omega^3(t)\|_{L^\infty} \le \|\omega_0^3\|_{L^\infty} + \int_0^t \|(\omega\cdot\nabla)v\cdot e_3](s)\|_{L^\infty}ds. \tag{2.2}$$

We now estimate the vortex stretching term $(\omega\cdot\nabla)v\cdot e_3$ pointwise. Using the Biot-Savart law, which follows from (1.5),

$$v(x,t) = \frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{y\times\omega(x+y,t)}{|y|^3}dy,$$

we can compute(See e.g. [19])

$$\begin{aligned}\frac{\partial v^i}{\partial x_j}(x,t) \quad &= \quad \frac{1}{4\pi}\sum_{l,m=1}^3 \epsilon_{jlm} PV\int_{\mathbb{R}^3}\left\{\frac{\delta_{il}}{|y|^3} - 3\frac{y_i y_l}{|y|^5}\right\}\omega_m(x+y,t)\,dy - \frac{1}{3}\sum_{l=1}^3 \epsilon_{ijl}\omega_l(x,t) \\ &:= \quad \mathcal{P}_{ij}(\omega)(x,t),\end{aligned}$$

where PV denotes the principal value of the integrals, and ϵ_{jlm} is the skew symmetric tensor with the normalization $\epsilon_{123}=1$. We note that $\mathcal{P}_{ij}(\cdot)$ is a matrix valued singular integral operator of the Calderon-Zygmund type. Hence, we compute explicitly the vortex stretching term as

$$\begin{aligned}
[(\omega\cdot\nabla)v\cdot e_3](x,t) &= \sum_{i,j=1}^3 \omega_i(x,t)\frac{\partial v^i}{\partial x_j}(x,t)(e_3)_j \\
&= \frac{1}{4\pi}PV\int_{\mathbb{R}^3}\left\{\frac{\omega(x,t)\times\omega(x+y,t)}{|y|^3}\cdot e_3 - 3\frac{y\times\omega(x+y,t)}{|y|^5}\cdot e_3\,(y\cdot\omega(x,t))\right\} \\
&= (\text{ Decomposing the vorticity into } \omega = \tilde{\omega} + \omega^3 e_3,) \\
&= \frac{1}{4\pi}PV\int_{\mathbb{R}^3}\left\{\frac{\tilde{\omega}(x,t)\times\tilde{\omega}(x+y,t)}{|y|^3}\cdot e_3 - 3\frac{y\times\tilde{\omega}(x+y,t)}{|y|^5}\cdot e_3\ y_3\omega_3(x,t)\right. \\
&\qquad\qquad\qquad \left. -3\frac{y\times\tilde{\omega}(x+y,t)}{|y|^5}\cdot e_3\,(y\cdot\tilde{\omega}(x,t))\right\}\,dy \\
&= \sum_{i,j=1}^3 \tilde{\omega}_i(x,t)\mathcal{P}_{ij}(\tilde{\omega})(x,t)(e_3)_j + \sum_{i,j=1}^3 \omega^3(x,t)(e_3)_i\mathcal{P}_{ij}(\tilde{\omega})(x,t)(e_3)_j.
\end{aligned}$$

We thus have the pointwise estimate

$$|[(\omega\cdot\nabla)v\cdot e_3](x,t)| \le C|\tilde{\omega}(x,t)||\mathcal{P}(\tilde{\omega})(x,t)| + C|\omega^3(x,t)||\mathcal{P}(\tilde{\omega})(x,t)|,$$

and from the embedding, $\dot{B}^0_{\infty,1} \hookrightarrow L^\infty$, we obtain

$$
\begin{aligned}
\|[\mathcal{D}\omega \cdot e_3]\|_{L^\infty} &\leq C\|\omega^3\|_{L^\infty}\|\mathcal{P}(\tilde{\omega})\|_{L^\infty} + C\|\tilde{\omega}\|_{L^\infty}\|\mathcal{P}(\tilde{\omega})\|_{L^\infty} \\
&\leq C\|\omega^3\|_{L^\infty}\|\mathcal{P}(\tilde{\omega})\|_{\dot{B}^0_{\infty,1}} + C\|\tilde{\omega}\|_{L^\infty}\|\mathcal{P}(\tilde{\omega})\|_{\dot{B}^0_{\infty,1}} \\
&\leq C\|\omega^3\|_{L^\infty}\|\tilde{\omega}\|_{\dot{B}^0_{\infty,1}} + C\|\tilde{\omega}\|^2_{\dot{B}^0_{\infty,1}}, \qquad (2.3)
\end{aligned}
$$

where we used the fact that the Calderon-Zygmund singular integral operator maps $\dot{B}^0_{\infty,1}$ into itself boundedly. Substituting (2.3) into (2.2), we have the estimate

$$
\begin{aligned}
\|\omega^3(t)\|_{L^\infty} &\leq \|\omega^3_0\|_{L^\infty} + C\int_0^t \|\omega^3(s)\|_{L^\infty}\|\tilde{\omega}(s)\|_{\dot{B}^0_{\infty,1}} ds \\
&\qquad + C\int_0^t \|\tilde{\omega}(s)\|^2_{\dot{B}^0_{\infty,1}} ds.
\end{aligned}
$$

The Gronwall lemma yields

$$
\begin{aligned}
\|\omega^3(t)\|_{L^\infty} &\leq \|\omega^3_0\|_{L^\infty} \exp\left(C\int_0^t \|\tilde{\omega}(s)\|_{\dot{B}^0_{\infty,1}} ds\right) \\
&\quad + C\int_0^t \|\tilde{\omega}(s)\|^2_{\dot{B}^0_{\infty,1}} \exp\left(C\int_s^t \|\tilde{\omega}(\tau)\|_{\dot{B}^0_{\infty,1}} d\tau\right) ds \\
&\leq \left(\|\omega^3_0\|_{L^\infty} + \int_0^t \|\tilde{\omega}(s)\|^2_{\dot{B}^0_{\infty,1}} ds\right) \exp\left(C\int_0^t \|\tilde{\omega}(s)\|_{\dot{B}^0_{\infty,1}} ds\right).
\end{aligned}
$$

Hence, denoting $\left(\int_0^T \|\tilde{\omega}(t)\|^2_{\dot{B}^0_{\infty,1}} dt\right)^{\frac{1}{2}} = A_T$, we deduce that

$$
\begin{aligned}
\int_0^T \|\omega(t)\|_{L^\infty} dt &\leq \int_0^T \|\tilde{\omega}(t)\|_{L^\infty} dt + \int_0^T \|\omega^3(t)\|_{L^\infty} dt \\
&\leq \sqrt{T} A_T + \left[\|\omega^3_0\|_{L^\infty} + CA_T^2\right] T \exp\left(C\sqrt{T} A_T\right).
\end{aligned}
$$

Combining this with (1.7) implies the necessity part of (1.8). The sufficiency part easily follows by trivial application of the imbedding, $H^m(\mathbb{R}^3) \hookrightarrow \dot{B}^0_{\infty,1}(\mathbb{R}^3)$ for $m > \frac{5}{2}$. This completes the proof of Theorem 1.1.□

Remark after Proof: The special structure of the vortex stretching term used in the above proof was emphasized and used previously in [9, 10].

Proof of Theorem 1.2: We will use the notations,

$$
\tilde{\nabla}\tilde{v} = \begin{pmatrix} \dfrac{\partial v^r}{\partial r} & \dfrac{\partial v^r}{\partial x_3} \\ \dfrac{\partial v^3}{\partial r} & \dfrac{\partial v^3}{\partial x_3} \end{pmatrix}, \qquad \nabla\tilde{v} = \left(\frac{\partial \tilde{v}_j}{\partial x_k}\right)^3_{j,k=1}.
$$

One can check easily(or, may see [6] for detailed computations.)

$$|\tilde{\nabla}\tilde{v}(x)| \leq |\nabla\tilde{v}(x)| \qquad \forall x \in \mathbb{R}^3. \tag{2.4}$$

As in the proof of Theorem 1.1 the elliptic system, (1.17) implies

$$\nabla\tilde{v}(x) = \mathcal{P}(\vec{\omega}_\theta)(x) + C_0\vec{\omega}_\theta(x),$$

where $\mathcal{P}(\cdot)$ is a matrix valued singular integral operator of the Calderon-Zygmund type, and C_0 is a constant matrix. Given $\tilde{v}(x,t)$, we consider the particle trajectory mapping $\tilde{X}(\alpha,t)$ defined by the system of ordinary differential equation,

$$\frac{\partial\tilde{X}(\alpha,t)}{\partial t} = \tilde{v}(\tilde{X}(\alpha,t),t), \quad \tilde{X}(\alpha,0) = \alpha.$$

Then, integrating (1.14)-(1.15) along $\tilde{X}(\alpha,t)$, we find that

$$\omega^r(\tilde{X}(\alpha,t),t) = \omega_0^r(\alpha) + \int_0^t (\tilde{\omega}\cdot\tilde{\nabla})v^r(\tilde{X}(\alpha,s),s)ds,$$

$$\omega^3(\tilde{X}(\alpha,t),t) = \omega_0^3(\alpha) + \int_0^t (\tilde{\omega}\cdot\tilde{\nabla})v^3(\tilde{X}(\alpha,s),s)ds.$$

Thus, taking supremum over $\alpha \in \mathbb{R}^3$, we infer

$$\begin{aligned}
\|\tilde{\omega}(t)\|_{L^\infty} &\leq \|\tilde{\omega}_0\|_{L^\infty} + \int_0^t \|\tilde{\omega}(s)\|_{L^\infty}\|\tilde{\nabla}\tilde{v}(s)\|_{L^\infty}ds \\
&\leq \|\tilde{\omega}_0\|_{L^\infty} + \int_0^t \|\tilde{\omega}(s)\|_{L^\infty}\|\nabla\tilde{v}(s)\|_{L^\infty}ds,
\end{aligned}$$

where we used (2.4). By Gronwall's lemma we obtain

$$\begin{aligned}
\|\tilde{\omega}(t)\|_{L^\infty} &\leq \|\tilde{\omega}_0\|_{L^\infty}\exp\left(\int_0^t \|\nabla\tilde{v}(s)\|_{L^\infty}ds\right) \\
&\leq \|\tilde{\omega}_0\|_{L^\infty}\exp\left(C\int_0^t \|\nabla\tilde{v}(s)\|_{\dot{B}^0_{\infty,1}}ds\right) \\
&\leq \|\tilde{\omega}_0\|_{L^\infty}\exp\left(C\int_0^t \|\vec{\omega}_\theta(s)\|_{\dot{B}^0_{\infty,1}}ds\right),
\end{aligned}$$

where we used the fact that the Calderon-Zygmund singular integral operator maps $\dot{B}^0_{\infty,1}$ into itself boundedly. Combining this estimate with the embeddig, $\dot{B}^0_{\infty,1} \hookrightarrow L^\infty$, we find

$$\begin{aligned}
\int_0^T \|\omega(t)\|_{L^\infty}dt &\leq \int_0^T \|\tilde{\omega}(t)\|_{L^\infty}dt + \int_0^T \|\vec{\omega}_\theta(t)\|_{L^\infty}dt \\
&\leq T\|\tilde{\omega}_0\|_{L^\infty}\exp\left(C\int_0^T \|\vec{\omega}_\theta(t)\|_{\dot{B}^0_{\infty,1}}dt\right) \\
&\quad +C\int_0^T \|\vec{\omega}_\theta(t)\|_{\dot{B}^0_{\infty,1}}dt.
\end{aligned}$$

Thus, the BKM criterion, (1.7) implies the necessity part of Theorem 1.2. Similarly to the proof of Theorem 1.1 the sufficiency part easily follows from the imbedding, $H^m(\mathbb{R}^3) \hookrightarrow \dot{B}^0_{\infty,1}(\mathbb{R}^3)$ for $m > \frac{5}{2}$. This completes the proof of Theorem 1.2. $\square$

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