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Energy and Helicity Preserving Schemes for Hydro- and Magnetohydro-dynamics Flows with Symmetry

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Abstract

We propose a class of simple and efficient numerical schemes for incompressible fluid equations with coordinate symmetry. With the introduction of a generalized vorticity-stream formulation, explicit treatment of the nonlinear terms and local vorticity boundary condition, the divergence free constraints are automatically satisfied and the Navier Stokes (MHD, respectivley) equation essentially decouples into 2 (4, respectivly) scalar equation and thus the scheme is very efficient. Moreover, with proper discretization of the nonlinear terms, the scheme preserves both energy and helicity identities numerically. This is achieved by recasting the nonlinear terms (convection, vorticity stretching, geometric source, Lorentz force and electro-motive force) in terms of Jacobians. This conservative property even holds true in the presence of the pole singularity for axisymmetric flows. The exact conservation of energy and helicity has effectively eliminated excessive numerical viscosity. Numerical examples have demonstrated both efficiency and accuracy of the scheme. In addition, local mesh refinement near the physical boundary can also be easily incorporated into the scheme without extra cost.

1 Introduction

In the numerical simulation of these flows, it is desirable to have exact numerical conservation of physically conserved quantities such as the energy and the helicity. The conservation of physical quantities not only provides a diagnostic check for physically relevant numerical solutions, it also guarantees that the numerical scheme is nonlinearly stable and free from excessive numerical viscosity. This is essential for large time direct numerical simulations as well as the numerical search for possible flow singularities.

Preserving energy numerically for incompressible Navier Stokes equation has been quite common in many numerical methods. For examples, a well known trick to obtain the conservation of energy is by averaging a conservative and a non-conservative discretization of the nonlinear convection term. However, satisfying numerically two or more physical conservation laws is usually difficult. The classical Arakawa scheme preserves both energy and enstrophy (mean square of the vorticity) for 2D incompressible Euler equation. This result was generalized recently to a high order discontinuous Galerkin method [17]. A strong convergence result is also obtained for this scheme [20] when the initial value of the vorticity is merely square-integrable. Some important flows such as vortex patches belong to this class.

For general three dimensional flows, enstrophy is no longer a conserved quantity. Instead, there is a conservation law for the helicity. Although the discovery of this conservation law (by Moreau in 1961 [24]) is only a recent event, it has played an important role in modern research on vortex dynamics for fluids and plasma. The helicity has an interesting topological interpretation in terms of total circulations and Gauss linking number of two interlocking vortex filaments. A comprehensive review of this subject can be found in Moffatt [22]. Although there is at present no numerical method preserving both helicity and energy, the conservation of the energy and cross helicity for three dimensional MHD has already been obtained in a recent work by the authors [19]. On a set of dual staggered grids, the classic MAC scheme for Navier Stokes equation and Yee's scheme for Maxwell equation is combined with particular care on discretization of the nonlinear terms. The divergence free condition for both the velocity field and magnetic field are maintained in the MAC-Yee scheme.

In this paper, we will first focus on three dimensional flows with coordinate symmetry. Pipe flows and axisymmetric flows are two typical examples. For such symmetric flows, it is possible to introduce a generalized vorticity-stream formulation, thus the divergence free constraint for the fluid velocity is trivially satisfied. Under this vorticity-stream formulation, all the nonlinear terms (convection, vorticity stretching, geometric source, Lorentz force and electro-motive force) for the Navier Stokes and MHD equation can be recast as Jacobians. Associated with these Jacobians we then introduce a trilinear form equipped with a set of permutation identities which leads naturally to the conservation of energy and helicities as well as all the first moments for both the Navier Stokes and MHD equation. We then device a recipe of preserving the permutation identities numerically and hence the energy and helicities. As an illustration, we implement a simple 2nd order finite difference scheme based on centered difference in space and high order Runge Kutta in time. The scheme is very efficient, since the nonlinears term are treated explicitly and a local vorticity boundary condition, namely Thom's formula [27, 7] is applied for time integration, the system essentially decouples into several scalar equations. On the other hand, since the energy and helicities are preserved exactly, there is no excess numerical viscosity introduced thus the scheme is also very accurate. Another advantage of the scheme is the flexibility of choosing coordinate system since our formulation is relatively coordinate-independent. Mesh refinement near the boundary can thus be built into the equation by stretching the coordinate with essentially no extra cost. Our treatment of the nonlinear terms can be generalized to higher order finite difference, finite element and spectral methods in a similar fashion.

In practical implementations of the scheme, there is another difficulty that needs to be resolved. The most natural coordinate systems associated with these symmetric flows often exhibits coordinate singularities such as the symmetry axis in cylindrical coordinate systems and the origin in polar coordinates. Usually these coordinate singularities are treated with artificial pole conditions to insure the stability of the scheme and the smoothness of the solution. Here we overcome this difficulty by shifting the grid points half grid length away from the singularity. Remarkably, the permutation identities and therefore the energy and helicity identities remain valid even in the presence of the pole singularity for axisymmetric flows. The validity of the permutation idenetities gives more than just the conservation of energy and helicities. We give a very simple and elementary error estimate based on the permutation idenetities and local truncation error analysis.

The rest of this paper is organized as follows: In section 2, we recall the energy and helicity identities first for general 3D flows. In section 3, we introduce the generalized vorticity-stream formulation for symmetric flows and derive the expression of the nonlinear trems as Jacobians. In section 4, we introduce the permutation identities for the Jacobian and rederive the energy and helicity identities for Navier Stokes and MHD from these permutation identities. In section 5, we devise our numerical scheme by decretizing the nonlinear terms in such a way that the permutation identities are preserved numerically. We also show how to handle the pole singularity and how to impose the physical boundary conditions that preserve the energy and helicity identities. Finally we give some numerical examples in section 6 and the error analysis in the Appendix.

2 Energy and Helicity Conservation Laws for 3D Flows

For $D \subseteq \mathbb{R}^3$ with boundary ∂D, the incompressible Navier-Stokes equation can be written as:

$$\begin{aligned} \boldsymbol{u}_t + \boldsymbol{\omega} \times \boldsymbol{u} + \nabla \tilde{p} &= -\nu \nabla \times \boldsymbol{\omega} && \text{(momentum)} \\ \nabla \cdot \boldsymbol{u} &= 0 && \text{(incompressibility)} \\ \boldsymbol{u}|_{\partial D} &= 0 && \text{(no-slip B.C.)} \end{aligned} \tag{2.0.1}$$

where $\tilde{p} = p + |\boldsymbol{u}|^2/2$ is the total pressure.

In this form, (2.0.1) involves only elementary grad, div, curl and the cross product of vector fields. Thus (2.0.1) is intrinsic and suitable to work with in any curvilinear orthogonal coordinate system. This gives us the freedom to choose a convenient coordinate system that fits the computational domain if necessary.

The energy identity follows easily from integration by parts the inner product of $\boldsymbol{u}$ with (2.0.1):

$$\frac{d}{dt}\frac{1}{2}\int_D |\boldsymbol{u}|^2 = -\nu \int_D |\boldsymbol{\omega}|^2 \,, \tag{2.0.2}$$

It is also interesting to examine the time evolution of the quantity $\int_D \boldsymbol{u} \cdot \boldsymbol{\omega}$, known as the helicity. This quantity has an intrinsic topological interpretation of the flow. For example, when the flow pattern is two knotted vortex tubes or vortex filaments, the helicity is then equal to $\pm 2n\Phi_1\Phi_2$, where Φ_1 and Φ_2 are the circulation in the cross section of the vortex tubes respectively, and n is the Gauss linking number [22]. For inviscid flow, this quantity is invariant in time. Indeed, we take the curl on both sides of the momentum equation (2.0.1),

$$\boldsymbol{\omega}_t + \nabla \times (\boldsymbol{\omega} \times \boldsymbol{u}) = -\nu \nabla \times \nabla \times \boldsymbol{\omega} \,, \tag{2.0.3}$$

or equivalently

$$(\partial_t + \boldsymbol{u} \cdot \nabla)\boldsymbol{\omega} = (\boldsymbol{\omega} \cdot \nabla)\boldsymbol{u} - \nu \nabla \times \nabla \times \boldsymbol{\omega} \, . \tag{2.0.4}$$

The first term in the right hand side is the vorticity stretching term and is well believed to be the source of possible singularity formation in the flows. A simple computation leads to the following equation

$$(\partial_t + \boldsymbol{u} \cdot \nabla)(\boldsymbol{u} \cdot \boldsymbol{\omega}) + \nabla \cdot [(\tilde{p} - |\boldsymbol{u}|^2)\boldsymbol{\omega}] = \nu(\boldsymbol{\omega} \cdot \triangle \boldsymbol{u} + \boldsymbol{u} \cdot \triangle \boldsymbol{\omega}).$$

If $\partial D(t)$ is any closed surface moving with the fluid on which the condition $\boldsymbol{\nu} \cdot \boldsymbol{\omega}|_{\partial D(t)} = 0$ is satisfied, we have the following conservation laws,

$$\frac{d}{dt} \int_{D(t)} \boldsymbol{u} \cdot \boldsymbol{\omega} = \nu \int_{D(t)} (\boldsymbol{\omega} \cdot \triangle \boldsymbol{u} + \boldsymbol{u} \cdot \triangle \boldsymbol{\omega}) \, . \tag{2.0.5}$$

When $D(t)$ is taken as D, the no-slip boundary condition insures $\boldsymbol{\nu} \cdot \boldsymbol{\omega}|_{\partial D} = 0$, and hence we have the following global helicity conservation law

$$\frac{d}{dt} \frac{1}{2} \int_{D} \boldsymbol{u} \cdot \boldsymbol{\omega} = -\nu \int_{D} \boldsymbol{\omega} \cdot (\nabla \times \boldsymbol{\omega}) \, . \tag{2.0.6}$$

3 Reformulation of Navier Stokes Equation for 3D Symmetric Flows

This paper is motivated by the work of R. Grauer and T. Sideris [12] in the numerical search of possible singularities for the axisymmetric solutions of the Euler equation. For axisymmetric flows, the velocity and the vorticity can be written as

$$\boldsymbol{u} = (0, 0, u) + \nabla \times (0, 0, \psi), \quad \boldsymbol{\omega} = (0, 0, \omega) + \nabla \times (0, 0, u) \, , \tag{3.0.1}$$

and the Euler equation reduces to

$$\begin{aligned} &\partial_t u + (u_x \partial_x + u_r \partial_r) u + \tfrac{u_r}{r} u = 0 \, , \\ &\partial_t \omega + (u_x \partial_x + u_r \partial_r) \omega - \tfrac{u_r}{r} \omega = \tfrac{1}{r} \partial_x (u^2) \, , \\ &-\triangle \psi + \tfrac{1}{r^2} \psi = \omega \, . \end{aligned} \tag{3.0.2}$$

There is a geometric singularity at $r = 0$ (the pole singularity). Handling this geometric singularity is essential in the numerical search for possible singularities in axisymmetric solutions of Navier Stokes equation. We will address the issue of numerical difficulties associated with the pole singularity in section 4. In [12], the authors observed the similarity between (3.0.2) and the 2D Boussinesq equation and conducted a numerical simulation of the 2D Boussinesq equation. This work has stimulate research interest in the numerical simulation of 2D Boussinesq equation [8, 2].

3.1 Generalized Vorticity-Stream Formulation for Symmetric Flows

The formulation (3.0.1,3.0.2) can be generalized to three dimensional flows with coordinate symmetry.

Let $\boldsymbol{X} = (x_1, x_2, x_3)$ be the Cartesian coordinate system and $\boldsymbol{Y} = (y_1, y_2, y_3)$ a curvilinear orthogonal coordinate system with unit vectors $(\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3)$. Denote by h_i, $i = 1, 2, 3$, the local stretching factors given by $d\boldsymbol{X} = \sum_j h_j dy_j \boldsymbol{e}_j$. The 3 basic differential operators are given by

$$\nabla f = (\frac{1}{h_1}\partial_1 f, \frac{1}{h_2}\partial_2 f, \frac{1}{h_3}\partial_3 f) \tag{3.1.1}$$

$$\nabla \cdot \boldsymbol{f} = \frac{1}{h_1 h_2 h_3} \left(\partial_1(h_2 h_3 f_1) + \partial_2(h_1 h_3 f_2) + \partial_3(h_1 h_2 f_3)\right) \tag{3.1.2}$$

$$\nabla \times \boldsymbol{f} = \frac{1}{h_1 h_2 h_3} \begin{vmatrix} h_1\boldsymbol{e}_1 & h_2\boldsymbol{e}_2 & h_3\boldsymbol{e}_3 \\ \partial_1 & \partial_2 & \partial_3 \\ h_1 f_1 & h_2 f_2 & h_3 f_3 \end{vmatrix} \tag{3.1.3}$$

By symmetry, we mean that the physical domain is of the form $D = \Omega \times \mathbb{R}$ or $D = \Omega \times S^1$, that $h_i = h_i(y_1, y_2)$, $i = 1, 2, 3$ and that the solutions are invariant under translation in the y_3 direction. For a symmetric incompressible velocity field $\boldsymbol{u} = \boldsymbol{u}(y_1, y_2)$

$$\nabla \cdot \boldsymbol{u} = \frac{1}{h_1 h_2 h_3} \left(\partial_1(h_2 h_3 u_1) + \partial_2(h_3 h_1 u_2)\right) = 0 \tag{3.1.4}$$

with $\boldsymbol{u} \cdot \boldsymbol{\nu}|_{\partial\Omega} = 0$, we can always introduce a potential ψ, the component of the stream vector in the symmetry direction, such that

$$\partial_2 (h_3 \psi) = h_2 h_3 u_1, \qquad \partial_1 (h_3 \psi) = -h_3 h_1 u_2, \tag{3.1.5}$$

and we can write

$$\boldsymbol{u} = \left(\frac{1}{h_3}\frac{\partial_2(h_3\psi)}{h_2}, -\frac{1}{h_3}\frac{\partial_1(h_3\psi)}{h_1}, u\right) = (0, 0, u) + \nabla \times (0, 0, \psi). \tag{3.1.6}$$

Here $u = u_3$ is the velocity component in the symmetry direction. For axisymmetric flows, u is known as the swirling velocity. Direct computation leads to

$$\nabla \times \nabla \times (0, 0, \psi) = (0, 0, \mathcal{L}\psi)$$

where

$$\mathcal{L}\psi = -\frac{1}{h_1 h_2}\left(\partial_1\left(\frac{h_2}{h_1 h_3}\partial_1(h_3\psi)\right) + \partial_2\left(\frac{h_1}{h_2 h_3}\partial_2(h_3\psi)\right)\right) \tag{3.1.7}$$

Denote by $\omega = -\mathcal{L}\psi$, the vorticity component in the symmetry direction, it follows similarly that

$$\boldsymbol{\omega} = (0, 0, \omega) + \nabla \times (0, 0, u)\,. \tag{3.1.8}$$

We next introduce the following identity:

$$-\mathcal{L}\psi = -\triangle\psi + V\psi \tag{3.1.9}$$

where $\triangle$ is the standard Laplacian in Ω:

$$\triangle\psi = \frac{1}{h_1 h_2 h_3}\left(\partial_1\left(\frac{h_2 h_3}{h_1}\partial_1\psi\right) + \partial_2\left(\frac{h_1 h_3}{h_2}\partial_2\psi\right)\right) = \nabla\cdot\nabla\psi \tag{3.1.10}$$

and V is the geometric source term

$$V = \frac{-1}{h_1 h_2}\left(\partial_1\left(\frac{h_2}{h_1 h_3}\partial_1 h_3\right) + \partial_2\left(\frac{h_1}{h_2 h_3}\partial_2 h_3\right)\right) = h_3\triangle(\frac{1}{h_3}) \tag{3.1.11}$$

We remark here that the identity (3.1.9) plays an important role in the treatment of the pole singularity in conjunction with our spatial discretization (5.2.3) below. In the axisymmetric case, $h_3 = 0$ on the axis of symmetry which poses difficulties in discretizing $\mathcal{L}$. This difficulty disappears with the equivalent operator $\triangle - V$ provided $r = 0$ is not a grid point. See section 5 for details.

Our next crucial observation is to write the nonlinear terms as Jacobians:

$$\boldsymbol{\omega}\times\boldsymbol{u} = \begin{vmatrix} \boldsymbol{e}_1 & \boldsymbol{e}_2 & \boldsymbol{e}_3 \\ \frac{1}{h_3}\frac{\partial_2(h_3 u)}{h_2} & -\frac{1}{h_3}\frac{\partial_1(h_3 u)}{h_1} & w \\ \frac{1}{h_3}\frac{\partial_2(h_3\psi)}{h_2} & -\frac{1}{h_3}\frac{\partial_1(h_3\psi)}{h_1} & u \end{vmatrix}$$

$$= \left(\frac{w}{h_3}\frac{\partial_1(h_3\psi)}{h_1} - \frac{u}{h_3}\frac{\partial_1(h_3 u)}{h_1}, \frac{w}{h_3}\frac{\partial_2(h_3\psi)}{h_2} - \frac{u}{h_3}\frac{\partial_2(h_3 u)}{h_2}, \frac{1}{h_3^2}(\frac{\partial_2(h_3\psi)}{h_2}\frac{\partial_1(h_3 u)}{h_1} - \frac{\partial_2(h_3\psi)}{h_1}\frac{\partial_2(h_3 u)}{h_2})\right).$$

$$(\boldsymbol{\omega}\times\boldsymbol{u})_3 = \frac{1}{h_3^2}\frac{1}{h_1h_2}J\left(h_3u, h_3\psi\right), \tag{3.1.12}$$

and

$$\begin{aligned}
&(\nabla\times(\boldsymbol{\omega}\times\boldsymbol{u}))_3\\
= &\frac{1}{h_1h_2}\begin{vmatrix} \partial_1 & \partial_2 \\ \frac{\omega\partial_1(h_3\psi)}{h_3}-\frac{u\partial_1(h_3u)}{h_3} & \frac{\omega\partial_2(h_3\psi)}{h_3}-\frac{u\partial_2(h_3u)}{h_3}\end{vmatrix}\\
= &\frac{1}{h_1h_2}\left(\partial_1\left(\frac{\omega}{h_3}\right)\partial_2\left(h_3\psi\right)-\partial_2\left(\frac{\omega}{h_3}\right)\partial_1\left(h_3\psi\right)-\partial_1\left(\frac{u}{h_3}\right)\partial_2\left(h_3\psi\right)+\partial_2\left(\frac{u}{h_3}\right)\partial_1\left(h_3u\right)\right)\\
= &\frac{1}{h_1h_2}J\left(\frac{\omega}{h_3}, h_3\psi\right)-\frac{1}{h_1h_2}J\left(\frac{u}{h_3}, h_3u\right)
\end{aligned}$$

Thus, we have the (ψ, u, ω) formulation of the Navier Stokes equation for the three dimensional symmetric flow:

$$\begin{aligned}
&u_t+\tfrac{1}{h_3^2}\tfrac{1}{h_1h_2}J\left(h_3u, h_3\psi\right)=\nu(\triangle-V)u\,,\\
&\omega_t+\tfrac{1}{h_1h_2}J\left(\tfrac{\omega}{h_3}, h_3\psi\right)=\nu(\triangle-V)\omega+\tfrac{1}{h_1h_2}J\left(\tfrac{u}{h_3}, h_3u\right)\,,\\
&\omega=-(\triangle-V)\psi\,,
\end{aligned} \tag{3.1.13}$$

Next, we come to the no-slip boundary condition $\boldsymbol{u}|_\Gamma=\boldsymbol{0}$. Clearly, the outer normal $\boldsymbol{\nu}$ is orthogonal to $\boldsymbol{e}_3$. Let us define $\boldsymbol{\tau}=\boldsymbol{\nu}\times\boldsymbol{e}_3$ and we have

$$\boldsymbol{u}\cdot\boldsymbol{\nu}=\partial_\tau(h_3\psi)=0,\quad \boldsymbol{u}\cdot\boldsymbol{\tau}=\partial_\nu(h_3\psi)=0,\quad \boldsymbol{u}\cdot\boldsymbol{e}_3=u=0$$

When the cross section Ω is simply connected, the no-slip boundary condition takes the form:

$$u=0,\quad \psi=0,\quad \partial_\nu(h_3\psi)=0\quad\text{on}\quad\partial\Omega\,. \tag{3.1.14}$$

4 Permutation identities and Conservation Laws Revisited

The Jacobian $J(a,b)=\nabla a\cdot\nabla^\perp b$ satisfies some nice identities. We can rewrite the Jacobian as

$$\nabla a\cdot\nabla^\perp b=\frac{1}{3}\left\{\nabla a\cdot\nabla^\perp b+\nabla\cdot\left(a\nabla^\perp b\right)+\nabla^\perp\cdot\left(b\nabla a\right)\right\} \tag{4.0.1}$$

We define the trilinear form:

$$T(a,b,c) = \frac{1}{3}\int_\Omega [c(\nabla a \cdot \nabla^\perp b) + a(\nabla b \cdot \nabla^\perp c) + b(\nabla c \cdot \nabla^\perp a)]\,, \tag{4.0.2}$$

the following permutation identities

$$T(a,b,c) = T(b,c,a) = T(c,a,b), \qquad T(a,b,c) = -T(b,a,c)\,, \tag{4.0.3}$$

leads naturally to the conservation laws for energy and helicity, see section 5 below. From (4.0.1) and (4.0.2), it follows that

$$\int_\Omega cJ(a,b) = T(a,b,c) - \int_{\partial\Omega} c(a\partial_\tau b - b\partial_\tau a)\,. \tag{4.0.4}$$

Proposition 1 *Assume a,b,c are sufficient smooth and $c(a\partial_\tau b - b\partial_\tau a) = 0$ on the boundary. Then*

$$\int_\Omega cJ(a,b) = T(a,b,c)\,. \tag{4.0.5}$$

The assumption in Proposition 1 is valid on the physical boundary provided at least one of a, b or c is either ψ or u. It is also valid on the axis of rotation for axisymmetric flows as all dependent variables are the swirling components of axisymmetric vector fields and satisfy odd extension across the axis of rotation. See also (5.2.4).

Next we express the energy and helicity identity in terms of u and ω. We first define the weighted L^2 and H^1 inner products

$$\langle \phi^1, \phi^2 \rangle = \int_\Omega h_1 h_2 h_3 \phi^1(y_1,y_2)\phi^2(y_1,y_2)\, dy_1 dy_2\,, \tag{4.0.6}$$

and

$$[\phi^1, \phi^2] = \langle \frac{1}{h_1}\partial_1\phi^1, \frac{1}{h_1}\partial_1\phi^2\rangle + \langle \frac{1}{h_2}\partial_2\phi^1, \frac{1}{h_2}\partial_2\phi^2\rangle + \langle \phi^1, V\phi^2\rangle\,. \tag{4.0.7}$$

It is useful to point out here that $\langle \phi^1, (-\triangle + V)\phi^2\rangle = [\phi^1, \phi^2]$ provided either $\phi^1 = 0$ or $\partial_\nu \phi^2 = 0$ on $\partial\Omega$.

Since $\boldsymbol{u}$ and h_i are independent of y_3, it suffice to consider the energy and helicity identities on the cross section Ω:

$$\int_\Omega |\boldsymbol{u}|^2 h_1 h_2 h_3 dy_1 dy_2 = \langle \begin{pmatrix} 0 \\ 0 \\ u \end{pmatrix} + \nabla \times \begin{pmatrix} 0 \\ 0 \\ \psi \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ u \end{pmatrix} + \nabla \times \begin{pmatrix} 0 \\ 0 \\ \psi \end{pmatrix} \rangle$$

$$= \langle \begin{pmatrix} 0 \\ 0 \\ u \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ u \end{pmatrix} \rangle + \langle \nabla \times \begin{pmatrix} 0 \\ 0 \\ \psi \end{pmatrix}, \nabla \times \begin{pmatrix} 0 \\ 0 \\ \psi \end{pmatrix} \rangle = \langle u, u \rangle + [\psi, \psi]$$

Similarly,

$$\int_\Omega |\boldsymbol{\omega}|^2 h_1 h_2 h_3 dy_1 dy_2 = [u, u] + \langle \omega, \omega \rangle$$

$$\int_\Omega \boldsymbol{u} \cdot \boldsymbol{\omega} h_1 h_2 h_3 dy_1 dy_2 = \langle \begin{pmatrix} 0 \\ 0 \\ u \end{pmatrix} + \nabla \times \begin{pmatrix} 0 \\ 0 \\ \psi \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ \omega \end{pmatrix} + \nabla \times \begin{pmatrix} 0 \\ 0 \\ u \end{pmatrix} \rangle$$

$$= \langle \begin{pmatrix} 0 \\ 0 \\ u \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ \omega \end{pmatrix} \rangle + \langle \nabla \times \begin{pmatrix} 0 \\ 0 \\ \psi \end{pmatrix}, \nabla \times \begin{pmatrix} 0 \\ 0 \\ u \end{pmatrix} \rangle$$

Since $u = 0$ on the physical boundary, we can integrate by parts the second term to get

$$\langle \nabla \times \begin{pmatrix} 0 \\ 0 \\ \psi \end{pmatrix}, \nabla \times \begin{pmatrix} 0 \\ 0 \\ u \end{pmatrix} \rangle = \langle \nabla \times \nabla \times \begin{pmatrix} 0 \\ 0 \\ \psi \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ u \end{pmatrix} \rangle = \langle \begin{pmatrix} 0 \\ 0 \\ \omega \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ u \end{pmatrix} \rangle$$

Therefore

$$\int_\Omega \boldsymbol{u} \cdot \boldsymbol{\omega} h_1 h_2 h_3 dy_1 dy_2 = 2\langle u, \omega \rangle$$

and

$$\int_D \boldsymbol{\omega} \cdot (\nabla \times \boldsymbol{\omega}) = \langle \begin{pmatrix} 0 \\ 0 \\ \omega \end{pmatrix} + \nabla \times \begin{pmatrix} 0 \\ 0 \\ u \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ -(\triangle - V)u \end{pmatrix} + \nabla \times \begin{pmatrix} 0 \\ 0 \\ \omega \end{pmatrix} \rangle = [u, \omega] - \langle \omega, (\triangle - V)u \rangle$$

We conclude with the energy and helicity identity in terms of u and ω:

$$\frac{d}{dt}\frac{1}{2}(\langle u, u \rangle + [\psi, \psi]) + \nu([u, u] + \langle \omega, \omega \rangle) = 0\,. \tag{4.0.8}$$

and

$$\frac{d}{dt}\langle u, \omega \rangle + \nu([u, \omega] - \langle \omega, \triangle_2 u \rangle) = 0\,. \tag{4.0.9}$$

Here we give an alternative derivation of (4.0.8, 4.0.9). We take the weighted inner product of the first equation in (2.0.1) with υ, the second with φ to get

$$\langle \upsilon, \partial_t u \rangle + T(h_3 u, h_3 \psi, \upsilon / h_3) = \nu \langle \upsilon, (\triangle - V)u \rangle$$

$$[\varphi, \partial_t \psi] + T(\omega / h_3, h_3 \psi, h_3 \varphi) = \nu \langle \varphi, (\triangle - V)\omega \rangle + T(u / h_3, h_3 u, h_3 \varphi) \tag{4.0.10}$$

$$\langle \xi, \omega \rangle = [\xi, \psi]$$

In view of (4.0.10), the conservation laws (4.0.8, 4.0.9) follow easily from the permutation identities (4.0.3) by taking $(v, \varphi) = (u, \psi)$ and (ω, u) respectively. We will derive a discrete analogue of the permutation identities (4.0.3) with a proper discretization of the nonlinear terms and implementation of boundary conditions. The discrete analogue of (4.0.8, 4.0.9) then follows as a direct consequence. See (5.1.15, 5.1.16, 5.1.17).

4.1 3D MHD Equations

The 3D dimensional MHD equation

$$\begin{aligned}
&\boldsymbol{u}_t + \boldsymbol{\omega} \times \boldsymbol{u} + \nabla p = -\nu \nabla \times \boldsymbol{\omega} + \alpha \boldsymbol{\jmath} \times \boldsymbol{b} \\
&\nabla \cdot \boldsymbol{u} = \boldsymbol{0} \\
&\boldsymbol{b}_t = -\eta \nabla \times \boldsymbol{\jmath} + \nabla \times (\boldsymbol{u} \times \boldsymbol{b}) \\
&\boldsymbol{\omega} = \nabla \times \boldsymbol{u}, \quad \boldsymbol{\jmath} = \nabla \times \boldsymbol{b},
\end{aligned} \tag{4.1.1}$$

with the no-slip and perfectly conducting wall conditions

$$\boldsymbol{u} = \boldsymbol{0}, \quad \boldsymbol{\jmath} \times \boldsymbol{\nu} = \boldsymbol{0} \quad \text{on } \partial D \tag{4.1.2}$$

shares some similarity with the Navier Stokes equation in the structure of the nonlinear terms. Here $\boldsymbol{u}$ is the fluid velocity, $\boldsymbol{\omega}$ is the vorticity, p is the total pressure, $\boldsymbol{b}$ is the magnetic field and $\boldsymbol{\jmath}$ is the electric current density. The parameters ν^{-1}, η^{-1} and $\alpha^{-1/2}$ are usually referred to as the fluid Reynolds number, the magnetic Reynolds number and the Alfven number respectively.

The conservation of energy and cross helicity follows easily from elementary calculations:

$$\int_\Omega \boldsymbol{u} \cdot [\boldsymbol{u}_t + \boldsymbol{\omega} \times \boldsymbol{u} + \nabla p] = \int_\Omega \boldsymbol{u} \cdot [-\nu \nabla \times \boldsymbol{\omega} + \alpha \boldsymbol{\jmath} \times \boldsymbol{b}] \tag{4.1.3}$$

$$\alpha \int_\Omega \boldsymbol{b} \cdot \boldsymbol{b}_t = \alpha \int_\Omega \boldsymbol{b} \cdot [-\eta \nabla \times \boldsymbol{\jmath} + \nabla \times (\boldsymbol{u} \times \boldsymbol{b})] \tag{4.1.4}$$

and the energy identity follows:

$$\frac{d}{dt} \frac{1}{2} \int_\Omega (|\boldsymbol{u}|^2 + \alpha |\boldsymbol{b}|^2) = -\nu \int_\Omega |\boldsymbol{\omega}|^2 - \alpha\eta \int_\Omega |\boldsymbol{\jmath}|^2 \tag{4.1.5}$$

Similarly,

$$\int_\Omega \boldsymbol{b} \cdot [\boldsymbol{u}_t + \boldsymbol{\omega} \times \boldsymbol{u} + \nabla p] = \int_\Omega \boldsymbol{b} \cdot [-\nu \nabla \times \boldsymbol{\omega} + \alpha \boldsymbol{\jmath} \times \boldsymbol{b}] \tag{4.1.6}$$

$$\int_\Omega \boldsymbol{u} \cdot \boldsymbol{b}_t = \int_\Omega \boldsymbol{u} \cdot [-\eta \nabla \times \boldsymbol{\jmath} + \nabla \times (\boldsymbol{u} \times \boldsymbol{b})] \tag{4.1.7}$$

and there follows the conservation of cross helicity:

$$\frac{d}{dt} \int_\Omega \boldsymbol{u} \cdot \boldsymbol{b} + \int_{\partial\Omega} p \boldsymbol{b} \cdot \boldsymbol{\nu} = -\nu \int_\Omega \boldsymbol{b} \cdot (\nabla \times \boldsymbol{\omega}) - \eta \int_\Omega \boldsymbol{\omega} \cdot \boldsymbol{\jmath} \tag{4.1.8}$$

The conservation of magnetic helicity involves the vector potential. By rewriting the Faraday equation as

$$\partial_t \boldsymbol{b} + \nabla \times \boldsymbol{e} = \mathbf{0}, \quad \boldsymbol{e} = \eta \boldsymbol{\jmath} - \boldsymbol{u} \times \boldsymbol{b}, \tag{4.1.9}$$

it is clear that the Faraday equation admits a potential formulation given by

$$\boldsymbol{a}_t = -\boldsymbol{e} + \nabla \chi, \quad \boldsymbol{a}|_{t=0} = \boldsymbol{a}_0, \quad \nabla \times \boldsymbol{a}_0 = \boldsymbol{b}_0 \tag{4.1.10}$$

with an arbitrary gauge function χ. Therefore

$$\partial_t (\boldsymbol{b} - \nabla \times \boldsymbol{a}) = \mathbf{0}, \quad \text{or} \quad \boldsymbol{b} = \nabla \times \boldsymbol{a} \tag{4.1.11}$$

If we further restrict $\chi|_{\partial\Omega} = 0$, then the magnetic helicity $\mathcal{M} = \frac{1}{2} \int_\Omega \boldsymbol{a} \cdot \boldsymbol{b}$ is gauge invariant and

$$\frac{d}{dt} \mathcal{M} = \frac{1}{2} \int_\Omega (\boldsymbol{a}_t \cdot \boldsymbol{b} + \boldsymbol{a} \cdot \boldsymbol{b}_t) = \frac{1}{2} \int_\Omega [(-\boldsymbol{e} + \nabla \chi) \cdot \boldsymbol{b} - \boldsymbol{a} \cdot (\nabla \times \boldsymbol{e})]. \tag{4.1.12}$$

After integration by parts using the boundary conditions $\chi = 0$ and $\boldsymbol{e} \times \boldsymbol{\nu} = \mathbf{0}$, we have

$$\frac{1}{2} \int_\Omega [(-\boldsymbol{e} + \nabla \chi) \cdot \boldsymbol{b} - \boldsymbol{a} \cdot (\nabla \times \boldsymbol{e})] = -\int_\Omega \boldsymbol{e} \cdot \boldsymbol{b} \tag{4.1.13}$$

Thus the conservation of magnetic helicity follows

$$\frac{d}{dt} \mathcal{M} = -\int_\Omega (\boldsymbol{b} \times \boldsymbol{u} + \eta \boldsymbol{\jmath}) \cdot \boldsymbol{b} = -\eta \int_\Omega \boldsymbol{b} \cdot \boldsymbol{\jmath} \tag{4.1.14}$$

For symmetric MHD, we can similarly write

$$\begin{aligned}
\boldsymbol{u} &= (0,0,u) + \nabla \times (0,0,\psi) \\
\boldsymbol{\omega} &= (0,0,\omega) + \nabla \times (0,0,u), \quad \omega = -(\triangle - V)\psi \\
\boldsymbol{b} &= (0,0,b) + \nabla \times (0,0,a) \\
\boldsymbol{\jmath} &= (0,0,\jmath) + \nabla \times (0,0,b), \quad \jmath = -(\triangle - V)a
\end{aligned} \tag{4.1.15}$$

and reformulate all the nonlinear terms as Jacobians:

$$\begin{aligned} (\boldsymbol{\omega}\times\boldsymbol{u})_3 &= \frac{1}{h_3^2}\frac{1}{h_1h_2}J\left(h_3u,h_3\psi\right)\\ (\boldsymbol{\jmath}\times\boldsymbol{b})_3 &= \frac{1}{h_3^2}\frac{1}{h_1h_2}J\left(h_3b,h_3a\right)\\ (\boldsymbol{u}\times\boldsymbol{b})_3 &= \frac{1}{h_3^2}\frac{1}{h_1h_2}J\left(h_3\psi,h_3a\right) \end{aligned} \tag{4.1.16}$$

$$\begin{aligned} (\nabla\times(\boldsymbol{\omega}\times\boldsymbol{u}))_3 &= \frac{1}{h_1h_2}J\left(\frac{\omega}{h_3},h_3\psi\right)-\frac{1}{h_1h_2}J\left(\frac{u}{h_3},h_3u\right)\\ (\nabla\times(\boldsymbol{\jmath}\times\boldsymbol{b}))_3 &= \frac{1}{h_1h_2}J\left(\frac{\jmath}{h_3},h_3a\right)-\frac{1}{h_1h_2}J\left(\frac{b}{h_3},h_3b\right)\\ (\nabla\times(\boldsymbol{u}\times\boldsymbol{b}))_3 &= \frac{1}{h_1h_2}J\left(\frac{u}{h_3},h_3a\right)-\frac{1}{h_1h_2}J\left(\frac{b}{h_3},h_3\psi\right) \end{aligned} \tag{4.1.17}$$

and the 3D symmetric MHD takes the form:

$$\begin{aligned} &\partial_t u+\frac{1}{h_3^2}\frac{1}{h_1h_2}J\left(h_3u,h_3\psi\right)=\nu(\triangle-V)u+\frac{\alpha}{h_3^2}\frac{1}{h_1h_2}J\left(h_3b,h_3a\right)\\ &\partial_t\omega+\frac{1}{h_1h_2}J\left(\frac{\omega}{h_3},h_3\psi\right)-\frac{1}{h_1h_2}J\left(\frac{u}{h_3},h_3u\right)=\nu(\triangle-V)\omega+\frac{\alpha}{h_1h_2}J\left(\frac{\jmath}{h_3},h_3a\right)-\frac{\alpha}{h_1h_2}J\left(\frac{b}{h_3},h_3b\right)\\ &\omega=-(\triangle-V)\psi\\ &\partial_t a=\eta(\triangle-V)a+\frac{1}{h_3^2}\frac{1}{h_1h_2}J\left(h_3\psi,h_3a\right)\\ &\partial_t b=\eta(\triangle-V)b+\frac{1}{h_1h_2}J\left(\frac{u}{h_3},h_3a\right)-\frac{1}{h_1h_2}J\left(\frac{b}{h_3},h_3\psi\right)\\ &\jmath=-(\triangle-V)a \end{aligned} \tag{4.1.18}$$

where we have chosen the zero gauge $\chi=0$ for convenience.

On a simply connected Ω, the perfectly conducting wall conditions $\boldsymbol{\jmath}\times\boldsymbol{\nu}=\mathbf{0}$ is given by

$$\jmath=0,\quad \partial_\nu(h_3b)=0\quad\text{on}\quad\partial\Omega \tag{4.1.19}$$

Since a is a computational variables, it is convenient to take the alternative form of $\jmath=0$ in terms of a, namely $\partial_t a=0$ which follows easily from the no-slip condition and the Faraday equation. Therefore we have the boundary condition for the symmetric MHD:

$$u=0,\quad \psi=0,\quad \partial_\nu(h_3\psi)=0,\quad \partial_t a=0,\quad \partial_\nu(h_3b)=0\quad\text{on}\quad\partial\Omega \tag{4.1.20}$$

This is also consistent with the boundary constraint $\partial_t(\boldsymbol{b}\cdot\boldsymbol{\nu})=0$ which is a direct consequence of the Faraday equation.

We can similarly express the conservation laws for symmetric MHD (4.1.5, 4.1.8, 4.1.14) in terms of the computational variables by

$$\frac{d}{dt}\frac{1}{2}(\langle u,u\rangle + [\psi,\psi] + \alpha\langle b,b\rangle + \alpha[a,a]) + \nu([u,u] + \langle\omega,\omega\rangle) + \eta\alpha([\jmath,\jmath] + \langle b,b\rangle) = 0 \quad (4.1.21)$$

$$\begin{aligned} & \tfrac{d}{dt}(\langle u,b\rangle + \langle\omega,a\rangle) \\ = \; & \nu(\langle b,(\triangle - V)u\rangle + \langle a,(\triangle - V)\omega\rangle) + \alpha\eta\left([u,b] + \langle\omega,(\triangle - V)u\rangle\right) \\ & + \tfrac{\alpha}{3}\int_{\partial\Omega}(b^2\partial_\tau(h_3 a) - ab\partial_\tau(h_3 b))\, ds \end{aligned} \quad (4.1.22)$$

and

$$\frac{d}{dt}\langle a,b\rangle + \eta[a,b] = 0. \quad (4.1.23)$$

5 Finite Difference Scheme and Axisymmetric Flows

5.1 Finite Difference Method, Discrete Permutation Identities, and Energy and Helicity Conservation Laws

With the standards notation:

$$D_1\phi(y_1,y_2) = \frac{\phi(y_1+\Delta y_1/2, y_2) - \phi(y_1 - \Delta y_1/2, y_2)}{\Delta y_1}. \quad (5.1.1)$$

$$\tilde{D}_1\phi(y_1,y_2) = \frac{\phi(y_1+\Delta y_1, y_2) - \phi(y_1 - \Delta y_1, y_2)}{2\Delta y_1}. \quad (5.1.2)$$

$$\tilde{\nabla}_h = \begin{pmatrix} \tilde{D}_1 \\ \tilde{D}_2 \end{pmatrix}, \quad \tilde{\nabla}_h^\perp = \begin{pmatrix} -\tilde{D}_2 \\ \tilde{D}_1 \end{pmatrix}, \quad (5.1.3)$$

the finite difference approximation of $\triangle$ and the Jacobians are given by

$$\triangle_h\psi = \frac{1}{h_1h_2h_3}\left(D_1\left(\frac{h_2h_3}{h_1}D_1\psi\right) + D_2\left(\frac{h_1h_3}{h_2}D_2\psi\right)\right) \quad (5.1.4)$$

and

$$J_h(f,g) = \frac{1}{3}\left\{\tilde{\nabla}_h f\cdot\tilde{\nabla}_h^\perp g + \tilde{\nabla}_h\cdot(f\tilde{\nabla}_h^\perp g) + \tilde{\nabla}_h^\perp\cdot(g\tilde{\nabla}_h f)\right\} \quad (5.1.5)$$

Altogether, we have our finite difference approximation of Navier Stokes equation:

$$\begin{aligned} & \partial_t u + \tfrac{1}{h_3^2}\tfrac{1}{h_1h_2}J_h(h_3u, h_3\psi) = \nu(\triangle_h - V)u \\ & \partial_t\omega + \tfrac{1}{h_1h_2}J_h\left(\tfrac{\omega}{h_3}, h_3\psi\right) = \nu(\triangle_h - V)\omega + \tfrac{1}{h_1h_2}J_h\left(\tfrac{u}{h_3}, h_3u\right) \\ & \omega = (-\triangle_h + V)\psi \end{aligned} \quad (5.1.6)$$

With the discretization given by (5.1.4, 5.1.5), we can easily recover the permutation identities (4.0.3) and hence preserve the energy and helicity identities numerically. To see this, let us first look at the quasi-2D flow with $\Omega = R^2$ or T^2. We begin with the following identity:

$$\sum_{j=1}^{N-1} f_j \left(g_{j+1} - g_{j-1}\right) = -\sum_{j=1}^{N-1} \left(f_{j+1} - f_{j-1}\right) g_j + f_{N-1} g_N + f_N g_{N-1} - f_0 g_1 - f_1 g_0 \tag{5.1.7}$$

When there is no boundary contribution, we can simply write (5.1.7) as

$$\sum_j f_j \left(g_{j+1} - g_{j-1}\right) = -\sum_j \left(f_{j+1} - f_{j-1}\right) g_j \tag{5.1.8}$$

and hence

$$\Delta y_1 \Delta y_2 \sum_j \sum_i c \tilde{\nabla}_h \cdot (a \tilde{\nabla}_h^\perp b) = -\Delta y_1 \Delta y_2 \sum_{i,j} a \tilde{\nabla}_h c \cdot \tilde{\nabla}_h^\perp b \tag{5.1.9}$$

$$\Delta y_1 \Delta y_2 \sum_i \sum_j c \tilde{\nabla}_h^\perp \cdot (b \tilde{\nabla}_h a) = -\Delta y_1 \Delta y_2 \sum_{i,j} b \tilde{\nabla}_h^\perp c \cdot \tilde{\nabla}_h a \tag{5.1.10}$$

Therefore

$$\sum_{i,j} c_{i,j} J_h(a,b)_{i,j} = \frac{1}{3} \sum_{i,j} \left(c \tilde{\nabla}_h a \cdot \tilde{\nabla}_h^\perp b + a \tilde{\nabla}_h b \cdot \tilde{\nabla}_h^\perp c + b \tilde{\nabla}_h c \cdot \tilde{\nabla}_h^\perp a \right)_{i,j} \equiv T_h(a,b,c)\,. \tag{5.1.11}$$

and the discrete analogue of the permutation identity (4.0.3) follows.

As to the viscous terms, we define the weighted inner products by

$$\langle \phi, \psi \rangle_h = \sum_{i,j} (h_1 h_2 h_3 \phi \psi)_{i,j} \, \Delta y_1 \Delta y_2 \tag{5.1.12}$$

$$\begin{aligned} [\phi, \psi]_h \;\; &= \sum_{i,j} \left(\frac{h_2 h_3}{h_1} (D_1 \phi)(D_1 \psi) \right)_{i-1/2,j} \Delta y_1 \Delta y_2 \\ &+ \sum_{i,j} \left(\frac{h_1 h_3}{h_2} (D_2 \phi)(D_2 \psi) \right)_{i,j-1/2} \Delta y_1 \Delta y_2 \\ &+ \langle \phi, V \psi \rangle_h \end{aligned} \tag{5.1.13}$$

It is easy to see that

$$\langle \phi, \triangle_h \psi \rangle_h = -[\phi, \psi]_h \tag{5.1.14}$$

when there is no boundary terms involved.

Upon taking the weighted inner product of the first equation in (5.1.6) with υ, the second with φ, it follows that

$$\langle \upsilon, \partial_t u\rangle_h + T_h(h_3 u, h_3\psi, \upsilon/h_3) = \nu\langle \upsilon, (\triangle_h - V)u\rangle_h$$

$$[\varphi, \partial_t\psi]_h + T_h(\omega/h_3, h_3\psi, h_3\varphi) = \nu\langle\varphi, (\triangle_h - V)\omega\rangle_h + T_h(u/h_3, h_3 u, h_3\varphi) \tag{5.1.15}$$

$$\langle \xi, \omega\rangle_h = [\xi, \psi]_h$$

and we get the discrete energy identity

$$\frac{d}{dt}\frac{1}{2}(\langle u, u\rangle_h + [\psi, \psi]_h) + \nu([u, u]_h + \langle\omega, \omega\rangle_h) = 0 \tag{5.1.16}$$

by taking $\upsilon = u$, $\varphi = \psi$ in (5.1.15). Also the discrete helicity identity

$$\frac{d}{dt}\langle u, \omega\rangle_h + \nu([u, \omega]_h - \langle\omega, (\triangle_h - V)u\rangle_h) = 0 \tag{5.1.17}$$

follows by taking $\upsilon = \omega$, $\varphi = u$.

Remark 1 *(a) In the 2D case, the approximation $J_h(a, b)$ is equivalent to the classical Arakawa scheme [1].*

(b) A straight forward generalization to finite element and spectral Galerkin method is to define the discrete approximation of the nonlinear term $J_h(a, b)$ through its pairing (the weighted inner product) with the test function c by

$$\langle c, J_h(a, b)\rangle \equiv T(a, b, c)$$

The permutation identities are preserved and hence the energy and helicities. The details will be reported elsewhere.

In the next subsection, we proceed with the treatment of the pole singularity in axisymmetric flow.

5.2 Polar Coordinate System and the Pole Singularity

For axisymmetric flows, the cylindrical coordinate system $(y_1, y_2, y_3) = (x, r, \theta)$ with $r^2 = y^2 + z^2$ and $\theta = \arctan(z/y)$ is a natural one. In this case, we have $(h_1, h_2, h_3) = (1, 1, r)$ and the Navier Stokes equation can be written as

$$\begin{aligned} &u_t + \tfrac{1}{r^2} J\left(ru, r\psi\right) = \nu(\triangle - \tfrac{1}{r^2})u\,, \\ &\omega_t + J\left(\tfrac{\omega}{r}, r\psi\right) = \nu(\triangle - \tfrac{1}{r^2})\omega + J\left(\tfrac{u}{r}, ru\right)\,, \\ &\omega = (-\triangle + \tfrac{1}{r^2})\psi\,, \end{aligned} \tag{5.2.1}$$

In [3], it is shown that the only possible singularity for axisymmetric flow of Navier Stokes equation is on the axis. Therefore it is desirable to have local mesh refinement near the axis. This is achievable by a simple stretching of coordinate system. For example, take

$$(y_1, y_2, y_3) = (x, s, \theta)$$

with $s = r^{1/2} = (y^2 + z^2)^{1/4}$. The stretching factor now becomes $(h_1, h_2, h_3) = (1, 2s, s^2)$ and the Navier Stokes equation reads

$$\begin{aligned} &u_t + \tfrac{1}{2s^5} J\left(s^2u, s^2\psi\right) = \nu(\triangle - \tfrac{1}{s^4})u\,, \\ &\omega_t + \tfrac{1}{2s} J\left(\tfrac{\omega}{s^2}, s^2\psi\right) = \nu(\triangle - \tfrac{1}{s^4})\omega + \tfrac{1}{2s} J\left(\tfrac{u}{s^2}, s^2u\right)\,, \\ &\omega = (-\triangle + \tfrac{1}{s^4})\psi\,, \end{aligned} \tag{5.2.2}$$

In either (5.2.1) or (5.2.2), the Navier Stokes equation has pole singularity $1/r$ ($1/s$, respectively) at the axis of rotation, a simple and effective treatment for finite difference scheme is to shift the grids half grid size off the axis to avoid placing the grid point on the pole.

$$y_2(j) = (j - \frac{1}{2})\Delta y_2, \quad j = 0, 1, 2, \cdots \tag{5.2.3}$$

That is, $r_j = (j - \frac{1}{2})\Delta r$, $j = 0, 1, 2, \cdots$ in (x, r, θ) coordinates and $s_j = (j - \frac{1}{2})\Delta s$, $j = 0, 1, 2, \cdots$ in (x, s, θ) coordinates.

Since u, ψ, ω are the swirling components of $\boldsymbol{u}, \boldsymbol{\psi}, \boldsymbol{\omega}$, they satisfy the reflection boundary condition, namely, odd extension across the axis of rotation:

$$u(i, 0) = -u(i, 1), \quad \psi(i, 0) = -\psi(i, 1), \quad \omega(i, 0) = -\omega(i, 1), \tag{5.2.4}$$

and

$$h_1(0, j) = h_1(1, j), \quad h_2(0, j) = h_2(1, j), \quad h_3(0, j) = h_3(1, j), \tag{5.2.5}$$

for the local stretching factors.

If we denote by j the index in the direction parallel to the axis, it follows from (5.1.7) that

$$\sum_{j=1}^{\infty} f_j(g_{j+1} - g_{j-1}) = -\sum_{j=1}^{\infty} g_j(f_{j+1} - f_{j-1}) - (f_0 g_1 + g_0 f_1). \tag{5.2.6}$$

When we repeat the procedure outlined in (5.1.8-5.1.11), we will encounter the boundary contribution at the pole: $\sum_i (f_{i,0} g_{i,1} + g_{i,0} f_{i,1})$, with $f = c$ and $g = b\tilde{D}_x a - a\tilde{D}_x b$. In view of the reflection boundary condition (5.2.4)

$$f_{i,0} = -f_{i,1}, \quad g_{i,0} = g_{i,1}. \tag{5.2.7}$$

The boundary contribution at the pole drops out automatically and consequently the permutation identities is valid even in the presence of the pole singularity. Since $r_{\frac{1}{2}} = s_{\frac{1}{2}} = 0$, it follows that (5.1.14) remains valid for axisymmetric flow in view of following identity

$$\sum_{j=1}^{\infty} f_j(g_{j+1/2} - g_{j-1/2}) = -{\sum_{j=1}^{\infty}}' (f_j - f_{j-1}) g_{j-1/2} - \frac{1}{2}(f_1 + f_0) g_{1/2} \tag{5.2.8}$$

where

$${\sum_{j=1}^{\infty}}' = \frac{1}{2}\sum_{j=1} + \sum_{j=2}^{\infty}$$

As a consequence, the discrete energy and helicity identity (5.1.16, 5.1.17) remain valid for axisymmetric flow in the whole space.

5.3 Treatment of Physical Boundary Conditions

In order to preserve the conservation laws for the energy and helicity in the presence of the physical boundary, the no-slip boundary condition needs to be realized in a proper way. We consider the flow confined in a cylinder $\{x_{min} < x < x_{max}, 0 < r < r_{max}\}$, and let i be the index in the axial direction. From (5.1.7)

$$\sum_{i=1}^{M-1} f_i\left(g_{i+1} - g_{i-1}\right) = -\sum_{i=1}^{M-1}\left(f_{i+1} - f_{i-1}\right) g_i + f_{M-1} g_M + f_M g_{M-1} - f_0 g_1 - f_1 g_0, \tag{5.3.1}$$

it follows that if we place the physical boundary in the middle of grid points

$$x_{\frac{1}{2}} = x_{min}, \cdots, x_{M-\frac{1}{2}} = x_{max}, \qquad r_{N-\frac{1}{2}} = r_{max} \tag{5.3.2}$$

a second order approximation for two of the no-slip boundary condition

$$\psi = \partial_\nu \psi = 0$$

is realized by simply imposing

$$\psi_{0,j} = \psi_{1,j} = 0, \qquad \psi_{M-1,j} = \psi_{M,j} = 0, \qquad \psi_{i,N-1} = \psi_{i,N} = 0 \tag{5.3.3}$$

Together with

$$u_{0,j} + u_{1,j} = 0, \qquad u_{M-1,j} + u_{M,j} = 0, \qquad u_{i,N-1} + u_{i,N} = 0 \tag{5.3.4}$$

as a second order approximation of the third no slip condition $u = 0$.

It is also easy to see that the boundary contributions in the permutation identity drops out. Indeed, by introducing the convolution operator

$$(f * g)_{i-\frac{1}{2}} = \frac{1}{2}(f_{i-1}g_i + f_i g_{i-1})$$

we can write (5.3.1) as

$$\sum_{i=1}^{M-1} f_i \left(g_{i+1} - g_{i-1}\right) = -\sum_{i=1}^{M-1} \left(f_{i+1} - f_{i-1}\right) g_i + 2(f * g)_{M-\frac{1}{2}} - 2(f * g)_{\frac{1}{2}} \tag{5.3.5}$$

therefore

$$\begin{aligned} & \Delta x \Delta r \sum_j \sum_{i=1}^{M-1} \left(c\tilde{D}_x(a\tilde{D}_r b)\right)_{i,j} \\ = \quad & -\Delta x \Delta r \sum_j \sum_{i=1}^{M-1} \left(a(\tilde{D}_x c)(\tilde{D}_r b)\right)_{i,j} + \Delta r \left(\sum_j (c *_x a\tilde{D}_r b)_{M-\frac{1}{2},j} - (c *_x a\tilde{D}_r b)_{\frac{1}{2},j}\right) \end{aligned} \tag{5.3.6}$$

where $*_x$ denotes convolution in x direction. Similarly,

$$\begin{aligned} & \Delta x \Delta r \sum_i \sum_{j=1}^{N-1} \left(c\tilde{D}_r(a\tilde{D}_x b)\right)_{i,j} \\ = \quad & -\Delta x \Delta r \sum_i \sum_{j=1}^{N-1} \left(a(\tilde{D}_r c)(\tilde{D}_x b)\right)_{i,j} + \Delta x \left(\sum_i (c *_r a\tilde{D}_x b)_{i,N-\frac{1}{2}} - (c *_r a\tilde{D}_x b)_{i,\frac{1}{2}}\right) \end{aligned} \tag{5.3.7}$$

From (5.3.6) and (5.3.7) it follows that

$$\Delta x\Delta r\sum_{i,j} c\tilde{\nabla}_h\cdot(a\tilde{\nabla}_h^{\perp}b) = -\Delta x\Delta r\sum_{i,j} a\tilde{\nabla}_h c\cdot\tilde{\nabla}_h^{\perp}b - \Delta y_\tau\sum_{\Gamma_h}\left(c *_\nu (a\tilde{D}_\tau b)\right) \tag{5.3.8}$$

$$\Delta x\Delta r\sum_{i,j} c\tilde{\nabla}_h^{\perp}\cdot(b\tilde{\nabla}_h a) = -\Delta x\Delta r\sum_{i,j} b\tilde{\nabla}_h^{\perp} c\cdot\tilde{\nabla}_h a + \Delta y_\tau\sum_{\Gamma_h}\left(c *_\nu (b\tilde{D}_\tau a)\right) \tag{5.3.9}$$

where for brevity, we have used $*_\nu$ to denote the convolution in the normal direction and y_τ the variable in the tangential direction.

We have the discrete analogue of (4.0.4):

$$\begin{aligned}\sum_{i,j} cJ_h(a,b)\Delta x\Delta r = \ & \sum_{i,j}(c\tilde{\nabla}_h a\cdot\tilde{\nabla}_h^{\perp}b + a\tilde{\nabla}_h b\cdot\tilde{\nabla}_h^{\perp}c + b\tilde{\nabla}_h c\cdot\tilde{\nabla}_h^{\perp}a)\Delta x\Delta r \\ & +\tfrac{1}{3}\sum_{\Gamma_h}\left(c *_\nu (a\tilde{D}_\tau b - b\tilde{D}_\tau a)\right)\Delta y_\tau\end{aligned} \tag{5.3.10}$$

For the discrete energy identity (5.1.16), the boundary contribution from the 3 nonlinear terms

$$\frac{1}{3}\sum_{\Gamma_h}\left(c *_\nu (a\tilde{D}_\tau b - b\tilde{D}_\tau a)\right) \tag{5.3.11}$$

corresponds to $(a,b,c) = (h_3u, h_3\psi, u/h_3)$, $(\omega/h_3, h_3\psi, h_3\psi)$ and $(u/h_3, h_3u, h_3\psi)$ respectively. From (5.3.3), the convolutions involving ψ drop out automatically. For the same reason, the only boundary contribution from the nonlinear terms in the derivation of the discrete helicity identity corresponds to $(a,b,c) = (u/h_3, h_3u, h_3u)$. This term is also identically zero since on $r = r_{max}$, we have

$$(a\tilde{D}_\tau b - b\tilde{D}_\tau a) = (u\tilde{D}_x u - u\tilde{D}_x u) = 0 \quad \text{on } j = N-1, N$$

while on $x = x_{min}$ and $x = x_{max}$,

$$\left(c *_\nu (a\tilde{D}_\tau b\right) = r\left(u *_x (u/r\tilde{D}_r(ru)\right) = 0$$

and

$$\left(c *_\nu (b\tilde{D}_\tau a)\right) = r\left(u *_x ru\tilde{D}_r(u/r))\right) = 0$$

from (5.3.4).

In the mean time, we have the following Lemma concerning the boundary contributions for the viscous term

Lemma 1 *If either ϕ^1 satisfies the homogeneous Dirichlet boundary condition*

$$\phi^1_{0,j} + \phi^1_{1,j} = 0, \qquad \phi^1_{M-1,j} + \phi^1_{M,j} = 0, \qquad \phi^1_{i,N-1} + \phi^1_{i,N} = 0$$

or ϕ^2 satisfies the homogeneous Neumann boundary condition

$$\phi^2_{0,j} = \phi^2_{1,j}, \qquad \phi^2_{M-1,j} = \phi^2_{M,j}, \qquad \phi^2_{i,N-1} = \phi^2_{i,N}$$

at the physical boundary, then

$$\langle \phi^1, (\triangle_h - V)\phi^2 \rangle_h = -[\phi^1, \phi^2]_h \tag{5.3.12}$$

The proof follows straight forward from the following identity

$$\sum_{i=1}^{M-1} f_i(g_{i+1/2} - g_{i-1/2}) = -{\sum_{i=1}^{M}}'(f_i - f_{i-1})g_{i-1/2} + \frac{1}{2}(f_{M-1} + f_M)g_{M-1/2} - \frac{1}{2}(f_1 + f_0)g_{1/2} \tag{5.3.13}$$

with

$${\sum_{i=1}^{M}}' = \frac{1}{2}\sum_{i=1} + \sum_{i=2}^{M-1} + \frac{1}{2}\sum_{i=M}$$

From the analysis above, we see that the energy and helicity identities (5.1.16) and (5.1.17) remains valid with the physical boundary condition (5.3.3, 5.3.4).

In the case of MHD equation, boundary contribution (5.3.11) does not drop out automatically. A simple remedy is to add a correction term to the Jacobians at points $(1, j)$, $(M-1, j)$ and $(i, N-1)$. Since these points are $O(\Delta x)$ and $O(\Delta r)$ from the boundary, this correction is of $O(\Delta x^2 + \Delta r^2)$ and the resulting scheme is still second order consistent with the equation. However, this approach is quite artificial and we do not favor it so we omit the details.

We are unable to find a simple and local numerical boundary condition that preserves the MHD energy and helicity identities in the presence of physical boundaries.

In practice, a more convenient way of realizing (4.1.20) is to place the grid points on the physical boundary as is usually done. In other words, we put $x = x_{min}$ on $i = 0$, $x = x_{max}$ on $i = M$ and $r = r_{max}$ on $j = N$ (The pole $r = 0$ is still located at $j = \frac{1}{2}$). The $u = \psi = 0$ condition are given by

$$\psi_{0,j} = \psi_{M,j} = \psi_{i,N} = 0, \tag{5.3.14}$$

$$u_{0,j} = u_{M,j} = u_{i,N} = 0, \tag{5.3.15}$$

The boundary condition $\partial_\nu(h_3\psi) = 0$, or equivalently $\partial_\nu\psi = 0$ since $\psi = 0$ on the boundary, can be realized as

$$\psi_{-1,j} = \psi_{1,j}, \qquad \psi_{M+1,j} = \psi_{M-1,j}, \qquad \psi_{i,N+1} = \psi_{i,N-1} \tag{5.3.16}$$

Similarly

$$\partial_t a_{0,j} = 0, \qquad \partial_t a_{M,j} = 0, \qquad \partial_t a_{i,N} = 0 \tag{5.3.17}$$

$$(h_3 b)_{-1,j} = (h_3 b)_{1,j}, \qquad (h_3 b)_{M+1,j} = (h_3 b)_{M-1,j}, \qquad (h_3 b)_{i,N+1} = (h_3 b)_{i,N-1} \tag{5.3.18}$$

The boundary conditions (5.3.16, 5.3.18) uniquely determines the values of ψ and b on the ghost points $(-1, j)$, $(M + 1, j)$ and $(i, N + 1)$. The vorticity boundary condition can be easily derived from (5.3.16), known as Thom's formula:

$$\omega_{0,j} = \frac{2\psi_{1,j}}{(\Delta x)^2}, \tag{5.3.19}$$

In this setting, the active computational variables are u, ω and a at interior points and b at interior and boundary points.

Notice that the vorticity boundary condition for (5.3.2) and (5.3.3) corresponds to

$$\omega_{1,j} = \frac{\psi_{2,j}}{(\Delta x)^2}, \tag{5.3.20}$$

(5.3.20) differs from Thom's formula by a factor of 2, also known as Fromm's formula. If the grid points were placed right on the boundary, Fromm's formula reduces to a first order scheme, see [25]. It is indeed a second order scheme when the boundary is placed between the grid points (5.3.2). The numerical results are shown in Table 1 and a convergence proof for this vorticity boundary condition will be given in a forthcoming paper.

When there is no physical boundary involved (the whole space problem), we can derive the following estimate

$$\|u - u_h\| + \|\nabla_h(\psi - \psi_h)\|_1 \leq C(\Delta x^2 + \Delta r^2\sqrt{|\log \Delta r|}) \text{ in } x, r, \theta \text{ coordinates}$$

and

$$\|u - u_h\| + \|\nabla_h(\psi - \psi_h)\|_1 \leq C(\Delta x^2 + \Delta s^2) \text{ in } x, s, \theta \text{ coordinates}$$

where $\|f\|^2 = \langle f, f\rangle_h$ and $\|f\|_1^2 = [f, f]_h$. This is done by a standard but somewhat lengthy truncation error analysis together with a clever use of the permutation identity. See the Appendix for detail.

Remark 2 *It is worth noting that this numerical conservation property is very similar to the classical Zabusky-Kruskal scheme for the KdV equation [16]:*

$$u_t + uu_x + 6u_{xxx} = 0 \tag{5.3.21}$$

in which the convection term is discretized as :

$$(uu_x)_h = \frac{1}{3}uD_xu + \frac{2}{3}D_x(u^2/2) = \frac{u_{j-1} + u_j + u_{j+1}}{3}D_xu_j \tag{5.3.22}$$

and it gives local conservation for both u_j and u_j^2.

Remark 3 *The reformulation of nonlinear terms as Jacobian is also valid for nonorthogonal z_1 and z_2 coordinates in the (y_1, y_2) plane, as long as they are both orthogonal to y_3. The Jacobian remains the same and the factor $1/(h_2h_3)$ is replaced by $\frac{\partial(y_1,y_2)}{\partial(z_1,z_2)}$. It is therefore straight forward to generalize our scheme to non-orthogonal coordinate system and can be applied to simulate flows in non-regular domain or combined with the moving mesh method. This topic is currently under investigation.*

6 Numerical Examples

Example 1: Accuracy check

We first check the accuracy of our scheme for axisymmetric Navier Stokes equation. We setup the problem in a cylinder $\{0 < x < \pi, 0 < r < \pi\}$ with $\nu = 0.001$ and exact solution

$$\psi(x, r, t) = \cos(t)\sin(r)\cos(r/2)\sin(x)^2, \quad u(x, r, t) = \cos(t)\sin(r)\sin(x)$$

The result at $t = 3$ is given in Table 1. Clear second order accuracy is verified.

Example 2: Orszag-Tang Vortex

In this example, we repeat the calculation done by Cordoba and Marliani in [6] for ideal 2D MHD equation ($b = u = 0$ in (4.1.18)) using local mesh refinement technique. The underlying scheme is the second order upwind scheme combined with projection method on

	mesh	L^2 error	order	L^∞ error	order
ψ	100×128	3.0929E-4	—	5.8085E-4	—
	200×256	7.8090E-5	1.995	1.3717E-4	1.995
	400×512	1.9597E-5	1.997	3.9290E-5	1.998
	800×1024	4.9066E-6	1.999	1.0497E-5	1.999
	1600×2048	1.2274E-6	1.999	2.7118E-6	1.999
u	100×128	6.1831E-5	—	4.9415E-4	—
	200×256	1.5371E-5	2.008	1.2108E-4	2.029
	400×512	3.8393E-6	2.001	3.0150E-5	2.006
	800×1024	9.5976E-7	2.000	7.5111E-6	2.005
	1600×2048	2.3995E-7	2.000	1.8774E-6	2.000
ω	100×128	3.0929E-4	—	5.8085E-4	—
	200×256	7.8090E-5	1.986	1.3717E-4	2.082
	400×512	1.9597E-5	1.995	3.9290E-5	1.804
	800×1024	4.9066E-6	1.998	1.0497E-5	1.904
	1600×2048	1.2274E-6	1.999	2.7118E-6	1.953

Table 1: Errors and orders of accuracy for example 1.

the primitive variable for the fluids part and potential formulation for the magnetic part. The initial data is given by $a(x, y) = \cos(x+1.4)+\cos(y+2.0)$, $\psi(x, y) = \frac{1}{3}[\cos(2x+2.3)+\cos(y+6.2)]$ on a 2π periodic box. This configuration typically develops singularity-like structure known as current sheets where the current density is observed to grow exponentially in time and thickness shrinks at exponential rate as well. This problem is a good test on the performance of the scheme by monitoring the growth of the maximum of the current sheet as excessive numerical viscosity can easily smear out the current sheet. In [6], the initial resolution is 256^2 and adaptively refined on regions where the solution develops large variation. At $t = 2.7$, the finest mesh corresponds to the resolution of 4096^2 grids.

As a comparison, we repeat the same calculation with fixed resolution 1024^2. The contour plot of the current density $\jmath$ at $t = 2.7$ is shown in Fig 1, which agree well with the calculation done in [6]. Fig 2 is a local close up view of the same plot and we see the strong current sheet is well resolved with only 7-8 grid points across the sheet.

In addition, we plot the history of time evolution of the current sheet maximum against the simulated result $\jmath_{fit}(t)$ reported in [6]. Compared with the same plot ([6], Fig 8) of the fixed resolution calculation done there, we can see that our scheme is much less dissipative. Overall, we can achieve the same resolution with about half the number of grids in each space direction. (NOTE to referees: relevant pages of [6] attached)

For 2D MHD, the magnetic helicity is identically zero and $\int a^2$ emerges as an additional conserved quantity. This quantity is also preserved numerically by our scheme.

Example 3: Axisymmetric Flow in a Cylinder

We setup another test problem on a cylindrical domain $0 < x < 3$, $0 < r < 3$, with initial data:

$$\psi(x,r) = 0, \quad u(x,r) = \frac{1}{2r}\left(1 - \tanh\left(100\left((r-1)^2 + (x-1.5)^2 - 1/4\right)\right)\right)$$

and the no-slip condition. The initial configuration corresponds to a tube of flow in a circular cross section region with uniform angular momentum and the flow outside is at rest. This flow configuration induces a strong vortex sheet at the boundary of the circular region (see Fig5). At $t > 0$, the flow closer to the axis is thus driven towards outside and generates complicated flow patterns at later time. This situation is very similar to a rising bubble in 2D Boussinesq flow. The simulation is done with 1536^2 grids. We should mention here that we can afford high resolution simulation on an ordinary desktop because of the combined effect of the vorticity-stream formulation, explicit time integration of the nonlinear terms, and the local boundary condition that effectively decouples the Navier Stokes equation into 2 scalar evolution equations.

Several contour plots of u are given in Fig 4 through Fig 9. The details of the complicated flow structure is well captured.

7 Conclusions

For 3D symmetric flow, we reformulated all the nonlinear terms in Navier Stokes equation and MHD in terms of Jacobian. The physical conservation laws for energy, helicity, etc., follow directly from the permutation identities associated with the Jacobian.

We designed a numerical scheme that preserves the permutation identities and hence the energy and helicity numerically. This scheme is nonlinearly stable and free from excess numerical viscosity, and hence is suitable for long time integration. This scheme also gives a clean way of handling geometric singularities such as the rotation axis in axisymmetric flows.

The procedure is quite general. Any type of spatial discretization such as finite difference, finite element, and spectral methods can be treated similarly by numerically realizing the permutation identity (4.0.3). Local mesh refinement near the physical boundary can also be incorporated into the scheme by stretching the coordinate accordingly at no extra cost. Numerical evidence has demonstrated both efficiency and accuracy of the scheme.

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8 Appendix: Error Analysis for the (x, r, θ) Coordinate System

Here we give a detail derivation of the truncation error analysis and error estimate of our scheme for axisymmetric flows. For simplicity, we will only consider the whole space case in (x, r, θ) coordinates with exact solution decaying at infinity. The proof for (x, s, θ) is quite similar so we omit it.

The proof makes use of the discrete permutation identities for the nonlinear terms of the error equation. In the presence of physical boundaries, extra care needs to be taken to handle the boundary contributions for the permutation identities. This technique, known as the Strang expansion, has been applied to the analysis of 2D flow, see Hou and Wetton [15], Liu and Wang [18] for details.

For local truncation error analysis near the pole, it is necessary to identify smooth functions in this coordinate system.

Proposition 2 *The swirling component u of a smooth, axisymmetric vector field $\boldsymbol{u}$ admits the following expansion for r small:*

$$u(x, r) = C_1(x)r + C_3(x)r^3 + C_5(x)r^5 + O(r^6) \qquad \text{(A.1)}$$

Proof: Since

$$u(x, y_2, y_3) = u_\theta(x, y_2, y_3) = -u_2(x, y_2, y_3)\sin\theta + u_3(x, y_2, y_3)\cos\theta,$$

we can expand u_2 and u_3 as Taylor series in $y_2 = r\cos\theta$ and $y_3 = r\sin\theta$ to get

$$u_\theta(x, y_2, y_3) = \sum_{k=1}^{6} r^{k-1} P_k(x; \cos\theta, \sin\theta) + O(r^6)$$

where P_k is a homogeneous polynomial of $\cos\theta$ and $\sin\theta$ with degree k. For axisymmetric flows $u_\theta(x, r\cos\theta, r\sin\theta)$ is independent of θ. It follows that P_k is a constant in θ. An easy analysis shows that $P_1 = P_3 = P_5 = 0$.

Lemma 2 *Let (ψ, u, ω) be an exact solution of the axisymmetric Navier Stokes equation (5.2.1) that decays sufficiently fast at infinity, then*

$$\omega = (-\triangle_h + V)\psi + \widehat{r}\Delta x^2 + \widehat{r^{-1}}\Delta r^2$$

$$\partial_t u + \frac{1}{r^2} J_h(ru, r\psi) = \nu(\triangle_h - V)u + \widehat{r}\Delta x^2 + \widehat{r^{-1}}\Delta r^2$$

$$\partial_t \omega + J_h\left(\frac{\omega}{r}, r\psi\right) = \nu(\triangle_h - V)\omega + J_h\left(\frac{u}{r}, ru\right) + \widehat{r}\Delta x^2 + \widehat{r^{-1}}\Delta r^2$$

where $\widehat{r^n} = O(r^n)$ for r small and decays sufficiently fast at infinity.

Proof: We analyze the truncation error term by term. For the Laplacian operator, we have

$$\begin{aligned} \triangle_h \psi_{i,j} &= \left(D_x^2 + D_r^2 + \tfrac{\widetilde{D}_r}{r}\right)\psi_{i,j} \\ &= \triangle\psi_{i,j} + \widehat{r}\Delta x^2 + O(\partial_r^4\psi\Delta r^2 + O(\tfrac{1}{r}\partial_r^3\psi\Delta r^2) \\ &= \triangle\psi_{i,j} + \widehat{r}\Delta x^2 + \widehat{r^{-1}}\Delta r^2 \end{aligned} \tag{A.2}$$

It is easy to verify that (A.2) is valid up to $j = 1$, or $r = r_1 = \Delta r/2$.

Next we analyze the nonlinear terms, for simplicity we omit the spatial index (i, j) from here on.

$$\begin{aligned} \tfrac{1}{r^2} J_h(ru, r\psi) = \tfrac{1}{r^2}\big(& \widetilde{D}_x(ru)\widetilde{D}_r(r\psi) - \widetilde{D}_r(ru)\widetilde{D}_x(r\psi) \\ + & \widetilde{D}_x(ru\widetilde{D}_r(r\psi)) - \widetilde{D}_r(ru\widetilde{D}_x(r\psi)) \\ + & \widetilde{D}_r(r\psi\widetilde{D}_x(ru)) - \widetilde{D}_x(r\psi\widetilde{D}_r(ru))\ \big) \end{aligned} \tag{A.3}$$

For the first two terms, we have

$$\widetilde{D}_x(ru) = \partial_x(ru) + \widehat{r^2}\Delta x^2 \tag{A.4}$$

and

$$\widetilde{D}_r(r\psi)_{i,j} = \partial_r(r\psi)_{i,j} + \begin{cases} O(\Delta r), & j = 1 \\ O(\Delta r^2), & j = 2, 3, \cdots \end{cases} \tag{A.5}$$

where we have used the local expansion (A.1) in (A.4) and (A.5). It follows that

$$\widetilde{D}_r(r\psi)_{i,j} = \partial_r(r\psi)_{i,j} + \widehat{r^{-1}}\Delta r^2$$

A similar estimate holds for $\widetilde{D}_r(ru)$ and $\widetilde{D}_x(r\psi)$ and we have

$$\frac{1}{r^2}\widetilde{D}_x(ru)\widetilde{D}_r(r\psi) = \frac{1}{r^2}\partial_x(ru)\partial_r(r\psi) + \widehat{r}\Delta x^2 + \widehat{r^{-1}}\Delta r^2 \tag{A.6}$$

The degeneracy at $j = 1$ in (A.5) is a result of the even extension for the stretching factor $h_3(r) = r$ with $h_3(r_0) = h_3(r_1) = \Delta r/2$. One might suspect that this extension introduces a kink at $r = 0$ and produces extra truncation error near the pole. In fact, it is easy to see that $\widetilde{D}_r(r\psi) = \partial_r(r\psi) + \widehat{r^0}\Delta r^2$ for all j had we chosen the odd extension for r: $r_0 = -r_1$. However, the local truncation error in (A.5) is no worse than that of $\triangle_h$ as shown in (A.2). Moreover, it is also comparable to those of $\widetilde{D}_r(ru\widetilde{D}_x(r\psi))$ and $\widetilde{D}_r(r\psi\widetilde{D}_x(ru))$ regardless of the extension of r. Indeed, we have

$$\begin{aligned} & \widetilde{D}_r(ru\widetilde{D}_x(r\psi)) \\ = \; & \widetilde{D}_r(ru\partial_x(r\psi) + O(ru\partial_x^3(r\psi)\Delta x^2)) \\ = \; & \partial_r(ru\partial_x(r\psi)) + \widehat{r}\Delta r^2 + \widehat{r^3}\Delta x^2 \end{aligned}$$

and similarly

$$\widetilde{D}_r(r\psi\widetilde{D}_x(ru)) = \partial_r(r\psi\partial_x(ru)) + \widehat{r}\Delta r^2 + \widehat{r^3}\Delta x^2$$

The remaining terms in (A.3) can be easily estimated by

$$\widetilde{D}_x(ru\widetilde{D}_r(r\psi)) = \widetilde{D}_x\left(ru\left(\partial_r(r\psi) + \widehat{r}\Delta r^2\right)\right) = \partial_x(ru\partial_r(r\psi)) + \widehat{r}\Delta r^2 + \widehat{r^3}\Delta x^2 \tag{A.7}$$

and

$$\widetilde{D}_x(r\psi\widetilde{D}_r(ru)) = \partial_x(r\psi\partial_r(ru)) + \widehat{r}\Delta r^2 + \widehat{r^3}\Delta x^2 \tag{A.8}$$

In summary, we have

$$\frac{1}{r^2}J_h(ru, r\psi) = \frac{1}{r^2}J(ru, r\psi) + \widehat{r}\Delta x^2 + \widehat{r^{-1}}\Delta r^2 \tag{A.9}$$

Similarly, we can estimate $J_h\left(\frac{\omega}{r}, r\psi\right)$ as follows:

$$\begin{aligned} J_h\left(\tfrac{\omega}{r}, r\psi\right) &= \widetilde{D}_x\left(\tfrac{\omega}{r}\right)\widetilde{D}_r(r\psi) - \widetilde{D}_r\left(\tfrac{\omega}{r}\right)\widetilde{D}_x(r\psi) \\ &+ \widetilde{D}_x\left(\tfrac{\omega}{r}\widetilde{D}_r(r\psi)\right) - \widetilde{D}_r\left(\tfrac{\omega}{r}\widetilde{D}_x(r\psi)\right) \\ &+ \widetilde{D}_r\left(r\psi\widetilde{D}_x(\tfrac{\omega}{r})\right) - \widetilde{D}_x\left(r\psi\widetilde{D}_r(\tfrac{\omega}{r})\right) \end{aligned} \tag{A.10}$$

For the first term in (A.10), we have

$$\widetilde{D}_x\left(\frac{\omega}{r}\right) = \partial_x\left(\frac{\omega}{r}\right) + \widehat{r^0}\Delta x^2, \tag{A.11}$$

$$\widetilde{D}_r(r\psi) = \partial_r(r\psi) + \widehat{r^{-1}}\Delta r^2, \tag{A.12}$$

therefore

$$\begin{aligned} \widetilde{D}_x\left(\tfrac{\omega}{r}\right)\widetilde{D}_r(r\psi) &= \left(\partial_x\left(\tfrac{\omega}{r}\right) + \widehat{r^0}\Delta x^2\right)\left(\partial_r(r\psi) + \widehat{r^{-1}}\Delta r^2\right) \\ &= \partial_x\left(\tfrac{\omega}{r}\right)\partial_r(r\psi) + \widehat{r}\Delta x^2 + \widehat{r^{-1}}\Delta r^2 \end{aligned} \tag{A.13}$$

Similarly, the second term can be estimated as

$$\widetilde{D}_r\left(\frac{\omega}{r}\right) = \partial_r\left(\frac{\omega}{r}\right) + \widehat{r^{-3}}\Delta r^2 \tag{A.14}$$

$$\widetilde{D}_x(r\psi) = \partial_x(r\psi) + \widehat{r^2}\Delta x^2 \tag{A.15}$$

and

$$\widetilde{D}_r\left(\frac{\omega}{r}\right)\widetilde{D}_x(r\psi) = \partial_x(r\psi)\partial_r\left(\frac{\omega}{r}\right) + \widehat{r}\Delta x^2 + \widehat{r^{-1}}\Delta r^2 \tag{A.16}$$

The remaining terms in (A.10) follow similarly:

$$\begin{aligned} \widetilde{D}_x\left(\tfrac{\omega}{r}\widetilde{D}_r(r\psi)\right) &= \widetilde{D}_x\left(\tfrac{\omega}{r}\left(\partial_r(r\psi) + \widehat{r^{-1}}\Delta r^2\right)\right) \\ &= (\partial_x + \Delta x^2\partial_x^3)\left(\tfrac{\omega}{r}\partial_r(r\psi) + \widehat{r^{-1}}\Delta r^2\right) \\ &= \partial_x\left(\tfrac{\omega}{r}\partial_r(r\psi)\right) + \widehat{r}\Delta x^2 + \widehat{r^{-1}}\Delta r^2 \end{aligned} \tag{A.17}$$

$$\begin{aligned} \widetilde{D}_r\left(\tfrac{\omega}{r}\widetilde{D}_x(r\psi)\right) &= \widetilde{D}_r(\omega\partial_x\psi + \widehat{r^2}\Delta x^2) \\ &= \partial_r(\omega\partial_x\psi) + \widehat{r}\Delta x^2 + \widehat{r^{-1}}\Delta r^2 \end{aligned} \tag{A.18}$$

$$\begin{aligned} \widetilde{D}_r\left(r\psi\widetilde{D}_x(\tfrac{\omega}{r})\right) &= \widetilde{D}_r\left(\left(r\psi\left(\partial_x\tfrac{\omega}{r}\right)\right) + \widehat{r^0}\Delta x^2\right) \\ &= \partial_r\left(r\psi(\partial_x\tfrac{\omega}{r})\right) + \widehat{r}\Delta x^2 + \widehat{r^{-1}}\Delta r^2 \end{aligned} \tag{A.19}$$

$$\begin{aligned} \widetilde{D}_x\left(r\psi\widetilde{D}_r(\tfrac{\omega}{r})\right) &= \widetilde{D}_x\left(r\psi\left(\partial_r\tfrac{\omega}{r}\right) + \widehat{r^{-1}}\Delta r^2\right) \\ &= \partial_x\left(r\psi\partial_r\left(\tfrac{\omega}{r}\right)\right) + \widehat{r}\Delta x^2 + \widehat{r^{-1}}\Delta r^2 \end{aligned} \tag{A.20}$$

In Summary, we have

$$J_h\left(\frac{\omega}{r}, r\psi\right) = J\left(\frac{\omega}{r}, r\psi\right) + \widehat{r}\Delta x^2 + \widehat{r^{-1}}\Delta r^2 \tag{A.21}$$

The same argument also leads to

$$J_h\left(\frac{u}{r}, ru\right) = J\left(\frac{u}{r}, ru\right) + \widehat{r}\Delta x^2 + \widehat{r^{-1}}\Delta r^2 \tag{A.22}$$

This completes the proof of Lemma 2.

To perform the energy estimate for the error, we first recall the weighted inner products (5.1.12) and (5.1.13) in (x, r, θ) coordinate:

$$\langle \phi, \psi \rangle_h = \sum_{i,j} (r\phi\psi)_{i,j}\, \Delta x \Delta r \tag{A.23}$$

$$[\phi, \psi]_h = \sum_{i,j} \left(r(D_x\phi)(D_x\psi)\right)_{i-1/2,j}\, \Delta x \Delta r + \sum_{i,j} \left(r(D_r\phi)(D_r\psi)\right)_{i,j-1/2}\, \Delta x \Delta r + \langle \phi, V\psi \rangle_h \tag{A.24}$$

and the corresponding norms

$$\|\phi\|^2 = \langle \phi, \phi \rangle_h, \qquad \|\phi\|_1^2 = [\phi, \phi]_h = \|\nabla_h \phi\|^2 + \langle \phi, V\psi \rangle_h$$

where

$$(D_x\phi)_{i+\frac{1}{2},j} = \frac{1}{\Delta x}(\phi_{i+1,j} - \phi_{i,j}), \qquad (D_r\phi)_{i,j+\frac{1}{2}} = \frac{1}{\Delta r}(\phi_{i,j+1} - \phi_{i,j})$$

and $\nabla_h \phi = (D_x\phi, D_r\phi)$.

Lemma 3 *For any $T > 0$, we have*

$$\sup_{[0,T]} \|u - u_h\| + \|\nabla_h(\psi - \psi_h)\|_1 \le C(\Delta x^2 + \Delta r^2 \sqrt{|\log \Delta r|}) \text{ in } x, r, \theta \text{ coordinates}$$

where $C = C(\psi, u, \nu, T)$

Proof: We denote by $\mathcal{T}_{i,j}$ the local truncation error. The numerical solution ψ_h, u_h, ω_h satisfies

$$\begin{aligned}
&\omega_h = (-\triangle_h + V)\psi_h \\
&\partial_t u_h + \frac{1}{r^2} J_h(u_h, r\psi_h) = \nu(\triangle_h - V)u_h \\
&\partial_t \omega_h + J_h\left(\frac{\omega_h}{r}, r\psi_h\right) = \nu(\triangle_h - V)\omega_h + J_h\left(\frac{u_h}{r}, ru_h\right)
\end{aligned}$$

while the exact solution ψ, u, ω satisfies

$$\omega = (-\triangle_h + V)\psi + \mathcal{T}$$
$$\partial_t u + \frac{1}{r^2} J_h(u, r\psi) = \nu(\triangle_h - V)u + \mathcal{T}$$
$$\partial_t \omega + J_h\left(\frac{\omega}{r}, r\psi\right) = \nu(\triangle_h - V)\omega + J_h\left(\frac{u}{r}, ru\right) + \mathcal{T}$$

therefore

$$(\omega - \omega_h) = (-\triangle_h + V)(\psi - \psi_h) + \mathcal{T} \tag{A.25}$$
$$\partial_t(u - u_h) + \frac{1}{r^2}\left(J_h(ru, r\psi) - J_h(ru_h, r\psi_h)\right) = \nu(\triangle_h - V)(u - u_h) + \mathcal{T} \tag{A.26}$$
$$\begin{aligned} & \partial_t(\omega - \omega_h) + \left(J_h\left(\tfrac{\omega}{r}, r\psi\right) - J_h\left(\tfrac{\omega_h}{r}, r\psi_h\right)\right) \\ = \ & \nu(\triangle_h - V)(\omega - \omega_h) + \left(J_h\left(\tfrac{u}{r}, ru\right) - J_h\left(\tfrac{u_h}{r}, ru_h\right)\right) + \mathcal{T} \end{aligned} \tag{A.27}$$

We take the weighted inner product of $u - u_h$ with (A.26) to get

$$\begin{aligned} & \tfrac{1}{2}\partial_t \|u - u_h\|^2 + \langle u - u_h, \tfrac{1}{r^2}\left(J_h(ru, r\psi) - J_h(ru_h, r\psi_h)\right)\rangle_h \\ = \ & \nu\langle u - u_h, (\triangle_h - V)(u - u_h)\rangle_h + \langle u - u_h, \mathcal{T}\rangle_h \end{aligned} \tag{A.28}$$

For the nonlinear terms, we have

$$\begin{aligned} & \langle u - u_h, \frac{1}{r^2}\left(J_h(ru, r\psi) - J_h(ru_h, r\psi_h)\right)\rangle_h \\ = \ & \langle u - u_h, \frac{1}{r^2}\left(J_h(r(u - u_h), r\psi) + J_h(ru_h, r(\psi - \psi_h))\right)\rangle_h \\ = \ & \langle u - u_h, \frac{1}{r^2}\left(J_h(r(u - u_h), r\psi) + J_h(r(u_h - u), r(\psi - \psi_h)) + J_h(ru, r(\psi - \psi_h))\right)\rangle_h \\ = \ & I + II + III \end{aligned}$$

From the reflection boundary condition (5.2.4), we have

$$(II) \ = \ T_h\left(\frac{u - u_h}{r}, r(u_h - u), r(\psi - \psi_h)\right)$$

The first and the third term can be estimated as follows:

$$\begin{aligned} & J_h(r(u - u_h), r\psi) \\ = \ & \widehat{r}\widetilde{D}_x(r(u - u_h)) + \widehat{r^2}\widetilde{D}_r(r(u - u_h)) \\ = \ & \widehat{r}\left((\widetilde{A}_x r)\widetilde{D}_x(u - u_h) + (\widetilde{D}_x r)\widetilde{A}_x(u - u_h)\right) + \widehat{r^2}\left((\widetilde{A}_r r)\widetilde{D}_r(u - u_h) + (\widetilde{D}_r r)\widetilde{A}_r(u - u_h)\right) \\ = \ & \widehat{r^2}\left(|\widetilde{\nabla}_h(u - u_h)| + |u - u_h|\right) \end{aligned}$$

where we have used the identity

$$\widetilde{D}(fg) = \widetilde{A}f\widetilde{D}g + \widetilde{A}g\widetilde{D}f$$

with

$$(\widetilde{A}_x f)_{i,j} = \frac{1}{2}(f_{i+1,j} + f_{i-1,j}), \qquad (\widetilde{A}_r f)_{i,j} = \frac{1}{2}(f_{i,j+1} + f_{i,j-1}).$$

Therefore

$$\begin{aligned}
(I) &= \langle u - u_h, \tfrac{1}{r^2} J_h(r(u-u_h), r\psi)\rangle_h \\
&= \langle u - u_h, \widehat{r^0}(|\widetilde{\nabla}_h(u-u_h)| + |u-u_h|)\rangle_h \\
&= \epsilon\|\widetilde{\nabla}_h(u-u_h)\|^2 + O(\tfrac{1}{\epsilon})\|u-u_h\|^2 \\
&\le \epsilon\|\nabla_h(u-u_h)\|^2 + O(\tfrac{1}{\epsilon})\|u-u_h\|^2
\end{aligned} \tag{A.29}$$

We have used the identity

$$\widetilde{\nabla}_h = (A_x D_x, A_r D_r),$$

$$(A_x f)_{i,j} = \frac{1}{2}(f_{i+\frac{1}{2},j} + f_{i-\frac{1}{2},j}), \qquad (A_x f)_{i,j} = \frac{1}{2}(f_{i,j+\frac{1}{2}} + f_{i,j-\frac{1}{2}})$$

together with the Cauchy Schwartz inequality in (A.29).

Similarly, we have

$$\begin{aligned}
(III) &= \langle u - u_h, \tfrac{1}{r^2} J_h(ru, r(\psi-\psi_h))\rangle_h \\
&= \langle u - u_h, \widehat{r^0}|\widetilde{\nabla}_h(\psi-\psi_h)| + \widehat{r^0}|\psi-\psi_h|\rangle_h \\
&= O(1)\left(\|u-u_h\|^2 + \|\psi-\psi_h\|_1^2\right)
\end{aligned} \tag{A.30}$$

For the viscosity term, we have

$$\nu\langle u - u_h, (\triangle_h - V)(u-u_h)\rangle_h = -\nu[u-u_h, u-u_h]_h = -\nu\|u-u_h\|_1^2 \tag{A.31}$$

For the truncation error term,

$$\langle u - u_h, \mathcal{T}\rangle_h \le \|u-u_h\|^2 + \|\mathcal{T}\|^2 \le \|u-u_h\|^2 + O(\Delta x^4 + |\log\Delta r|\Delta r^4)$$

Therefore, we have

$$\begin{aligned}
&\tfrac{1}{2}\partial_t\|u-u_h\|^2 + T_h\left(\tfrac{u-u_h}{r}, r(u_h-u), r(\psi-\psi_h)\right) + \tfrac{\nu}{2}\|u-u_h\|_1^2 \\
\le\ & C\left(\|\psi-\psi_h\|_1^2 + \|u-u_h\|^2\right) + O(\Delta x^4 + |\log\Delta r|\Delta r^4)
\end{aligned} \tag{A.32}$$

Similarly, we have from (A.27) that

$$\begin{aligned}
&\tfrac{1}{2}\partial_t\|\psi-\psi_h\|_1^2 + \langle\psi-\psi_h, J_h(\tfrac{\omega}{r}, r\psi) - J_h(\tfrac{\omega_h}{r}, r\psi_h)\rangle_h + \langle\psi-\psi_h, \partial_t\mathcal{T} - \mathcal{T}\rangle_h \\
=\ & \nu\langle\psi-\psi_h, (\triangle_h - V)(\omega-\omega_h)\rangle_h + \langle\psi-\psi_h, J_h(\tfrac{u}{r}, ru) - J_h(\tfrac{u_h}{r}, ru_h)\rangle_h
\end{aligned} \tag{A.33}$$

For the nonlinear terms, we have

$$
\begin{aligned}
&\langle\psi-\psi_h, J_h(\frac{\omega}{r}, r\psi) - J_h(\frac{\omega}{r}, r\psi_h)\rangle_h \\
= \ &\langle\psi-\psi_h, J_h\left(\frac{\omega-\omega_h}{r}, r\psi\right) + J_h\left(\frac{(\omega_h-\omega)}{r}, r(\psi-\psi_h)\right) + J_h\left(\frac{\omega}{r}, r(\psi-\psi_h)\right)\rangle_h \\
= \ &IV + V + VI
\end{aligned}
$$

Again from the reflection BC, we have

$$
(V) = T_h\left(r(\psi-\psi_h), \frac{(\omega_h-\omega)}{r}, r(\psi-\psi_h)\right) = 0 \tag{A.34}
$$

also

$$
\langle\psi-\psi_h, J_h(\frac{\omega-\omega_h}{r}, r\psi)\rangle_h = \langle\frac{(\omega-\omega_h)}{r^2}, J_h(r\psi, r(\psi-\psi_h))\rangle_h \tag{A.35}
$$

Then we proceed as before to get

$$
\begin{aligned}
(IV) = \langle\frac{(\omega-\omega_h)}{r^2}, J_h(r\psi, r(\psi-\psi_h))\rangle_h &\le \langle|\omega-\omega_h|, \widehat{r^0}(|\psi-\psi_h| + |\nabla_h(\psi-\psi_h)|)\rangle_h \\
&\le \epsilon\|\omega-\omega_h\|^2 + O(\frac{1}{\epsilon})\|\psi-\psi_h\|_1^2
\end{aligned}
$$

Direct calculation gives

$$
J_h\left(\frac{\omega}{r}, r(\psi-\psi_h))\right) \le \widehat{r^0}|\psi-\psi_h| + \widehat{r}|\nabla_h(\psi-\psi_h)|
$$

Similarly

$$
\begin{aligned}
(VI) = \langle\psi-\psi_h, J_h\left(\frac{\omega}{r}, r(\psi-\psi_h))\right)\rangle_h &\le \langle|\psi-\psi_h|, \widehat{r}|\nabla_h(\psi-\psi_h)| + \widehat{r^0}(|\psi-\psi_h|\rangle_h \\
&\le O(1)\|\psi-\psi_h\|_1^2
\end{aligned}
$$

$$
\begin{aligned}
&\langle\psi-\psi_h, J_h(\frac{u}{r}, ru) - J_h(\frac{u_h}{r}, ru_h)\rangle_h \\
= \ &\langle\psi-\psi_h, J_h\left(\frac{u-u_h}{r}, ru\right) + J_h\left(\frac{u_h-u}{r}, r(u-u_h)\right) + J_h\left(\frac{u}{r}, r(u-u_h)\right)\rangle_h \\
= \ &VII + VIII + IX
\end{aligned}
$$

Again from the reflection BC, we have

$$(VIII) = T_h\left(r(\psi-\psi_h), \frac{(u_h-u)}{r}, r(u-u_h)\right) \tag{A.36}$$

$$(VII) = \langle r(\psi-\psi_h), J_h(\frac{u-u_h}{r}, ru)\rangle_h \le \langle|\psi-\psi_h|, \widehat{r^0}|u-u_h| + \widehat{r}|\nabla_h(u-u_h)|\rangle_h$$
$$\le \epsilon\|\nabla_h(u-u_h)\|^2 + O(\frac{1}{\epsilon})\|\psi-\psi_h\|_1^2 + O(1)\|u-u_h\|^2$$

Similarly

$$(IX) = \langle\psi-\psi_h, J_h(\frac{u}{r}, r(u-u_h))\rangle_h \le \langle|\psi-\psi_h|, \widehat{r^0}|u-u_h| + \widehat{r}|\nabla_h(u-u_h)|\rangle_h$$

$$\le \epsilon\|\nabla_h(u-u_h)\|^2 + O(\frac{1}{\epsilon})\|\psi-\psi_h\|_1^2 + O(1)\|u-u_h\|^2$$

For the viscosity term, we sum by part twice to get

$$\nu\langle(\psi-\psi_h), (\triangle_h - V)(\omega-\omega_h)\rangle_h = \nu\langle(\triangle_h - V)(\psi-\psi_h), \omega-\omega_h\rangle_h$$

$$= -\nu\|\omega-\omega_h\|^2 + \nu\langle\omega-\omega_h, \mathcal{T}\rangle_h \le -\frac{\nu}{2}\|\omega-\omega_h\|^2 + O(\frac{1}{\nu})(\Delta x^4 + |\log\Delta r|\Delta r^4)$$

and finally

$$|\langle\psi-\psi_h, \partial_t\mathcal{T} - \mathcal{T}\rangle_h| = \|\psi-\psi_h\|^2 + O(\Delta x^4 + |\log\Delta r|\Delta r^4)$$

In summary, we have

$$\frac{1}{2}\partial_t\|\psi-\psi_h\|_1^2 + \frac{\nu}{2}\|\omega-\omega_h\|^2$$
$$\le T_h\left(r(\psi-\psi_h), \frac{(u_h-u)}{r}, r(u-u_h)\right) + O(\Delta x^4 + \Delta r^4|\log\Delta r|)$$
$$+C\|\psi-\psi_h\|_1^2 + \epsilon\|\nabla_h(u-u_h)\|^2 + C\|u-u_h\|^2$$

Let

$$\mathcal{H}(t) = \frac{1}{2}\left(\|u-u_h\|^2 + \|\psi-\psi_h\|_1^2\right)$$

with suitably chosen ϵ, we have

$$\frac{d\mathcal{H}}{dt} + \frac{\nu}{2}(\|u-u_h\|_1^2 + \|\omega-\omega_h\|^2) \le C\mathcal{H} + O(\Delta x^4 + \Delta r^4|\log\Delta r|)$$

It follows from Gronwall's inequality that

$$\sup_{[0,T]}\mathcal{H}(t) + \frac{\nu}{2}\int_0^T(\|u-u_h\|_1^2 + \|\omega-\omega_h\|^2)dt \le C(\Delta x^4 + \Delta r^4|\log\Delta r|)$$

References

[1] A. Arakawa, *Computational design for long-term numerical integration of the equations of fluid motion: two dimensional incompressible flow. Part I*, J. Compute. Phys. **1** (1966) 119-143.

[2] H. Ceniceros and T. Y. Hou, *An Efficient Dynamically Adaptive Mesh for Potentially Singular Solutions* in J. Comput. Phys., **172**, 1-31 2001.

[3] L. Caffarelli, R. Kohn and L. Nirenberg, *Partial regularity of suitable weak solutions of the Navier-Stokes equations*, Comm. Pure Appl. Math. **35** (1982) 771–831.

[4] R. Caflisch, I. Klapper and G. Steele, *Remarks on singularities, dimension and energy dissipation for idea hydrodynamics and MHD,* Comm. Math. Phys. **184** (1997), 443-455.

[5] A.J. Chorin, *Numerical solution of the Navier-Stokes equations*, Math. Comp. **22** (1968) 745–762.

[6] D. Cordoba and C. Marliani *Evolution of current sheets and regularity of ideal incompressible magnetic fluids in 2D,* Comm. Pure and Appl. Math. **53** (2000), 512-524

[7] W. E and J.-G. Liu, *Vorticity boundary condition and related issues for finite difference schemes,* J. Comput. Phys., **124** (1996) 368–382.

[8] W. E and C.-W. Shu, *Small-scale structures in Boussinesq convection.* Phys. Fluids 6 (1994), no. 1, 49–58

[9] B. Fornberg, *A Practical Guide to Pseudo-spectral Methods*, Cambridge University Press, New York, 1998.

[10] R. Grauer and C. Marliani, *Geometry of singular structures in magneto-hydrodynamics flows,* Physics of Plasma **5** (1998), 2544

[11] R. Grauer and C. Marliani, *Current-sheet formation in 3D ideal incompressible magneto-hydrodynamics,* Physical Review Letters **84** (2000), 4850-4853.

[12] R. Grauer and T.C. Sideris, *Numerical computation of 3D incompressible ideal fluids with Swirl*, Physical Review Letters **67** (1991) 3511–3514.

[13] P.M. Gresho, *Incompressible fluid dynamics: some fundamental formulation issues*, Rev. Fluid Mech. **23** (1991) 413–453.

[14] F.H. Harlow and J.E. Welch, *Numerical calculation of time-dependent viscous incompressible flow of fluid with free surface*, Phys. Fluids **8** (1965) 2182–2189.

[15] Thomas. Y. Hou and T.R. Wetton, *Convergence of a finite difference scheme for the Navier-Stokes equations using vorticity boundary conditions.* SIAM J. Numer. Anal. **29** (1992), no. 3, 615–639.

[16] C.D. Levermore and J.-G. Liu, *Large oscillations arising from a dispersive numerical scheme,* Physica D, **99** (1996) 191–216.

[17] J.-G. Liu and C.-W. Shu, *A high order discontinuous Galerkin method for incompressible flows,* J. Comput. Phys., **160** (2000) 577-596.

[18] J.-G. Liu and C. Wang, *Analysis of finite difference schemes for unsteady Navier-Stokes equations in vorticity formulation*, Numer. Math., **91** (2002), 543–576.

[19] J.-G. Liu and W.C. Wang, *An energy preserving MAC-Yee scheme for the incompressible MHD equation*, J. Comput. Phys., **174** (2001) 12-37.

[20] J.-G. Liu and Z. Xin, *Convergence of a Galerkin method for 2-D discontinuous Euler flows,* Comm. Pure Appl. Math., **53** (2000) 786-798.

[21] H.K. Moffatt, *The degree of knotedness of tangled vortex lines*, J. Fluid Mech. **35**, (1969) 117–129.

[22] H.K. Moffatt and A. Tsinober, *Helicity in laminar and turbulent flow*, Ann. Rev. Fluid Mech. **24**, (1992) 281-312.

[23] K. Mohseni and T. Colonius, *Numerical treatment of polar coordinate singularities*, J. Comput. Phys., **157**, 787 (2000).

[24] J.J. Moreau, *Constantes d'un îlot tourbillionnaire en fluide parfait barotrope*, C.R. Acad. Sci. Paris, **252**, (1961) 2810-2812.

[25] S.A. Orszag and M. Israeli, *Numerical simulation of viscous incompressible flows,* Ann. Rev. Fluid Mech. **6** (1974), 281–318.

[26] S.A. Orszag and C. Tang, *Small-scale structure of two-dimensional magneto-hydrodynamic turbulence,* J. Fluid Mech. **90** (1979), 129.

[27] A. Thom, *The flow past circular cylinders at low speeds*, Proc. R. Soc. London A **141** (1933) 651–669.

[28] K. Yee, *Numerical solution of initial boundary value problems involving Maxwell's equations in isotropic media,* IEEE Trans. Antennas and Propagation, **AP-16** (1966), 302-307.

[29] N.J. Zabusky and M.D. Kruskal, *Interaction of "solitons" in a collisionless plasma and the recurrence of initial states*, Phys. Rev. Lett. **15** (1965) 240–243.

Figure 1: Example 2, contour plot of the current density j at $t = 2.7$

Figure 2: Example 2, close up of Figure 1. The current sheet is resolved with about 8 grid points.

Figure 3: Example 2, 3D plot of Figure 1

Figure 4: Example 2, time evolution history of maximum current sheet with different resolutions. j_{fit}: data computed in [6] using equivalence of 4096^2 resolution

Figure 5: Example 3, contour plot of u at $t = 0$. horizontal axis: x, vertical axis: r

Figure 6: Example 3, contour plot of u at $t = 2$

Figure 7: Example 3, contour plot of u at $t = 2.5$

Figure 8: Example 3, contour plot of u at $t = 3.0$

Figure 9: Example 3, contour plot of u at $t = 3.5$

Figure 10: Example 3, close up of Figure 9

Figure 11: Pole patterns in [6] for