A note on the norm of oblique projections

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Abstract The purpose of this note is to give a somewhat simplified version of T. Katos [2, Proof of Lemma 4]. As shown by J. Xu and L. Zikatanov [5], the lemma is of interest to the approximation theory of the finite element method.

Keywords oblique projection · finite element method · quasi-optimality

Mathematics Subject Classification (2000) MSC 65N30

Introduction and proof of lemma

D. B. Szyld collected in [4] several proofs of the identity $\|P\| = \|I - P\|$ for nontrivial projections P on a Hilbert space, see also [3] and [1, Example 5.8]. J. Xu and L. Zikatanov exposed in [5] the utility of this result to remove the notorious "1+" in the quasi-optimality estimate for the finite element method. We provide here a simplified version of T. Katos proof [2, Lemma 4]. The difference is in the choice of the vector y.

Lemma 1 *Let H be a Hilbert space. Let $P : H \to H$ be a linear idempotent operator such that $0 \neq P^2 = P \neq I$. Then $\|P\| = \|I - P\|$.*

Proof Since $P^2 = P$ and $(I - P)^2 = I - P$, both norms are no less than one. If $\|P\| = 1 = \|I - P\|$, there is nothing to prove, so let $x \in H$ be nonzero with, say, $\alpha := \|Px\|^2/\|x\|^2 > 1$. Then $y := \alpha x - Px \neq 0$ due to $Py \neq 0$. By direct computation, $\|(I - P)y\|\|x\| = \|Px\|\|y\|$. Since $x \neq 0$ was arbitrary (subject to $\alpha > 1$) dividing by $\|x\|\|y\|$ and taking the supremum over x shows $\|I - P\| \geq \|P\| > 1$. Swapping the roles of P and $I - P$ concludes the proof.

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