A RELAXATION THEORY WITH POLYCONVEX ENTROPY FUNCTION CONVERGING TO ELASTODYNAMICS

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Abstract. The equations of polyconvex elastodynamics can be embedded to an augmented symmetric hyperbolic system. This property provides a stability framework between solutions of the viscosity approximation of polyconvex elastodynamics and smooth solutions of polyconvex elastodynamics. We devise here a model of stress relaxation motivated by the format of the enlargement process which formally approximates the equations of polyconvex elastodynamics. The model is endowed with an entropy function which is not convex but rather of polyconvex type. Using the relative entropy we prove a stability estimate and convergence of the stress relaxation model to polyconvex elastodynamics in the smooth regime.

1. Introduction

The mechanical motion of a continuous medium with nonlinear elastic response is described by the system of partial differential equations

$$\frac{\partial^2 y}{\partial t^2} = \nabla \cdot T(\nabla y) \tag{1.1}$$

where $y : \mathbb{R}^3 \times \mathbb{R}^+ \to \mathbb{R}^3$ describes the motion and T is the Piola–Kirchhoff stress tensor. The system (1.1) may be recast as a system of conservation laws, for the velocity $v = \partial_t y$ and the deformation gradient $F = \nabla y$, in the form

$$\begin{aligned} \partial_t F_{i\alpha} &= \partial_\alpha v_i \\ \partial_t v_i &= \partial_\alpha T_{i\alpha}(F)\,, \end{aligned} \tag{1.2}$$

$i, \alpha = 1, \dots, 3$. The equivalence holds for solutions (v, F) with $F = \nabla y$, i.e. subject to the set of differential constraints

$$\partial_\beta F_{i\alpha} - \partial_\alpha F_{i\beta} = 0\,. \tag{1.3}$$

Equation (1.3) is an involution: if it holds initially it is propagated by $(1.2)_1$ to hold for all times.

Motivated by the requirements imposed on the theory of thermoelasticity from consistency with the Clausius-Duhem inequality of thermodynamics, one often imposes the assumption of hyperelasticity, i.e. that T is expressed as the gradient of a stored energy function $W : \mathrm{Mat}^{3\times 3} \to [0,\infty)$

$$T(F) = \frac{\partial W}{\partial F}(F)\,. \tag{s}$$

The principle of material frame indifference dictates that W remains invariant under rotations

$$W(OF) = W(F) \quad \text{for all orthogonal matrices } O \in O(3).$$

Convexity of the stored energy W is too restrictive and even incompatible with certain physical requirements. It conflicts with frame indifference in conjunction with the requirement that the energy increase without bound as $\det F \to 0^+$. In addition, convexity of the energy together with the axiom of frame indifference impose restrictions on the induced Cauchy stresses that rule out certain naturally occurring states of stress (*e.g.* [5, Sec 8], [3, Sec 4.8]). As a result, it has been replaced in the theory of elastostatics by various weaker notions such as quasi-convexity, rank-1 convexity or polyconvexity, see [1] or [2] for a recent survey. Here, we adopt the assumption of polyconvexity which postulates that

$$W(F) = g(F, \operatorname{cof} F, \det F)\,,$$

where g is a strictly convex function of $\Phi(F) = (F, \operatorname{cof} F, \det F)$, and encompasses various interesting models (*e.g.* [3]).

Convexity of the entropy is known to provide a stabilizing mechanism for thermomechanical processes, and entropy inequalities for convex entropies have been employed in the theory of hyperbolic conservation laws as an admissibility criterion for weak solutions [14] and provide stability frameworks for classical solutions [6], [11]. On the other hand, for many physical systems convexity is an unnatural assumption and the question arises to understand the mechanisms that provide stability in such contexts. In the elastodynamics model, for instance, the lack of convexity in the stored energy makes the mechanical energy $E = \frac{1}{2}v^2 + W(F)$ non-convex, and induces an array of questions regarding the stability the model and its various approximating theories. Our objective is to contribute to a program [15, 10, 13] of understanding such issues and suggest remedies especially as it pertains with the stable approximation of elastodynamics by stress relaxation theories.

It is instructive to compare the properties of the elastodynamics system for the cases of a convex and a polyconvex stored energy. For convex stored energies, the theory of relative entropy [6] yields stability of smooth solutions of (1.2) within the approximating theory of viscoelasticity of the rate type

$$\begin{aligned} \partial_t F_{i\alpha} &= \partial_\alpha v_i \\ \partial_t v_i &= \partial_\alpha T_{i\alpha}(F) + \varepsilon \partial_\alpha \partial_\alpha v_i \end{aligned} \tag{1.4}$$

Convexity of the entropy has a stabilizing effect for general relaxation approximations [4], [18], and a relative entropy computation [13] shows that convex elastodynamics is stable within the theory of stress relaxation

$$\begin{aligned} \partial_t F_{i\alpha} &= \partial_\alpha v_i \\ \partial_t v_i &= \partial_\alpha S_{i\alpha} \\ \partial_t (S_{i\alpha} - f_{i\alpha}(F)) &= -\frac{1}{\varepsilon}(S_{i\alpha} - T_{i\alpha}(F)). \end{aligned} \tag{1.5}$$

The latter model may be visualized within the framework of viscoelasticity with memory

$$S = f(F) + \int_{-\infty}^{t} \frac{1}{\varepsilon} e^{-\frac{1}{\varepsilon}(t-\tau)} h(F(\cdot,\tau))\, d\tau$$

with the equilibrium stress $T(F)$ decomposed into an elastic and viscoelastic contribution, $T(F) = f(F) + h(F)$, $f = \frac{\partial W_I}{\partial F}$ and $T = \frac{\partial W}{\partial F}$, and a kernel exhibiting a single relaxation time $\frac{1}{\varepsilon}$. The approximation (1.5) is consistent with the second law of thermodynamics, provided the potential of the instantaneous elastic response W_I dominates the potential of the equilibrium response W.

As convexity is largely incompatible with material frame indifference, the effect of adopting weaker notions of convexity on the stability of thermomechanical processes needs to be understood. The elastodynamics system has been a test ground to study such issues. An analog of the Lax-entropy admissibility that exploits the structure of involutions and is applicable to rank-1 convex energies has been proposed in [7]. Insight was recently obtained on the structure of polyconvex elastodynamics [10], where, due to kinematic constraints on the null-Lagrangians [15], (1.2) can be embedded into the symmetric hyperbolic system

$$
\begin{aligned}
\partial_t v_i &= \partial_\alpha \left(\frac{\partial g}{\partial \Xi^A}(\Xi) \frac{\partial \Phi^A}{\partial F_{i\alpha}}(F) \right) \\
\partial_t \Xi^A &= \partial_\alpha \left(\frac{\partial \Phi^A}{\partial F_{i\alpha}}(F) v_i \right)
\end{aligned}
\tag{1.6}
$$

and be visualized as constrained evolution thereof. The augmented system admits the convex entropy $\eta = \frac{1}{2}|v|^2 + g(\Xi)$ and is symmetrizable. A relative entropy calculation shows stability of smooth solutions [9], while an analogous embedding of the viscoelasticity system (1.4) yields stability of polyconvex elastodynamics within viscoelasticity of the rate type [13].

Convexity of the entropy is a dictum of stability for relaxation approximations; at the same time it is not a consequence of thermodynamical consistency of relaxation theories with the Clausius-Duhem inequality [16, 13]. A natural question then arises whether relaxation theories that forego convexity can approximate in a stable way limit theories of polyconvex elastodynamics. This question is pursued here for the paradigm

$$
\begin{aligned}
\partial_t v_i - \partial_\alpha \left(T^A \frac{\partial \Phi^A}{\partial F_{i\alpha}} \right) &= 0 \\
\partial_t F_{i\alpha} - \partial_\alpha v_i &= 0 \\
\partial_t \left(T^A - \frac{\partial \sigma_I}{\partial \Xi^A}(\Phi(F)) \right) &= -\frac{1}{\varepsilon} \left(T^A - \frac{\partial \sigma_E}{\partial \Xi^A}(\Phi(F)) \right) \\
\partial_\beta F_{i\alpha} - \partial_\alpha F_{i\beta} &= 0\,.
\end{aligned}
\tag{1.7}
$$

The format of this stress-relaxation model is motivated by the embedding of (1.2) to (1.6), and system (1.7) formally approximates as $\varepsilon \to 0$ the equations of polyconvex elastodynamics. The system (1.7) has the property that it can be embedded to an augmented relaxation system (see (3.6)) and the latter is endowed with an entropy inequality for a convex entropy. The reduced entropy inherited by (1.7) is of the form

$$
\mathcal{E} = \frac{1}{2}|v|^2 + \Psi(\Phi(F), \tau)
$$

with Ψ is a convex function and $\Phi(F) = (F, \operatorname{cof} F, \det F)$, and $\mathcal{E}$ is not convex but rather of polyconvex type. We prove using a relative entropy computation and the null-Lagrangian structure that this theory approximates in a stable way smooth solutions of (1.2) with polyconvex stored energy.

The article is organized as follows. In Section 2 we present the embedding of (1.2) into the augmented system (1.6), define the relative entropy for the augmented system and outline how it is used in [13] to obtain stability and convergence of smooth solutions of (1.4) to smooth solutions of (1.2). In section 3 we define the augmented relaxation system (3.6), show that the augmented system is endowed with a convex entropy, and exhibit the inherited relative entropy calculation (3.24) for the system (1.7). This culminates into the stability and convergence Theorem 4.1 between solutions of the relaxation model (1.7) and the polyconvex elastodynamics system (1.2).

2. Polyconvex elastodynamics

The system of elastodynamics (1.1) can be expressed in the form of a system of conservation laws

$$\begin{aligned} \partial_t F_{i\alpha} &= \partial_\alpha v_i \\ \partial_t v_i &= \partial_\alpha T_{i\alpha}(F)\,. \end{aligned} \tag{2.1}$$

where $v = y_t$ and $F = \nabla y$. The equivalence of the two formulations holds for functions F that are gradients. Note that $F = \nabla y$ if and only if it satisfies

$$\partial_\beta F_{i\alpha} - \partial_\alpha F_{i\beta} = 0 \tag{2.2}$$

and, technically, the system (1.1) is equivalent to both (2.1) and (2.2). The latter relation may be viewed as a constraint in the initial data that is propagated by solutions of $(2.1)_1$.

2.1. **The symmetrizable extension of polyconvex elastodynamics.**

We consider a theory of polyconvex hyperelasticity, that is the Piola-Kirchoff stress is derived from a potential $T(F) = \frac{\partial W(F)}{\partial F}$ where the stored energy $W : \mathrm{Mat}^{3\times 3} \to [0,\infty)$ factorizes as a convex function of the minors of F:

$$W(F) = \big(g \circ \Phi\big)(F)\,, \quad \text{where } \Phi(F) = (F, \operatorname{cof} F, \det F) \tag{2.3}$$

and $g : \mathrm{Mat}^{3\times 3} \times \mathrm{Mat}^{3\times 3} \times \mathbb{R} \to \mathbb{R}$ is convex. The cofactor matrix $\mathrm{cof}\, F$ and the determinant $\det F$ are

$$
\begin{aligned}
(\mathrm{cof}\, F)_{i\alpha} &= \frac{1}{2}\epsilon_{ijk}\epsilon_{\alpha\beta\gamma}F_{j\beta}F_{k\gamma}\,, \\
\det F &= \frac{1}{6}\epsilon_{ijk}\epsilon_{\alpha\beta\gamma}F_{i\alpha}F_{j\beta}F_{k\gamma} = \frac{1}{3}(\mathrm{cof}\, F)_{i\alpha}F_{i\alpha}\,.
\end{aligned}
$$

We review a symmetrizable extension of polyconvex elastodynamics [10], based on certain kinematic identities on $\det F$ and $\mathrm{cof}\, F$ from [15]. The components of $\Phi(F)$ are null Lagrangians and satisfy the identities

$$
\frac{\partial}{\partial x^\alpha}\left(\frac{\partial \Phi^A}{\partial F_{i\alpha}}(\nabla y)\right) \equiv 0
$$

for any smooth map $y(x,t)$. Equivalently, this is expressed as

$$
\partial_\alpha\left(\frac{\partial \Phi^A}{\partial F_{i\alpha}}(F)\right) = 0\,, \quad \forall F \text{ with } \partial_\beta F_{i\alpha} - \partial_\alpha F_{i\beta} = 0\,. \tag{2.4}
$$

The kinematic compatibility equation $(2.1)_1$ implies

$$
\partial_t \Phi^A(F) - \partial_\alpha\left(\frac{\partial \Phi^A}{\partial F_{i\alpha}}(F)v_i\right) = 0\,. \tag{2.5}
$$

Strictly speaking (2.5) do not form what is called in the theory of conservation laws entropy - entropy flux pairs, as they hold only for F that are gradients, i.e. $\forall F$ with $\partial_\beta F_{i\alpha} - \partial_\alpha F_{i\beta} = 0$.

This motivates to embed (2.1) into the system of conservation laws

$$
\begin{aligned}
\partial_t v_i &= \partial_\alpha\left(\frac{\partial g}{\partial \Xi^A}(\Xi)\frac{\partial \Phi^A}{\partial F_{i\alpha}}(F)\right) \\
\partial_t \Xi^A &= \partial_\alpha\left(\frac{\partial \Phi^A}{\partial F_{i\alpha}}(F)v_i\right).
\end{aligned} \tag{2.6}
$$

Note that $\Xi = (F, Z, w)$ takes values in $\mathrm{Mat}^{3\times 3} \times \mathrm{Mat}^{3\times 3} \times \mathbb{R} \simeq \mathbb{R}^{19}$ and is treated as a new dependent variable. The extension has the following properties:

(i) If $F(\cdot, 0)$ is a gradient then $F(\cdot, t)$ remains a gradient $\forall t$.

(ii) If $\Xi(\cdot, 0) = \Phi(F(\cdot, 0))$ with $F(\cdot, 0) = \nabla y_0$, then $\Xi(\cdot, t) = \Phi(F(\cdot, t))$ where $F(\cdot, t) = \nabla y(\cdot, t)$. In other words, the system of elastodynamics can be visualized as constrained evolution of (2.6).

(iii) The enlarged system admits a strictly convex entropy

$$
\eta(v, \Xi) = \frac{1}{2}|v|^2 + g(\Xi)
$$

and is thus symmetrizable (along solutions that are gradients).

Property (iii) is again based on the null-Lagrangian structure and η is not an entropy in the usual sense of the theory of conservation laws. Rather, the identity

$$\partial_t\left[\frac{1}{2}|v|^2+g(\Xi)\right]-\partial_\alpha\left[\sum_{i,A}v_i\frac{\partial g(\Xi)}{\partial\Xi^A}\frac{\partial\Phi^A(F)}{\partial F_{i\alpha}}\right]=0$$

is again based on (2.4).

2.2. **Relative energy for viscosity approximations.** Let $\widehat{y}$ be a smooth solution of (1.1) and y^ε a solution of the viscosity approximation (1.4). Then $(\widehat{v},\widehat{F})$ satisfy (2.1) and $(v^\varepsilon,F^\varepsilon)$ the viscous approximation

$$\begin{aligned}\partial_t F_{i\alpha}&=\partial_\alpha v_i\\ \partial_t v_i&=\partial_\alpha T_{i\alpha}(F)+\varepsilon\partial_\alpha\partial_\alpha v_i\end{aligned}\tag{2.7}$$

We outline a strategy [13, 9] for comparing the two systems. As already noted the function $(\widehat{v},\widehat{\Xi})$ with $\widehat{\Xi}=\Phi(\widehat{F})\in\mathbb{R}^D$, $D=19$ for $d=3$ while $D=5$ for $d=2$, solves the enlarged elastodynamics system (2.6). Similarly, the function (v,Ξ) with $\Xi=\Phi(F)$ solves the extended viscosity approximation

$$\begin{cases}\partial_t\Xi^A=\partial_\alpha\left(\frac{\partial\Phi^A(F)}{\partial F_{i\alpha}}v_i\right)\\ \partial_t v_i=\partial_\alpha\left(\frac{\partial g(\Xi)}{\partial\Xi^A}\frac{\partial\Phi^A(F)}{\partial F_{i\alpha}}\right)+\varepsilon\partial_\alpha\partial_\alpha v_i\,.\end{cases}\tag{2.8}$$

Smooth solutions of (2.6) and (2.8) can be compared using a relative energy identity. Define the relative energy

$$\eta_r(v,\Xi\mid\widehat{v},\widehat{\Xi}):=\frac{1}{2}|v-\widehat{v}|^2+g(\Xi)-g(\widehat{\Xi})-\frac{\partial g(\widehat{\Xi})}{\partial\Xi^A}(\Xi^A-\widehat{\Xi}^A)$$

and the associated (relative) flux

$$q_r^\alpha(v,\Xi\mid\widehat{v},\widehat{\Xi}):=\left(\frac{\partial g(\Xi)}{\partial\Xi^A}-\frac{\partial g(\widehat{\Xi})}{\partial\Xi^A}\right)(v_i-\widehat{v}_i)\frac{\partial\Phi^A(F)}{\partial F_{i\alpha}}\,,$$

$\alpha=1,2,3$. Using (2.4), it follows that

$$\begin{aligned}&\partial_t\eta_r-\nabla\cdot q_r+\varepsilon|\nabla(v-\widehat{v})|^2\\ &\quad=Q+\frac{\varepsilon}{2}\triangle|v-\widehat{v}|^2+\varepsilon(v-\widehat{v})\cdot\triangle\widehat{v}\,,\end{aligned}\tag{2.9}$$

where Q is a quadratic error term of the form

$$\begin{aligned}
Q := & \frac{\partial^2 g(\widehat{\Xi})}{\partial \Xi^A \partial \Xi^B} \partial_\alpha \widehat{\Xi}^B \left(\frac{\partial \Phi^A(F)}{\partial F_{i\alpha}} - \frac{\partial \Phi^A(\widehat{F})}{\partial F_{i\alpha}} \right) (v_i - \widehat{v}_i) \\
& + \partial_\alpha \widehat{v}_i \left(\frac{\partial g(\Xi)}{\partial \Xi^A} - \frac{\partial g(\widehat{\Xi})}{\partial \Xi^A} \right) \left(\frac{\partial \Phi^A(F)}{\partial F_{i\alpha}} - \frac{\partial \Phi^A(\widehat{F})}{\partial F_{i\alpha}} \right) \\
& + \partial_\alpha \widehat{v}_i \left(\frac{\partial g(\Xi)}{\partial \Xi^A} - \frac{\partial g(\widehat{\Xi})}{\partial \Xi^A} - \frac{\partial^2 g(\widehat{\Xi})}{\partial \Xi^A \partial \Xi^B} (\Xi^B - \widehat{\Xi}^B) \right) \frac{\partial \Phi^A(\widehat{F})}{\partial F_{i\alpha}} \, .
\end{aligned} \tag{2.10}$$

The details of the lengthy computation can be found in [13] and use in a substantial way the null-Lagrangian identity (2.4). The reader may consult the following section where a similar computation is performed for a relaxation approximation of (2.1).

The identity (2.9) concerns general solutions of the enlarged systems (2.6) and (2.8). It is then restricted to functions $(v, \Xi = \Phi(F))$ and $(\widehat{v}, \widehat{\Xi} = \Phi(\widehat{F}))$ as emerge from the embeddings of (2.1) to (2.6) and (2.7) to (2.8). The resulting relative energy and corresponding flux are

$$\begin{aligned}
H_r &= \eta_r(v, \Phi(F) \mid \widehat{v}, \Phi(\widehat{F})) \\
&= \frac{1}{2}|v - \widehat{v}|^2 + g(\Phi(F)) - g(\Phi(\widehat{F})) - \frac{\partial g}{\partial \Xi^A}(\Phi(\widehat{F}))(\Phi(F)^A - \Phi(\widehat{F})^A) \, , \\
Q_r^\alpha &= q_r^\alpha(v, \Phi(F) \mid \widehat{v}, \Phi(\widehat{F})) \\
&= \left(\frac{\partial g}{\partial \Xi^A}(\Phi(F)) - \frac{\partial g}{\partial \Xi^A}(\Phi(\widehat{F})) \right) (v_i - \widehat{v}_i) \frac{\partial \Phi^A(F)}{\partial F_{i\alpha}}
\end{aligned} \tag{2.11}$$

Under a uniform convexity assumption for g, one can control the norm

$$\Psi_d(t) := \int_{\mathbb{R}^d} \Big(|v - \widehat{v}|^2 + |\Phi(F) - \Phi(\widehat{F})|^2 \Big)(x,t) \, dx \, , \tag{2.12}$$

$d = 2, 3$, for $0 < t \le T$. This norm is stronger than L^2 with respect to the growth in F. We show:

Theorem 2.1. [13]. *Let $\{y^\varepsilon\}$ be a family of smooth solutions to* (1.4) *and $\widehat{y}$ a smooth solution of* (1.1)*, defined on $\mathbb{R}^d \times [0,T]$, $d = 2, 3$ and decaying sufficiently fast at infinity. Assume that g satisfies*

$$0 < \gamma I \le \nabla^2_\Xi g(\Xi) \le \Gamma I \, , \qquad |\nabla^3_\Xi g(\Xi)| \le M \, . \tag{2.13}$$

There exists a constant $C = C(T, \gamma, \Gamma, M, \widehat{v}, \widehat{\Xi}) > 0$ *such that*

$$\Psi_d(t) \le C \left(\Psi_d(0) + \varepsilon^2\right) . \tag{2.14}$$

If moreover the data satisfy $\Psi_d^\varepsilon(0) \to 0$ *as* $\varepsilon \downarrow 0$*, then*

$$\sup_{t\in[0,T]} \left(\|v^\varepsilon(\cdot,t) - \widehat{v}(\cdot,t)\|_{L^2(\mathbb{R}^d)} + \|\Phi(F^\varepsilon(\cdot,t)) - \Phi(\widehat{F}(\cdot,t))\|_{L^2(\mathbb{R}^d)}\right) \to 0\,,$$

as $\varepsilon \downarrow 0$.

3. A relaxation scheme for polyconvex elastodynamics

We next consider the stress relaxation model

$$\begin{aligned} \partial_t v_i - \partial_\alpha \left(T^A \frac{\partial \Phi^A}{\partial F_{i\alpha}} \right) &= 0 \\ \partial_t F_{i\alpha} - \partial_\alpha v_i &= 0 \\ \partial_t \left(T^A - \frac{\partial \sigma_I}{\partial \Xi^A}(\Phi(F)) \right) &= -\frac{1}{\varepsilon} \left(T^A - \frac{\partial \sigma_E}{\partial \Xi^A}(\Phi(F)) \right) \\ \partial_\beta F_{i\alpha} - \partial_\alpha F_{i\beta} &= 0 \end{aligned} \tag{3.1}$$

and wish to compare the equations of elastodynamics

$$\begin{aligned} \partial_t v_i - \partial_\alpha \left(\frac{\partial \sigma_E}{\partial \Xi^A}(\Phi(F)) \frac{\partial \Phi^A}{\partial F_{i\alpha}} \right) &= 0 \\ \partial_t F_{i\alpha} - \partial_\alpha v_i &= 0 \end{aligned} \tag{3.2}$$

Note that the stress in the model (3.2) satisfies

$$S_\infty = \frac{\partial}{\partial F} \sigma_E(\Phi(F))$$

and thus, when σ_E is convex, the model (3.2) corresponds to polyconvex elasticity.

The model (3.1) corresponds to a stress relaxation theory where the stress is decomposed into an instantaneous and a viscoelastic part

$$S = T^A \frac{\partial \Phi^A}{\partial F} = \frac{\partial(\sigma_I \circ \Phi)}{\partial F} + \tau^A \frac{\partial \Phi^A}{\partial F} \tag{3.3}$$

where the instantaneous elasticity is derived from a polyconvex potential $\sigma_I(\Phi(F))$ while the viscoelastic part is determined by internal variables τ^A evolving according to the model

$$\partial_t \tau^A = -\frac{1}{\varepsilon}\Big(\tau^A - \frac{\partial(\sigma_E - \sigma_I)}{\partial \Xi^A}(\Phi(F))\Big) \tag{3.4}$$

Note that when expressed in terms of the motion y the model (3.1) takes the form

$$
\begin{aligned}
\frac{\partial^2 y}{\partial t^2} &= \nabla \cdot \Big(\frac{\partial(\sigma_I \circ \Phi)}{\partial F}(\nabla y) + \tau^A \frac{\partial \Phi^A}{\partial F}(\nabla y) \Big) \\
\frac{\partial \tau^A}{\partial t} &= -\frac{1}{\varepsilon}\Big(\tau^A - \frac{\partial(\sigma_E - \sigma_I)}{\partial \Xi^A}(\Phi(\nabla y)) \Big)
\end{aligned}
\tag{3.5}
$$

Of course it may recast in the form of a theory with memory by integrating (3.4). We will see that the model (3.1) has very interesting structural properties.

3.1. **The augmented relaxation system.** The somewhat unconventional form of the above stress relaxation theory can be motivated (and was guided) by an attempt to mimick the structure of the polyconvex elastodynamics system described in section 2.

Consider the relaxation system

$$
\begin{aligned}
\partial_t v_i - \partial_\alpha \Big(\frac{\partial \Phi^A}{\partial F_{i\alpha}} T^A \Big) &= 0 \\
\partial_t \Xi^A - \partial_\alpha \Big(\frac{\partial \Phi^A}{\partial F_{i\alpha}} v_i \Big) &= 0 \\
\partial_t \Big(T^A - \frac{\partial \sigma_I}{\partial \Xi^A}(\Xi) \Big) &= -\frac{1}{\varepsilon}\Big(T^A - \frac{\partial \sigma_E}{\partial \Xi^A}(\Xi) \Big)
\end{aligned}
\tag{3.6}
$$

For this model the stress is

$$
S_{i\alpha} = T^A \frac{\partial \Phi^A}{\partial F_{i\alpha}}
$$

ans formally as $\varepsilon \to 0$

$$
S_{i\alpha} = T^A(\Xi)\Big|_{eq} \frac{\partial \Phi^A}{\partial F_{i\alpha}} = \frac{\partial \sigma_E}{\partial \Xi^A}(\Xi) \frac{\partial \Phi^A}{\partial F_{i\alpha}}
$$

it approximates the extended elastodynamics system

$$
\begin{aligned}
\partial_t v_i - \partial_\alpha \Big(\frac{\partial \sigma_E}{\partial \Xi^A}(\Xi) \frac{\partial \Phi^A}{\partial F_{i\alpha}} \Big) &= 0 \\
\partial_t \Xi^A - \partial_\alpha \Big(\frac{\partial \Phi^A}{\partial F_{i\alpha}} v_i \Big) &= 0
\end{aligned}
\tag{3.7}
$$

Therefore, formally, the $\varepsilon \to 0$ relaxation limit of (3.6) produces the extended elastodynamics system (3.7). Observe that solutions of (3.1) satisfy the kinematic constraints (2.5) and thus, for a polyconvex stored energy, the relaxation system (3.1) enjoys the same relation with the system (3.6) as the equations of elastodynamics (2.1) have with the system (3.2).

Next, we develop the Chapman-Enskog expansion for the relaxation limit from (3.6) to (3.7). Introduce the expansion for the internal variable T^A

$$T^{A,\varepsilon} = T_0^A + \varepsilon T_1^A + O(\varepsilon^2)$$

and, accordingly,

$$S_{i\alpha}^{\varepsilon} = T_0^A \frac{\partial \Phi^A}{\partial F_{i\alpha}} + \varepsilon T_1^A \frac{\partial \Phi^A}{\partial F_{i\alpha}} + O(\varepsilon^2)$$

to (3.6) in order to obtain

$$\begin{aligned} T_0^A &= \frac{\partial \sigma_E}{\partial \Xi^A}(\Xi) \\ \partial_t \Big(\frac{\partial \sigma_E}{\partial \Xi^A}(\Xi) - \frac{\partial \sigma_I}{\partial \Xi^A}(\Xi) \Big) &= -T_1^A + O(\varepsilon) \end{aligned}$$

The effective momentum equation becomes

$$\begin{aligned} \partial_t v_i - \partial_\alpha \Big(T_0^A \frac{\partial \Phi^A}{\partial F_{i\alpha}} \Big) &= \varepsilon \partial_\alpha \Big(T_1^A \frac{\partial \Phi^A}{\partial F_{i\alpha}} \Big) + O(\varepsilon^2) \\ &= \varepsilon \partial_\alpha (D_{i\alpha}^{j\beta} \partial_\beta v_j) + O(\varepsilon^2) \end{aligned}$$

where

$$D_{i\alpha}^{j\beta} := \frac{\partial^2 (\sigma_I - \sigma_E)}{\partial \Xi^A \partial \Xi^B} \frac{\partial \Phi^A}{\partial F_{i\alpha}} \frac{\partial \Phi^B}{\partial F_{j\beta}} \tag{3.8}$$

Thus, as $\varepsilon \to 0$, the relaxation process is approximated by the system

$$\begin{aligned} \partial_t \Xi^A - \partial_\alpha \Big(\frac{\partial \Phi^A}{\partial F_{i\alpha}} v_i \Big) &= 0 \\ \partial_t v_i - \partial_\alpha \Big(T_0^A \frac{\partial \Phi^A}{\partial F_{i\alpha}} \Big) &= \varepsilon \partial_\alpha (D_{i\alpha}^{j\beta} \partial_\beta v_j) \end{aligned}$$

Note that for $\Sigma := \sigma_I - \sigma_E$ convex the diffusivity tensor D satisfies the ellipticity condition $D_{i\alpha}^{j\beta} M_{i\alpha} M_{j\beta} \geq 0\,,\ \forall M \in \mathbb{R}^{3\times 3}$. The latter is stronger than the Legendre-Hadamard condition, and can be achieved for both the instantaneous potential $\sigma_I \circ \Phi$ and the equilibrium potential $\sigma_E \circ \Phi$ polyconvex.

3.2. **Entropy of the augmented relaxation system.** We next construct an entropy for the augmented relaxation system. Note that, if a function $\Psi(\Xi, \tau)$ can be constructed defined $\forall\, (\Xi, \tau)$ and satisfying

$$\begin{aligned} \frac{\partial \Psi}{\partial \Xi^A}(\Xi, \tau) = T^A &= \frac{\partial \sigma_I(\Xi)}{\partial \Xi^A} + \tau^A \\ \frac{\partial \Psi}{\partial \tau^A} \Big(\tau^A - \frac{\partial (\sigma_E - \sigma_I)}{\partial \Xi^A} \Big) &\geq 0 \qquad \forall\, (\Xi, \tau)\,, \end{aligned} \tag{3.9}$$

then the relaxation system is endowed with an H-theorem

$$\partial_t\Big(\frac{1}{2}|v|^2 + \Psi(\Xi,\tau)\Big) - \partial_\alpha\big(v_i S_{i\alpha}\big) + \frac{1}{\varepsilon}\frac{\partial\Psi}{\partial\tau^A}\big(\tau^A - \frac{\partial(\sigma_E - \sigma_I)}{\partial\Xi^A}\big) = 0\,. \quad (3.10)$$

This entropy identity is based on the null-Lagrangian property (2.4) and follows, using (3.6), (2.4) and (3.9), by the computation

$$\begin{aligned}
\partial_t\Big(\frac{1}{2}|v|^2 + \Psi(\Xi,\tau)\Big) &= v_i\partial_t v_i + \frac{\partial\Psi}{\partial\Xi^A}\partial_t\Xi^A + \frac{\partial\Psi}{\partial\tau^A}\partial_t\tau^A \\
&= v_i\partial_\alpha S_{i\alpha} + \frac{\partial\Psi}{\partial\Xi^A}\partial_\alpha\big(\frac{\partial\Phi^A}{\partial F_{i\alpha}}v_i\big) + \frac{\partial\Psi}{\partial\tau^A}\partial_t\tau^A \\
&= v_i\partial_\alpha S_{i\alpha} + \frac{\partial\Psi}{\partial\Xi^A}\frac{\partial\Phi^A}{\partial F_{i\alpha}}\partial_\alpha v_i + \frac{\partial\Psi}{\partial\tau^A}\partial_t\tau^A \\
&= \partial_\alpha(v_i S_{i\alpha}) - \frac{1}{\varepsilon}\frac{\partial\Psi}{\partial\tau^A}\big(\tau^A - \frac{\partial(\sigma_E - \sigma_I)}{\partial\Xi^A}\big)
\end{aligned}$$

Our next objective is to examine the solvability of (3.9) and study the convexity of the entropy. In this matter we follow the analysis in [13, 17]. Integrating $(3.9)_1$, we see that

$$\Psi(\Xi,\tau) = \sigma_I(\Xi) + \Xi\cdot\tau + G(\tau) \quad (3.11)$$

where the integrating factor $G(\tau)$ has to be selected so that it satisfies the inequality

$$\big(\Xi^A + \frac{\partial G}{\partial\tau^A}\big)\big(\tau^A + \frac{\partial\Sigma}{\partial\tau^A}\big) \geq 0 \quad \forall\,(\Xi,\tau) \quad (3.12)$$

where $\Sigma = \sigma_I - \sigma_E$.

For the solvability of (3.12) we have

Lemma 3.1. *The functions $G(\tau)$ and $\Sigma(\Xi)$ satisfy*

$$(\Xi + \nabla_\tau G)\cdot(\tau + \nabla_\Xi\Sigma) \geq 0 \quad \forall\,(\Xi,\tau)\,, \quad (3.13)$$

if and only if

$$\begin{cases} \Xi + \nabla_\tau G = 0 \quad \textit{iff} \quad \tau + \nabla_\Xi\Sigma = 0 \\ G \textit{ is convex} \\ \Sigma \textit{ is convex} \end{cases} \quad (3.14)$$

Proof. We first show that (3.13) implies (3.14). Fix Ξ^0, τ^0 such that $\Xi^0 + \nabla_\tau G(\tau^0) = 0$. Consider a fixed direction e^A and the increment along this direction $\Xi = \Xi^0 + te^A$. Then (3.13) implies that $e^A\cdot(\tau^0 + \nabla_\Xi\Sigma(\Xi^0)) = 0$ for every direction e^A and thus $\tau^0 + \nabla_\Xi\Sigma(\Xi^0) = 0$. Similarly, if Ξ^0, τ^0 are such that $\tau^0 + \nabla_\Xi \Sigma(\Xi^0) = 0$ then also $\Xi^0 + \nabla_\tau G(\tau^0) = 0$. This proves the first statement in (3.14).

Fix now Ξ_1, Ξ_2 and let $\tau_2 = -\nabla_\Xi \Sigma(\Xi_2)$. Then $\Xi_2 = -\nabla_\tau G(\tau_2)$, (3.13) is rewritten as

$$(\Xi_1 - \Xi_2) \cdot \big(\nabla_\Xi \Sigma(\Xi_1) - \nabla_\Xi \Sigma(\Xi_2)\big) \geq 0 \tag{3.15}$$

and Σ is convex. A similar argument shows that G is convex.

The converse is proved by re-expressing the convexity inequality (3.15) in the form (3.13) by using the first statement in the right of (3.14). $\square$

Lemma 3.1 indicates that the solvability of (3.9) is equivalent to the convexity of $\Sigma := \sigma_I - \sigma_E$. To complete the details of the construction of Ψ, we assume for simplicity that

$$\nabla^2_\Xi \Sigma > 0 \quad \text{and} \quad \nabla_\Xi \Sigma : \mathbb{R}^D \to \mathbb{R}^D \text{ is onto,} \tag{h$_0$}$$

with $D = 19$ for $d = 3$ and $D = 5$ for $d = 2$. Define the inverse map $(\nabla_\Xi \Sigma)^{-1} : \mathbb{R}^D \to \mathbb{R}^D$, and let $h(\tau) = -(\nabla_\Xi \Sigma)^{-1}(-\tau)$. Then $\nabla_\tau h$ is symmetric and the differential system $\nabla_\tau G = h$ is solvable. Its solution G is a convex function and satisfies

$$\begin{aligned} \nabla_\tau G(\tau) &= -\left(\nabla_\Xi \Sigma\right)^{-1}(-\tau) \\ \nabla^2_\tau G(\tau) &= \left[\nabla^2_\Xi \Sigma(-\nabla_\tau G)\right]^{-1} \end{aligned} \tag{3.16}$$

Ψ is defined by (3.11) with G as above. Observe that, by (3.9) and (3.14),

$$\begin{aligned} &\frac{\partial \Psi}{\partial \Xi^A}(\Xi, -\nabla_\Xi \Sigma) = \frac{\partial \sigma_E}{\partial \Xi^A}(\Xi) \\ &\frac{\partial \Psi}{\partial \tau^A}(\Xi, -\nabla_\Xi \Sigma) = \Xi^A + \frac{\partial G}{\partial \tau^A}\Big|_{\tau^A = -\frac{\partial(\sigma_I - \sigma_E)}{\partial \Xi^A}} = 0 \end{aligned} \tag{3.17}$$

and, by selecting a normalization constant,

$$\Psi(\Xi, -\nabla_\Xi \Sigma) = \sigma_E(\Xi) \tag{3.18}$$

We next consider the convexity of $\Psi(\Xi, \tau)$ determined by the matrix

$$\nabla^2_{(\Xi,\tau)} \Psi = \begin{bmatrix} \nabla^2_\Xi \sigma_I & \mathbb{I} \\ \mathbb{I} & \nabla^2_\tau G \end{bmatrix}$$

Lemma 3.2. *Let $\Sigma = \sigma_I - \sigma_E$ satisfy* (h$_0$) *and assume that σ_I, Σ satisfy for $\gamma_I > \gamma_v > 0$*

$$\nabla^2_\Xi \sigma_I \geq \gamma_I > \gamma_v \geq \nabla^2_\Xi \Sigma > 0 \tag{h$_1$}$$

Then for some $\delta > 0$ we have

$$\nabla^2_{(\Xi,\tau)} \Psi \geq \delta \, \mathbb{I}_{(\Xi,\tau)}$$

Proof. By differentiating the relation $\nabla_\Xi \Sigma = -(\nabla_\tau G)^{-1}(-\Xi)$ we get

$$\big(\nabla^2_\Xi \Sigma\big)(-\nabla_\tau G) \cdot \nabla^2_\tau G(\tau) = \mathbb{I}$$

Hence

$$\begin{aligned}\Big(\nabla^2_{(\Xi,\tau)} \Psi\Big) \ (\Xi,\tau)\cdot(\Xi,\tau) &= (\nabla^2_\Xi \sigma_I)\Xi\cdot\Xi + 2\Xi\cdot\tau + \big(\nabla^2_\Xi \Sigma\big)^{-1}\tau\cdot\tau \\ &\geq \gamma_I |\Xi|^2 + 2\Xi\cdot\tau + \frac{1}{\gamma_v}|\tau|^2 \\ &\geq (\gamma_I - \delta)|\Xi|^2 + \Big(\frac{1}{\gamma_v} - \frac{1}{\delta}\Big)|\tau|^2\end{aligned}$$

which can be made positive definite by selecting $\gamma_I > \delta > \gamma_v$. □

Remark 3.3. Hypothesis (h_1) implies that σ_E must be convex, which dictates that the limiting equations arise from a polyconvex energy.

3.3. **Relative entropy for the augmented system.** Next we compare a solution (v,Ξ,τ) of the system (3.6) with a solution $(\hat{v},\widehat{\Xi})$ of the extended elastodynamics system (3.7), using a relative entropy calculation in the spirit of [13, 18].

The relative entropy is defined by taking the Taylor polynomial of a nonequilibrium relative to a Maxwellian solution

$$\begin{aligned}\mathcal{E}_r &:= \frac{1}{2}|v-\hat{v}|^2 + \Psi(\Xi,\tau) - \Psi\Big(\widehat{\Xi}, \frac{\partial(\sigma_E-\sigma_I)}{\partial\Xi}(\widehat{\Xi})\Big) \\ &\quad - \frac{\partial\Psi}{\partial\Xi}(\widehat{\Xi}, -\frac{\partial\Sigma}{\partial\Xi}(\widehat{\Xi}))\cdot(\Xi-\widehat{\Xi}) - \frac{\partial\Psi}{\partial\tau}\Big(\widehat{\Xi}, -\frac{\partial\Sigma}{\partial\Xi}(\widehat{\Xi})\Big)\cdot\Big(\tau - \frac{\partial(\sigma_E-\sigma_I)}{\partial\Xi}(\widehat{\Xi})\Big)\end{aligned}$$

where $\Sigma = \sigma_I - \sigma_E$. By (3.17), (3.18), $\mathcal{E}_r$ has the simple form

$$\mathcal{E}_r = \frac{1}{2}|v-\hat{v}|^2 + \Psi(\Xi, T - \frac{\partial\sigma_I}{\partial\Xi}) - \sigma_E(\widehat{\Xi}) - \frac{\partial\sigma_E}{\partial\Xi}(\widehat{\Xi})\cdot(\Xi-\widehat{\Xi}) \tag{3.19}$$

We now recall the identities: The H-theorem for the relaxation approximation

$$\partial_t\Big(\frac{1}{2}|v|^2 + \Psi(\Xi,\tau)\Big) - \partial_\alpha(v_i S_{i\alpha}) + \frac{1}{\varepsilon}\frac{\partial\Psi}{\partial\tau^A}\big(\tau^A - \frac{\partial(\sigma_E-\sigma_I)}{\partial\Xi^A}\big) = 0 \tag{3.20}$$

and the energy equation for the extended elastodynamics system

$$\partial_t\Big(\frac{1}{2}|\widehat{v}|^2 + \sigma_E(\widehat{\Xi})\Big) - \partial_\alpha\Big(\frac{\partial\sigma_E}{\partial\Xi^A}(\widehat{\Xi})\frac{\partial\Phi^A}{\partial F_{i\alpha}}(\widehat{F})\widehat{v}_i\Big) = 0 \tag{3.21}$$

Finally we form the difference equations

$$\begin{aligned}\partial_t(v_i - \widehat{v}_i) - \partial_\alpha\left(T^A \frac{\partial\Phi^A}{\partial F_{i\alpha}}(F) - \frac{\partial\sigma_E}{\partial\Xi^A}(\widehat{\Xi})\frac{\partial\Phi^A}{\partial F_{i\alpha}}(\widehat{F})\right) = 0,\\ \partial_t(\Xi^A - \widehat{\Xi}^A) - \partial_\alpha\left(\frac{\partial\Phi^A}{\partial F_{i\alpha}}(F)\, v_i - \frac{\partial\Phi^A}{\partial F_{i\alpha}}(\widehat{F})\,\widehat{v}_i\right) = 0\end{aligned}$$

and compute using (3.6) and (3.7) to obtain

$$\begin{aligned}
&\partial_t\Big[\widehat{v}_i(v_i - \widehat{v}_i) + \frac{\partial\sigma_E}{\partial\Xi^A}(\widehat{\Xi})\,(\Xi^A - \widehat{\Xi}^A)\Big]\\
&\qquad - \partial_\alpha\bigg[\widehat{v}_i\left(T^A\frac{\partial\Phi^A}{\partial F_{i\alpha}}(F) - \frac{\partial\sigma_E}{\partial\Xi^A}(\widehat{\Xi})\frac{\partial\Phi^A}{\partial F_{i\alpha}}(\widehat{F})\right)\\
&\qquad\qquad\qquad + \frac{\partial\sigma_E}{\partial\Xi^A}(\widehat{\Xi})\left(\frac{\partial\Phi^A}{\partial F_{i\alpha}}(F)\, v_i - \frac{\partial\Phi^A}{\partial F_{i\alpha}}(\widehat{F})\,\widehat{v}_i\right)\bigg]\\
&= (\partial_t\widehat{v}_i)(v_i - \widehat{v}_i) + \partial_t\left(\frac{\partial\sigma_E}{\partial\Xi^A}(\widehat{\Xi})\right)(\Xi^A - \widehat{\Xi}^A)\\
&\qquad - \partial_\alpha\widehat{v}_i\left(T^A\frac{\partial\Phi^A}{\partial F_{i\alpha}}(F) - \frac{\partial\sigma_E}{\partial\Xi^A}(\widehat{\Xi})\frac{\partial\Phi^A}{\partial F_{i\alpha}}(\widehat{F})\right)\\
&\qquad\qquad\qquad - \partial_\alpha\left(\frac{\partial\sigma_E}{\partial\Xi^A}(\widehat{\Xi})\right)\left(\frac{\partial\Phi^A}{\partial F_{i\alpha}}(F)\, v_i - \frac{\partial\Phi^A}{\partial F_{i\alpha}}(\widehat{F})\,\widehat{v}_i\right)\\
&= -\partial_\alpha\left(\frac{\partial\sigma_E}{\partial\Xi^A}(\widehat{\Xi})\right)\left(\frac{\partial\Phi^A}{\partial F_{i\alpha}}(F) - \frac{\partial\Phi^A}{\partial F_{i\alpha}}(\widehat{F})\right) v_i\\
&\quad - \partial_\alpha\widehat{v}_i\Big[T^A\frac{\partial\Phi^A}{\partial F_{i\alpha}}(F) - \frac{\partial\sigma_E}{\partial\Xi^A}(\widehat{\Xi})\frac{\partial\Phi^A}{\partial F_{i\alpha}}(\widehat{F}) - \frac{\partial^2\sigma_E(\widehat{\Xi})}{\partial\Xi^A\partial\Xi^B}\frac{\partial\Phi^A}{\partial F_{i\alpha}}(\widehat{F})(\Xi^B - \widehat{\Xi}^B)\Big]\\
&=: I
\end{aligned} \tag{3.22}$$

By rearranging the terms and using the null-Lagrangian property (2.4) we may rewrite I in the form

$$\begin{aligned}
I &= -\partial_\alpha\Big[\widehat{v}_i\frac{\partial\sigma_E}{\partial\Xi^A}(\widehat{\Xi})\left(\frac{\partial\Phi^A}{\partial F_{i\alpha}}(F) - \frac{\partial\Phi^A}{\partial F_{i\alpha}}(\widehat{F})\right)\Big]\\
&\quad - \partial_\alpha\left(\frac{\partial\sigma_E}{\partial\Xi^A}(\widehat{\Xi})\right)\left(\frac{\partial\Phi^A}{\partial F_{i\alpha}}(F) - \frac{\partial\Phi^A}{\partial F_{i\alpha}}(\widehat{F})\right)(v_i - \widehat{v}_i)\\
&\quad - (\partial_\alpha\widehat{v}_i)\frac{\partial\Phi^A}{\partial F_{i\alpha}}(\widehat{F})\Big(\frac{\partial\sigma_E}{\partial\Xi^A}(\Xi) - \frac{\partial\sigma_E}{\partial\Xi^A}(\widehat{\Xi}) - \frac{\partial^2\sigma_E}{\partial\Xi^A\partial\Xi^B}(\widehat{\Xi})(\Xi^B - \widehat{\Xi}^B)\Big)\\
&\quad - (\partial_\alpha\widehat{v}_i)\left(\frac{\partial\sigma_E}{\partial\Xi^A}(\Xi) - \frac{\partial\sigma_E}{\partial\Xi^A}(\widehat{\Xi})\right)\left(\frac{\partial\Phi^A}{\partial F_{i\alpha}}(F) - \frac{\partial\Phi^A}{\partial F_{i\alpha}}(\widehat{F})\right)\\
&\quad - (\partial_\alpha\widehat{v}_i)\left(T^A - \frac{\partial\sigma_E}{\partial\Xi^A}(\Xi)\right)\frac{\partial\Phi^A}{\partial F_{i\alpha}}(F)\\
&= -\partial_\alpha\left[\widehat{v}_i\frac{\partial\sigma_E}{\partial\Xi^A}(\widehat{\Xi})\left(\frac{\partial\Phi^A}{\partial F_{i\alpha}}(F) - \frac{\partial\Phi^A}{\partial F_{i\alpha}}(\widehat{F})\right)\right] - Q_1 - Q_2 - Q_3 - L
\end{aligned} \tag{3.23}$$

That is the term I is written as the sum of a divergence term plus the quadratic terms Q_i plus a linear term L that is controlled by the distance from equilibrium.

Combining (3.20), (3.21), (3.22) and (3.23) we arrive at the relative entropy identity

$$\partial_t \mathcal{E}_r - \partial_\alpha \mathcal{F}_{\alpha,r} + \frac{1}{\varepsilon} D = Q_1 + Q_2 + Q_3 + L \tag{3.24}$$

where the flux is

$$\mathcal{F}_{\alpha,r} := \left(T^A - \frac{\partial \sigma_E}{\partial \Xi^A}(\widehat{\Xi}) \right) (v_i - \widehat{v}_i) \frac{\partial \Phi^A}{\partial F_{i\alpha}}(F) \tag{3.25}$$

the dissipation is

$$\frac{1}{\varepsilon} D = \frac{1}{\varepsilon} \frac{\partial \Psi}{\partial \tau^A}\Big(\Xi, T - \frac{\partial \sigma_I}{\partial \Xi} \Big) \left(T^A - \frac{\partial \sigma_E}{\partial \Xi^A} \right) \tag{3.26}$$

the quadratic errors Q_i are

$$\begin{aligned}
Q_1 &= \partial_\alpha \left(\frac{\partial \sigma_E}{\partial \Xi^A}(\widehat{\Xi}) \right) \left(\frac{\partial \Phi^A}{\partial F_{i\alpha}}(F) - \frac{\partial \Phi^A}{\partial F_{i\alpha}}(\widehat{F}) \right) (v_i - \widehat{v}_i) \\
Q_2 &= (\partial_\alpha \widehat{v}_i) \frac{\partial \Phi^A}{\partial F_{i\alpha}}(\widehat{F}) \Big(\frac{\partial \sigma_E}{\partial \Xi^A}(\Xi) - \frac{\partial \sigma_E}{\partial \Xi^A}(\widehat{\Xi}) - \frac{\partial^2 \sigma_E(\widehat{\Xi})}{\partial \Xi^A \partial \Xi^B} (\Xi^B - \widehat{\Xi}^B) \Big) \\
Q_3 &= (\partial_\alpha \widehat{v}_i) \left(\frac{\partial \sigma_E}{\partial \Xi^A}(\Xi) - \frac{\partial \sigma_E}{\partial \Xi^A}(\widehat{\Xi}) \right) \left(\frac{\partial \Phi^A}{\partial F_{i\alpha}}(F) - \frac{\partial \Phi^A}{\partial F_{i\alpha}}(\widehat{F}) \right)
\end{aligned} \tag{3.27}$$

and the linear error L is

$$L = (\partial_\alpha \widehat{v}_i) \left(T^A - \frac{\partial \sigma_E}{\partial \Xi^A}(\Xi) \right) \frac{\partial \Phi^A}{\partial F_{i\alpha}}(F) \tag{3.28}$$

Identity (3.24) is the key on which the stability and convergence analysis of section 4 is based.

4. Stability theorem

Consider a family of smooth solution $\{(v^\varepsilon, F^\varepsilon, \tau^\varepsilon)\}_{\varepsilon > 0}$ to the relaxation system

$$\begin{aligned}
\partial_t v_i - \partial_\alpha \left(T^A \frac{\partial \Phi^A}{\partial F_{i\alpha}} \right) &= 0 \\
\partial_t F_{i\alpha} - \partial_\alpha v_i &= 0 \\
\partial_t \left(T^A - \frac{\partial \sigma_I}{\partial \Xi^A}(\Phi(F)) \right) &= -\frac{1}{\varepsilon} \left(T^A - \frac{\partial \sigma_E}{\partial \Xi^A}(\Phi(F)) \right)
\end{aligned} \tag{4.1}$$

We wish to compare them with a smooth solution $(\widehat{v}, \widehat{F})$ of the equations of elastodynamics

$$
\begin{aligned}
\partial_t v_i - \partial_\alpha \left(\frac{\partial \sigma_E}{\partial \Xi^A}(\Phi(F)) \frac{\partial \Phi^A}{\partial F_{i\alpha}} \right) &= 0 \\
\partial_t F_{i\alpha} - \partial_\alpha v_i &= 0
\end{aligned}
\tag{4.2}
$$

The stress in model (3.2) satisfies

$$
S_{i\alpha} = \frac{\partial}{\partial F_{i\alpha}} \sigma_E(\Phi(F))
$$

and thus, when σ_E is convex, the model (3.2) corresponds to polyconvex elasticity.

The data F_0 and $\widehat{F}_0$ are taken gradients; the property is preserved and both F and $\widehat{F}$ are gradients for all times. The function $(v, \Phi(F), \tau)$ is a smooth solution of the augmented relaxation system (3.6) while the function $(\widehat{v}, \Phi(\widehat{F}))$ satisfies the extended elastodynamics equations (3.7). From the results of section 3.3, smooth solutions of (3.6) and (3.7) satisfy (3.24).

The identity is of course inherited by $(v, \Phi(F), \tau)$ and $(\widehat{v}, \Phi(\widehat{F}))$. The resulting relative energy and associated flux,

$$
\begin{aligned}
e_r &= \mathcal{E}_r\Big(v, \Phi(F), \tau \;\Big|\; \widehat{v}, \Phi(\widehat{F}), \frac{\partial(\sigma_E - \sigma_I)}{\partial \Xi}(\Phi(\widehat{F})) \Big) \\
&= \frac{1}{2}|v - \widehat{v}|^2 + \Psi\Big(\Phi(F), T - \frac{\partial \sigma_I}{\partial \Xi}(\Phi(F)) \Big) - \sigma_E(\Phi(\widehat{F})) \\
&\qquad - \frac{\partial \sigma_E}{\partial \Xi^A}(\Phi(\widehat{F}))(\Phi(F)^A - \Phi(\widehat{F})^A) ,
\end{aligned}
\tag{4.3}
$$

$$
\begin{aligned}
f_\alpha &= \mathcal{F}_{\alpha,r}\Big(v, \Phi(F), \tau \;\Big|\; \widehat{v}, \Phi(\widehat{F}), \frac{\partial(\sigma_E - \sigma_I)}{\partial \Xi}(\Phi(\widehat{F})) \Big) \\
&= \Big(T^A - \frac{\partial \sigma_E}{\partial \Xi^A}(\Phi(\widehat{F})) \Big) (v_i - \widehat{v}_i) \frac{\partial \Phi^A}{\partial F_{i\alpha}}(F) ,
\end{aligned}
\tag{4.4}
$$

satisfy

$$
\partial_t e_r - \partial_\alpha f_\alpha + \frac{1}{\varepsilon} D = Q_1 + Q_2 + Q_3 + L \tag{4.5}
$$

where the Q_i, L and D are now computed for $\Xi = \Phi(F)$ and $\widehat{\Xi} = \Phi(\widehat{F})$.

We prove convergence of the relaxation system to polyconvex elastodynamics so long as the limit solution is smooth.

Theorem 4.1. *Let* $(v^\varepsilon, F^\varepsilon, T^\varepsilon)$*,* $F^\varepsilon = \nabla y^\varepsilon$*, be smooth solutions of* (3.1) *and* $(\widehat{v}, \widehat{F})$*,* $\widehat{F} = \nabla \widehat{y}$*, a smooth solution of* (3.2)*, defined on* $\mathbb{R}^d \times [0, T]$ *and decaying fast as* $|x| \to \infty$*. The relative energy* e_r *defined in* (4.3) *satisfies*

(4.5). Assume that σ_I, σ_E satisfy for some constants $\gamma_I > \gamma_v > 0$ and $M > 0$ the hypotheses

$$\nabla^2 \sigma_I \geq \gamma_I \mathbb{I} > \gamma_v \mathbb{I} \geq \nabla^2(\sigma_I - \sigma_E) > 0 \,, \tag{h$_1$}$$

$$|\nabla^2 \sigma_E| \leq M \,, \quad |\nabla^3 \sigma_E| \leq M \,. \tag{h$_2$}$$

There exists a constant s and $C = C(T, \gamma_I, \gamma_v, M, \nabla \hat{v}, \nabla \widehat{F}) > 0$ independent of ε such that

$$\int_{\mathbb{R}^d} e_r(x,t) dx \leq C \left(\int_{\mathbb{R}^d} e_r(x,0) dx + \varepsilon \right) .$$

In particular, if the data satisfy

$$\int_{\mathbb{R}^d} e_r^\varepsilon(x,0) dx \longrightarrow 0 \,, \quad \text{as } \varepsilon \downarrow 0 \,,$$

then

$$\sup_{t \in [0,T]} \int_{\mathbb{R}^d} |v^\varepsilon - \hat{v}|^2 + |F^\varepsilon - \widehat{F}|^2 + |\tau^\varepsilon - \tau_\infty(\widehat{F})|^2 dx \longrightarrow 0 \,,$$

where $\tau_\infty(\widehat{F}) = \frac{\partial(\sigma_E - \sigma_I)}{\partial \Xi}(\Phi(\widehat{F}))$.

Proof. The equation (4.5),

$$\partial_t e_r + \partial_\alpha f_\alpha + \frac{1}{\varepsilon} D = J \,,$$

is integrated on $\mathbb{R}^d \times (0,t)$ and gives

$$\begin{aligned} \int_{\mathbb{R}^d} e_r(x,t) dx - \int_{\mathbb{R}^d} e_r(x,0) dx & \\ + \frac{1}{\varepsilon} \int_0^t \int_{\mathbb{R}^d} D dx d\tau &= \int_0^t \int_{\mathbb{R}^d} J dx d\tau \end{aligned} \tag{4.6}$$

From lemma 3.2 and (3.19) we see that there exists a positive constant $c = c(\gamma_I, \gamma_v)$ such that

$$\mathcal{E}_r \geq c \left(|v - \hat{v}|^2 + |\Xi - \widehat{\Xi}||^2 + |\tau - \frac{\partial(\sigma_E - \sigma_I)}{\partial \Xi}(\widehat{\Xi})|^2 \right)$$

and thus, by (4.3),

$$e_r \geq c\big(|v - \hat{v}|^2 + |\Phi(F) - \Phi(\widehat{F})|^2 + |\tau - \tau_\infty(\widehat{F})|^2 \big) \,.$$

Note that

$$
\begin{aligned}
D &:= \frac{\partial \Psi}{\partial \tau_A}\Big(\tau^A - \frac{\partial(\sigma_E - \sigma_I)}{\partial \Xi^A}\Big) \\
&= (\Xi + \nabla_\tau G)\cdot(\tau + \nabla_\Xi \Sigma) \\
&= (\nabla_\tau G(\tau) - \nabla_\tau G(-\nabla_\Xi \Sigma))\cdot(\tau + \nabla_\Xi \Sigma) \qquad (4.7) \\
&\geq \big(\min \nabla_\tau^2 G\big)\, |\tau + \nabla_\Xi \Sigma|^2 \\
&\geq \frac{1}{\gamma_v}|\tau - \nabla_\Xi(\sigma_E - \sigma_I)|^2
\end{aligned}
$$

Let now C be a positive constant depending on the L^∞-norm of $\widehat{v}$, $\widehat{F}$, $\partial_{alpha}\widehat{v}$, $\partial_\alpha \widehat{F}$ and the constants γ_I, γ_v and M. Using (3.27), (h$_2$), and (3.28) we have

$$
\begin{aligned}
\int_{\mathbb{R}^d} |Q_1| dx &\leq C \int_{\mathbb{R}^d} |v - \widehat{v}|^2 + \Big|\frac{\partial \Phi}{\partial F}(F) - \frac{\partial \Phi}{\partial F}(\widehat{F})\Big|^2 dx\,, \\
\int_{\mathbb{R}^d} |Q_2| dx &\leq C \int_{\mathbb{R}^d} |\Phi(F) - \Phi(\widehat{F})|^2 dx \\
\int_{\mathbb{R}^d} |Q_3| dx &\leq C \int_{\mathbb{R}^d} |\Phi(F) - \Phi(\widehat{F})|^2 + \Big|\frac{\partial \Phi}{\partial F}(F) - \frac{\partial \Phi}{\partial F}(\widehat{F})\Big|^2 dx\,.
\end{aligned}
$$

and

$$
\int_{\mathbb{R}^d} |L| dx \leq \frac{1}{\varepsilon}\frac{1}{2\gamma_v}\int_{\mathbb{R}^d} |\tau - \nabla_\Xi(\sigma_E - \sigma_I)|^2 dx + C\varepsilon \int_{\mathbb{R}^d} |\frac{\partial \Phi}{\partial F}(F)|^2 dx
$$

From the identities

$$
\frac{\partial \det F}{\partial F_{i\alpha}} = (\operatorname{cof} F)_{i\alpha}\,, \qquad \frac{\partial (\operatorname{cof} F)_{i\alpha}}{\partial F_{j\beta}} = \varepsilon_{ijk}\varepsilon_{\alpha\beta\gamma} F_{k\gamma}\,,
$$

we have

$$
\Big|\frac{\partial \Phi}{\partial F}(F) - \frac{\partial \Phi}{\partial F}(\widehat{F})\Big| \leq C|\Phi(F) - \Phi(\widehat{F})|\,.
$$

Combining with (4.7) and (4.6) we obtain

$$
\begin{aligned}
\int_{\mathbb{R}^d} e_r(x,t) dx &+ \frac{1}{2\varepsilon\gamma_v}\int_{\mathbb{R}^d} |\tau - \nabla_\Xi(\sigma_E - \sigma_I)|^2 dx \\
&= \int_{\mathbb{R}^d} e_r(x,0) dx + C\int_0^t \int_{\mathbb{R}^d} e_r(x,\tau) dx d\tau \\
&\quad + \varepsilon C \int_0^t \int_{\mathbb{R}^d} |\frac{\partial \Phi}{\partial F}(F)|^2 dx d\tau \qquad (4.8)
\end{aligned}
$$

The H-estimate implies that solution of the relaxation system (3.1) satisfy the uniform (in ε) bounds

$$
\int_{\mathbb{R}^d} |v|^2 + |\Phi(F)|^2 + |\tau|^2 dx + \frac{1}{\varepsilon\gamma_v}\int_{\mathbb{R}^d} |\tau - \nabla_\Xi(\sigma_E - \sigma_I)|^2 dx
$$

$$\leq C \int_{\mathbb{R}^d} |v_0|^2 + \Psi(\Phi(F_0), \tau_0) dx \leq O(1) \tag{4.9}$$

The result then follows from (4.8) via Gronwall's inequality. □

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References

[1] J.M. Ball, Convexity conditions and existence theorems in nonlinear elasticity, *Arch. Rational Mech. Anal.* **63** (1977), 337-403.
[2] J.M. Ball, Some open problems in elasticity In: *Geometry, mechanics and dynamics*, Springer, New York, 2002, pp. 3-59.
[3] P.G. Ciarlet, *Mathematical Elasticity*, Vol. 1, North Holland, (1993).
[4] G.-Q. Chen, C.D. Levermore, and T.-P. Liu, Hyperbolic conservation laws with stiff relaxation terms and entropy, *Comm. Pure and Appl. Math.* **47** (1994), 787–830.
[5] B. Coleman and W. Noll, On the Thermostatics of Continuous Media. *Arch. Rat. Mech. Anal.*, **4** (1959), 97-128.
[6] C.M. Dafermos, The second law of thermodynamics and stability, *Arch. Rational Mech. Analysis* **70** (1979), 167-179.
[7] C.M. Dafermos, Quasilinear hyperbolic systems with involutions, *Arch. Rational Mech. Anal.* **94** (1986), 373-389.
[8] C.M. Dafermos, *Hyperbolic conservation laws in continuum physics*, Grundlehren der Mathematischen Wissenschaften, vol. 325, Springer-Verlag, Berlin, 2000.
[9] C.M. Dafermos, Ch. V of 2nd edition of [8], (forthcoming).
[10] S. Demoulini, D.M.A. Stuart, and A.E. Tzavaras, A variational approximation scheme for three–dimensional elastodynamics with polyconvex energy, *Arch. Ration. Mech. Anal.* **157** (2001), 325–344.
[11] R.J. DiPerna, *Uniqueness of solutions to hyperbolic conservation laws*, Indiana Univ. Math. J. **28** (1979), 137-188.
[12] C. Faciu and M. Mihailescu-Suliciu, The energy in one-dimensional rate-type semilinear viscoelasticity, *Int. J. Solids Structures* **23** (1987), 1505-1520.
[13] C. Lattanzio and A. Tzavaras Structural properties of stress relaxation and convergence from viscoelasticity to polyconvex elastodynamics. *Arch. Rational Mech. Anal.* (to appear).
[14] P.D. Lax, Shock waves and entropy, in: "Contributions to Nonlinear Functional Analysis." E.H. Zarantonello, ed. New York: Academic Press, 1971, pp. 603-634.
[15] T. Qin, Symmetrizing nonlinear elastodynamic system, *J. Elasticity* **50** (1998), 245-252.
[16] I. Suliciu, On the thermodynamics of rate-type fluids and phase transitions. I. Rate-type fluids, *Internat. J. Engrg. Sci.* **36** (1998), no. 9, 921–947.
[17] A.E. Tzavaras, Materials with internal variables and relaxation to conservation Laws, *Arch. Rational Mech. Anal.* **146** (1999), 129–155.
[18] A. Tzavaras, Relative entropy in hyperbolic relaxation *Comm. Math. Sci* **3** (2005), 119-132.

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