Existence of the Semilocal Chern-Simons Vortices

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Abstract

We consider the Bogomol'nyi equations of the Abelian Chern-Simons-Higgs model with $SU(N)_{\mathrm{global}} \otimes U(1)_{\mathrm{local}}$ symmetry. This is a generalization of the well-known Abelian Chern-Simons-Higgs model with $U(1)_{\mathrm{local}}$ symmetry. We prove existence of both topological and nontopological multivortex solutions of the system on the plane.

1 Introduction

The Abelian Chern-Simons-Higgs model with $SU(N)_{\mathrm{global}} \otimes U(1)_{\mathrm{local}}$ symmetry is defined by the Lagrangian,

$$\mathcal{L} = \frac{\kappa}{4}\epsilon^{\mu\nu\lambda}F_{\mu\nu}A_\lambda + (D_\mu\Phi)^\dagger(D^\mu\Phi) - \frac{1}{\kappa^2}|\Phi|^2(|\Phi|^2-1)^2,$$

where $A_\mu(\mu = 0,1,2)$ is the gauge field on $\mathbb{R}^3, F_{\mu\nu} = \frac{\partial}{\partial x^\mu}A_\nu - \frac{\partial}{\partial x^\nu}A_\mu$ is the corresponding gauge curvature tensor, $D_\mu = \frac{\partial}{\partial x^\mu} - iA_\mu$ is the gauge covariant derivative, $\Phi = (\Phi_1, \Phi_2, \cdots, \Phi_N)$ is a $\mathbb{C}^N$ valued function on $\mathbb{R}^3$, called the Higgs multiplet, $\epsilon_{\mu\nu\rho}$ is the totally skewsymmetric tensor with $\epsilon_{012} = 1$, and finally $\kappa > 0$ is the Chern-Simons coupling constant. Our metric on $\mathbb{R}^3$ is $(g_{\mu\nu}) = diag(1,-1,-1)$. This model was suggested by Khare[6], generalizing the original Abelian Chern-Simons-Higgs model, due to Hong-Kim-Pac[4] and Jackiw-Weinberg[5]. We also mention that there are also studies of the corresponding Abelian Higgs model with $SU(N)_{\text{global}} \otimes U(1)_{\text{local}}$ symmetry in [14], [3]. Similarly to the case of Abelian Chern-Simons-Higgs model, in the static case, the following Bogomol'nyi system in $\mathbb{R}^2$ is obtained[6],

$$\begin{cases} (D_1 \pm iD_2)\Phi_k = 0, \quad \forall k = 1, \cdots N \\ F_{12} = \pm \frac{2}{\kappa^2}|\Phi|^2(|\Phi|^2 - 1). \end{cases} \tag{1.1}$$

This system is equipped with one of the following boundary conditions; either

$$|\Phi(z)|^2 \to 1 \quad \text{as } |z| \to \infty, \tag{1.2}$$

or

$$|\Phi(z)|^2 \to 0 \quad \text{as } |z| \to \infty, \tag{1.3}$$

Following the standard Jaffe-Taubes reduction procedure[13], we introduce new variable $(u_1, \cdots, u_N)$ by

$$\Phi_k(z) = \exp\left[\frac{1}{2}u_k + i\sum_{j=1}^{M_k} Arg(z - z_{k,j})\right], \quad z = x_1 + ix_2 \in \mathbb{C}^1 = \mathbb{R}^2,$$

where $\mathbb{Z}_k = \{z_{k,j}\}_{j=1}^{M_k}$ is the set of zeros of $\Phi_k(z)$. Then, the system (1.1) becomes the following semilinear elliptic system for $(u_1, \cdots, u_N)$ in $\mathbb{R}^2$.

$$\Delta u_k = \left(\sum_{j=1}^{N} e^{u_j}\right)\left(\sum_{j=1}^{N} e^{u_j} - 1\right) + 4\pi\sum_{j=1}^{M_k}\delta(z - z_{k,j}), \quad k = 1, \cdots, N, \tag{1.4}$$

where we set $\kappa = 2$ for simplicity. In terms of $(u_1, \cdots, u_N)$, the boundary condition (1.2) reads

$$\begin{cases} e^{u_k} \to \sigma_k \quad \text{as } |z| \to \infty \\ \text{with } \sigma_k \geq 0 \text{ for all } k = 1, \cdots N, \text{ and } \sum_{k=1}^{N}\sigma_k = 1, \end{cases} \tag{1.5}$$

while (1.3) reads

$$e^{u_k} \to 0 \quad \text{as } |z| \to \infty \quad \text{ for all } k = 1, \cdots, N. \tag{1.6}$$

The boundary condition (1.5) is called topological, while the boundary condition (1.6) is called nontopological. We observe that when $N = 1$, the system (1.4) reduces to the well-known (scalar) Chern-Simons equation, for which there are many studies for topological vortices([11, 15]), nontopological vortices([10, 1, 2]), periodic vortex condensates([12, 9, 8, 16]) respectively. We first consider the nontopological case. In the system (1.4) equipped with (1.3), without loss of generality, we assume $M_1 \geq M_k$ for all $k = 1, \cdots N$. Let us define

$$f_k(z) = (M_k + 1)\prod_{j=1}^{M_k}(z - z_{k,j}), \quad F_k(z) = \int_0^z f_k(\xi)d\xi. \tag{1.7}$$

Given $\varepsilon > 0, a = a_1 + ia_2 \in \mathbb{C}$, let us introduce the functions $\rho^{(k)}_{\varepsilon,a}(z)$ by

$$\rho^{(k)}_{\varepsilon,a}(z) = \frac{8\varepsilon^{2M_k+2}|f_k(z)|^2}{\left(1 + \varepsilon^{2M_1+2}|F_1(z) + \frac{a}{\varepsilon^{M_1+1}}|^2\right)^2}. \tag{1.8}$$

We note that for any $\varepsilon > 0$ and $a \in \mathbb{C}^1$, $\ln \rho^{(1)}_{\varepsilon,a}(z)$ is a solution of the Liouville equation.

$$\Delta \ln \rho^{(1)}_{\varepsilon,a}(z) = -\rho^{(1)}_{\varepsilon,a}(z) + 4\pi \sum_{j=1}^{M_1} \delta(z - z_{1,j}). \tag{1.9}$$

We state the existence theorem for the nontopological vortices.

Theorem 1.1 (Existence of nontopological vortices) *Let $N \geq 2$. For each $k = 1, \cdots N$ let $M_k \in \mathbb{N}$ with $M_1 \geq M_k$ for all $k = 1, \cdots N$, and let $\mathbb{Z}_1, \cdots \mathbb{Z}_N$ be given with $\mathbb{Z}_k = \{z_{k,j}\}_{j=1}^{M_k} \in \mathbb{R}^2$. Then, there exists a constant $\varepsilon_1 > 0$ such that for any $\varepsilon \in (0, \varepsilon_1)$ there exists a family of solutions to 1.4, $(u_1, u_2, \cdots, u_N)$ equipped with the boundary condition 1.6). Moreover, the solutions we constructed have the following representations:*

$$u_1(z) = \ln \rho^{(1)}_{\varepsilon,a^*_\varepsilon}(z) + \varepsilon^2 w(\varepsilon|z|) + \varepsilon^2 v^*_\varepsilon(\varepsilon z), \tag{1.10}$$

$$u_k(z) = \ln \rho^{(k)}_{\varepsilon,a^*_\varepsilon}(z) + \varepsilon^2 w(\varepsilon|z|) + \varepsilon^2 v^*_\varepsilon(\varepsilon z) + \ln \varepsilon^4 \tag{1.11}$$

for all $k = 2, \cdots N$.
In (1.10) and (1.11), the function $\varepsilon \mapsto a^*_\varepsilon$ is a continuous in a neighborhood of 0, and $|a^*_\varepsilon| \to 0$ as $\varepsilon \to 0$. The radial function w in (1.10) and (1.11) has the following asymptotic behavior.

$$w(|z|) = -C_0 \ln |z| + O(1), \tag{1.12}$$

as $|z| \to \infty$ with the constant $C_0 > 0$ defined by

$$C_0 = \frac{4\pi M_1^2 (2M_1 + 1)^6}{15(M_1 + 1)^5 \sin\left(\frac{\pi M_1}{M_1+1}\right)} \tag{1.13}$$

The function v^*_ε in (1.10) and (1.11) satisfies

$$\sup_{z\in\mathbb{R}^2} \frac{|v^*_\varepsilon(\varepsilon z)|}{\ln(1 + |z|)} \leq o(1) \qquad \text{as } \varepsilon \to 0. \tag{1.14}$$

Next, we consider the system (1.4) equipped with the topological boundary condition (1.5). Without loss of generality, we assume that for $m \in \{1, \cdots N\}$

$$e^{u_k} \to \sigma_k \quad \text{as } |z| \to \infty \qquad \text{for } k = 1, \cdots, m. \tag{1.15}$$

$$e^{u_k} \to 0 \quad \text{as } |z| \to \infty \qquad \text{for all } k = m + 1, \cdots, N, \tag{1.16}$$

where $\sum_{k=1}^m \sigma_k = 1$, and $\sigma_k \in (0, 1]$ for each $k = 1, \cdots, m$. The following is our second main theorem.

Theorem 1.2 (Existence of topological vortices) *In order to have solution to the system (1.4) equipped with (1.15) and (1.16), it is necessary that*

$$M_1 = M_2 = \cdots = M_m (\equiv M), \quad \text{and} \quad M_k < M \text{ for all } k = m+1, \cdots, N \tag{1.17}$$

If the condition (1.17) is satisfied, then there exists a solution $(u_1, \cdots, u_N)$ to the problem. Moreover, the solutions we constructed have the following representations:

$$u_k(z) = \ln\left(\sigma_k \prod_{j=1}^{M} \frac{|z - z_{k,j}|^2}{(\mu + |z - z_{1,j}|^2)}\right) + v \quad \text{for } k = 1, \cdots, m, \tag{1.18}$$

while

$$u_k(z) = \ln\left(\frac{\prod_{j=1}^{M_k} |z - z_{k,j}|^2}{\prod_{j=1}^{M}(\mu + |z - z_{1,j}|^2)}\right) + v \quad \text{for } k = m+1, \cdots, N \tag{1.19}$$

for a function $v \in \cap_{q=1}^{\infty} H^q(\mathbb{R}^2)$.

2 Existence of Nontopological Vortices

In this section our aim is to prove Theorem 1.1. From the equation, $\Delta \ln|z - z_0|^2 = 4\pi\delta(z - z_0)$ in $\mathbb{R}^2$ we find that

$$\Delta\left(u_k - u_1 - \sum_{j=1}^{M_k} \ln|z - z_{k,j}|^2 + \sum_{j=1}^{M_1} \ln|z - z_{1,j}|^2\right) = 0.$$

Hence, we obtain the relations between u_1 and u_k's

$$u_k = u_1 + \ln\left(\frac{\prod_{j=1}^{M_k} |z - z_{k,j}|^2}{\prod_{j=1}^{M_1} |z - z_{1,j}|^2}\right) + h_k(z) \tag{2.1}$$

for all $k = 1, \cdots N$, where $h_k(z)$ is a harmonic function in $\mathbb{R}^2$. We choose

$$h_k(z) \equiv \ln \varepsilon^{4+2M_k-2M_1}.$$

Then, (2.1) becomes

$$u_k = u_1 + \ln\left(\frac{\varepsilon^{4+2M_k-2M_1} \prod_{j=1}^{M_k} |z - z_{k,j}|^2}{\prod_{j=1}^{M_1} |z - z_{1,j}|^2}\right). \tag{2.2}$$

We introduce $g_{\varepsilon,a}^{(k)}(z), \rho_k(r)$, $k = 1, \cdots N$ as follows

$$g_{\varepsilon,a}^{(k)}(z) = \frac{1}{\varepsilon^2} \rho_{\varepsilon,a}^{(k)}\left(\frac{z}{\varepsilon}\right), \qquad \rho_k(r) = \frac{8(M_k+1)^2 r^{2M_k}}{(1 + r^{2M_1+2})^2} \left(= \lim_{\varepsilon \to 0} g_{\varepsilon,0}^{(k)}(z)\right). \tag{2.3}$$

Let us make a change of variables from u_1 to v by the following formula

$$u_1(z) = \ln \rho^{(1)}_{\varepsilon,a}(z) + \varepsilon^2 w(\varepsilon|z|) + \varepsilon^2 v(\varepsilon x), \tag{2.4}$$

where $w(\cdot)$ is a radial function to be determined below. Then, after elementary computations we find that combination of (2.2) and (2.4) implies the following representation formula for u_k,

$$u_k(z) = \ln \rho^{(k)}_{\varepsilon,a}(z) + \varepsilon^2 w(\varepsilon|z|) + \varepsilon^2 v(\varepsilon x) + \ln \varepsilon^4 \tag{2.5}$$

for all $k = 2, \cdots N$.

Then, the equation for u_1 in (1.4) can be written as the functional equation $P(v, a, \varepsilon) = 0$, where

$$\begin{aligned} P(v,a,\varepsilon) = \Delta v &+ \frac{1}{\varepsilon^2} g^{(1)}_{\varepsilon,a}(z)(e^{\varepsilon^2(v+w)} - 1) - [g^{(1)}_{\varepsilon,a}(z)]^2 e^{2\varepsilon^2(v+w)} \\ &+ \varepsilon^2 \sum_{k=2}^{N} g^{(k)}_{\varepsilon,a}(z) e^{\varepsilon^2(v+w)} - \varepsilon^4 \sum_{k=2}^{N} g^{(1)}_{\varepsilon,a}(z) g^{(k)}_{\varepsilon,a}(z) e^{2\varepsilon^2(v+w)} \\ &- \varepsilon^8 \sum_{k,l=2}^{N} g^{(l)}_{\varepsilon,a}(z) g^{(k)}_{\varepsilon,a}(z) e^{2\varepsilon^2(v+w)}. \end{aligned} \tag{2.6}$$

Now we introduce the functions spaces introduced in [1]. Let us fix $\alpha \in (0, \frac{1}{2})$ throughout this paper. Following [1], we introduce the Banach spaces X_α and Y_α as

$$X_\alpha = \{u \in L^2_{loc}(\mathbb{R}^2) \mid \int_{\mathbb{R}^2} (1 + |x|^{2+\alpha})|u(x)|^2 dx < \infty\}$$

equipped with the norm $\|u\|^2_{X_\alpha} = \int_{\mathbb{R}^2} (1 + |x|^{2+\alpha})|u(x)|^2 dx$, and

$$Y_\alpha = \{u \in W^{2,2}_{loc}(\mathbb{R}^2) \mid \|\Delta u\|^2_{X_\alpha} + \left\| \frac{u(x)}{1 + |x|^{1+\frac{\alpha}{2}}} \right\|^2_{L^2(\mathbb{R}^2)} < \infty\}$$

equipped with the norm $\|u\|^2_{Y_\alpha} = \|\Delta u\|^2_{X_\alpha} + \left\| \frac{u(x)}{1+|x|^{1+\frac{\alpha}{2}}} \right\|^2_{L^2(\mathbb{R}^2)}$. We recall the following propositions proved in [1].

Proposition 2.1 *Let Y_α be the function space introduced above. Then we have the followings.*

(i) If $v \in Y_\alpha$ is a harmonic function, then $v \equiv$ constant.

(ii) There exists a constant $C_1 > 0$ such that for all $v \in Y_\alpha$

$$|v(x)| \leq C_1 \|v\|_{Y_\alpha} \ln(e + |x|), \qquad \forall x \in \mathbb{R}^2.$$

Proposition 2.2 *Let $\alpha \in (0, \frac{1}{2})$, and let us set*

$$L = \Delta + \rho_1 : Y_\alpha \to X_\alpha. \tag{2.7}$$

We have

$$KerL = Span\{\varphi_+, \varphi_-, \varphi_0\}, \tag{2.8}$$

where we denoted

$$\varphi_+(r,\theta) = \frac{r^{M_1+1}\cos(M_1+1)\theta}{1+r^{2M_1+2}}, \quad \varphi_-(r,\theta) = \frac{r^{M_1+1}\sin(M_1+1)\theta}{1+r^{2M_1+2}}, \tag{2.9}$$

and

$$\varphi_0 = \frac{1-r^{2M_1+2}}{1+r^{2M_1+2}}. \tag{2.10}$$

Moreover, we have

$$ImL = \{f \in X_\alpha | \int_{\mathbb{R}^2} f\varphi_\pm = 0\}. \tag{2.11}$$

We can check easily that P is a well defined continuous mapping from $B_{\varepsilon_0} \subset Y_\alpha \times \mathbb{C} \times \mathbb{R}_+$ into X_α, where $B_{\varepsilon_0} = \{\|v\|_{Y_\alpha} + |a| \le \varepsilon < \varepsilon_0\}$ for sufficiently small ε_0. In order to have a continuous extension to $\varepsilon = 0$ of $P(\cdot)$ we require that $\lim_{\varepsilon\to 0} P(0,0,\varepsilon) = 0$, which implies the following equation for w,

$$\Delta w + \rho_1 w - \rho_1^2 = 0. \tag{2.12}$$

We first note the following lemma about asymptotic behaviors of the solutions $w \in Y_\alpha$, the proof of which is in Appendix.

Lemma 2.1 *Let C_0 be the number introduced in (1.13). Then, there exist radial solutions $w(|z|)$ of (2.12) belonging to Y_α, and satisfying the asymptotic formula in (1.12).*

In order to obtain the linearized operator $P'_{(v,a)}(0,0,0) = \mathcal{A}$ we first compute,

$$\left.\frac{\partial g^{(k)}_{\varepsilon,a}(z)}{\partial a_1}\right|_{\varepsilon=0,a=0} = -4\rho_k\varphi_+, \qquad \left.\frac{\partial g^{(k)}_{\varepsilon,a}(z)}{\partial a_2}\right|_{\varepsilon=0,a=0} = -4\rho_k\varphi_-,$$

for all $k = 1, \cdots, N$, where

Using these we find

$$\mathcal{A}[\nu,\beta] = L\nu - 4\left(\rho_1 w - 2\rho_1^2\right)(\varphi_+\beta_1 + \varphi_-\beta_2).$$

For the linearized operator $\mathcal{A}[\cdot]$ we need the following key lemma lemma.

Lemma 2.2 *The operator $\mathcal{A} : Y_\alpha \times \mathbb{R}^2 \to X_\alpha^2$ defined above is onto. Moreover, kernel of $\mathcal{A}$ is given by*

$$Ker\mathcal{A} = Span\{\varphi_+, \varphi_-, \varphi_0\} \times \{(0,0)\}. \tag{2.13}$$

Thus, if we decompose $Y_\alpha \times \mathbb{R}^2 = U_\alpha \oplus Ker\mathcal{A}$, where we set $U_\alpha = (Ker\mathcal{A})^\perp$, then $\mathcal{A}$ is an isomorphism from U_α onto X_α.

For the proof of Lemma 2.2 we need the following proposition, the proof of which is in Appendix.

Proposition 2.3

$$I_\pm := \int_{\mathbb{R}^2} (\rho_1 w - 2\rho_1^2)\varphi_\pm^2 dx \neq 0.$$

With Proposition 2.3 equipped, the proof of Lemma 2.2 is the same as the one in [1], since the linearized operator, $\mathcal{A}$ is the same as the one in it. We are now ready to prove our main theorem.

Proof of Theorem 1.1: Let us set

$$U_\alpha = (KerL)^\perp \times \mathbb{R}^2.$$

Then, Lemma 2.2 shows that $P'_{(v,\xi,\beta)}(0,0,0,0) : U_\alpha \to X_\alpha \times X_\alpha$ is an isomorphism for $\alpha \in (0, \frac{1}{2})$. Then, the standard implicit function theorem(See e.g. [17]), applied to the functional $P : U_\alpha \times (-\varepsilon_0, \varepsilon_0) \to X_\alpha \times X_\alpha$, implies that there exists a constant $\varepsilon_1 \in (0, \varepsilon_0)$ and a continuous function $\varepsilon \mapsto \psi_\varepsilon^* := (v_\varepsilon^*, a_\varepsilon^*)$ from $(0, \varepsilon_1)$ into a neighborhood of 0 in U_α such that

$$P(v_\varepsilon^*, a_\varepsilon^*, \varepsilon) = 0 \quad \text{for all } \varepsilon \in (0, \varepsilon_1).$$

This completes the proof of Theorem 1.1. Since $M_1 \geq M_k$ for all $k = 1, \cdots N$, the representation of solutions u_k, and the explicit form of

$$\ln[\rho^{(k)}_{\varepsilon, a_\varepsilon^*}(z)] = -[(4M_1 - 2M_k) + 4] \ln|z| + O(1)$$

as $|z| \to \infty$, together with the asymptotic behaviors of $w(\cdot)$ described in Lemma 2.1, the fact that $v_\varepsilon^* \in Y_\alpha$, combined with Proposition 2.1, implies that the solutions satisfy the boundary condition in (1.6). Now, from Proposition 2.1 we obtain that

$$|v_\varepsilon^*(z)| \leq C\|v_\varepsilon^*\|_{Y_\alpha}(\ln^+|z| + 1) \leq C\|\psi_\varepsilon\|_{U_\alpha}(\ln^+|z| + 1).$$

This implies then

$$|v_\varepsilon^*(\varepsilon x)| \leq C\|\psi_\varepsilon\|_{U_\alpha}(\ln^+|\varepsilon x| + 1) \leq C\|\psi_\varepsilon\|_{U_\alpha}(\ln^+|x| + 1). \tag{2.14}$$

From the continuity of the function $\varepsilon \mapsto \psi_\varepsilon$ from $(0, \varepsilon_0)$ into U_α and the fact $\psi_0^* = 0$ we have

$$\|\psi_\varepsilon\|_{U_\alpha} \to 0 \qquad \text{as } \varepsilon \to 0. \tag{2.15}$$

The proof of (1.14) follows from (2.15) combined with (2.14). This completes the proof of Theorem 1.1. $\square$

3 Existence of Topological Vortices

Our aim in this section is to prove Theorem 1.2.

Proof of Theorem 1.2: We first establish that in order to have existence of solution $(u_1, \cdots u_N)$ satisfying (1.4) and (1.15)-(1.16), it is necessary to have (1.17). Without loss of generality we may assume $M_1 \geq M_k$ for all $k = 1, \cdots m$. Suppose that there exists $M_k < M_1$ for some $k \in \{1, \cdots, m\}$. Then, from (1.4) we have

$$\Delta \left(u_k - u_1 - \sum_{j=1}^{M_k} \ln |z - z_{k,j}|^2 + \sum_{j=1}^{M_1} \ln |z - z_{l,j}|^2 \right) = 0,$$

and

$$u_k = u_1 + \ln \left(\frac{\prod_{j=1}^{M_k} |z - z_{k,j}|^2}{\prod_{j=1}^{M_1} |z - z_{l,j}|^2} \right) + h_k(z) \tag{3.1}$$

for some harmonic function $h_k(z)$. Since

$$u_k \to \ln \sigma_k, \quad u_1 \to \ln \sigma_1, \quad \ln \left(\frac{\prod_{j=1}^{M_k} |z - z_{k,j}|^2}{\prod_{j=1}^{M_1} |z - z_{l,j}|^2} \right) \to O((M_k - M_1) \ln |z|),$$

as $|z| \to \infty$, (3.1) implies $h_k \equiv C_k$(constant), and provides an absurd relation. Hence, $M_1 = \cdots = M_m \equiv M$. Similarly, the relation (3.1) with $k = m + 1, \cdots N$ implies $M_k < M$ for all $k = m + 1, \cdots, N$, since for all $k = m + 1, \cdots N$, $u_k \to -\infty$, while $u_1 \to \ln \sigma_1$ as $|z| \to \infty$.

Then, choosing $C_k = \ln(\sigma_k/\sigma_1)$ for all $k = 1, \cdots, m$, (2.1) becomes

$$u_k = u_1 + \ln \left(\frac{\sigma_k}{\sigma_1} \prod_{j=1}^{M} \frac{|z - z_{k,j}|^2}{|z - z_{1,j}|^2} \right). \tag{3.2}$$

for $k = 1, \cdots N$. Let us set

$$\eta_k(z) = \begin{cases} \sigma_k \displaystyle\prod_{j=1}^{M} \frac{|z - z_{k,j}|^2}{(\mu + |z - z_{1,j}|^2)} & \text{for } k \in \{1, \cdots, m\} \\ \dfrac{\prod_{j=1}^{M_k} |z - z_{k,j}|^2}{\prod_{j=1}^{M} (\mu + |z - z_{1,j}|^2)} & \text{for } k \in \{m + 1, \cdots, N\}, \end{cases}$$

where $\mu >$ is a sufficiently large parameter. We introduce new unknown v by

$$u_1 = v + \ln \eta_1. \tag{3.3}$$

Then, (3.2) combined with (3.3) implies the representation for u_k, $k = 1, \cdots, N$ by

$$u_k = v + \ln \eta_k.$$

We introduce

$$g(z) = \sum_{j=1}^{M} \frac{4\mu}{(\mu + |z - z_{1,j}|^2)^2}.$$

We note that

$$\Delta \ln \eta_1(z) = 4\pi \sum_{j=1}^{M} \delta(z - z_{1,j}) - g(z). \tag{3.4}$$

We also introduce the function u_0 defined by

$$\sum_{k=1}^{N} \eta_k = e^{u_0}. \tag{3.5}$$

Note that since $e^{u_0} \to \sum_{k=1}^{m} \sigma_k = 1$, we have $u_0 \to 0$ as $|z| \to \infty$.

Using (3.4) and (3.5), we can rewrite the equation for u_1 in (1.4) as follows

$$\Delta v = e^{v+u_0}(e^{v+u_0} - 1) + g,$$

which is the Euler-Lagrange equation of the functional

$$F(v) = \int_{\mathbb{R}^2} \left[\frac{1}{2}|\nabla v|^2 + \frac{1}{2}(e^{u+u_0} - 1)^2 + gv \right] dx. \tag{3.6}$$

After this step the arguments for the existence of solution by minimization of the functional $F(v)$ by showing the coercivity and the weak lower semicontinuity in $H^1(\mathbb{R}^2)$ for sufficiently large μ, is exactly the same as in [15], [11] or [16], and we do not repeat them here. This finishes the proof of Theorem 1.2 $\square$

4 Appendix

Here we prove Lemma 2.1 and Proposition 2.3. We begin with the following elementary integration lemma

Lemma 4.1 *Let $m \geq k+1$, then we have*

$$\int_0^\infty \frac{r^{k(2N+2)-3}}{(1+r^{2N+2})^m} dr = \frac{\pi \prod_{j=1}^{k-1}(Nj + j - 1) \prod_{j=1}^{m-k}[Nj + j - 1]}{2(m-1)!(N+1)^m \sin\left(\frac{\pi M_1}{M_1+1}\right)}. \tag{4.1}$$

Proof:

$$\begin{aligned}
\int_0^\infty \frac{r^{k(2N+2)-3}}{(1+r^{2N+2})^m}dr &= \frac{1}{2N+2}\int_0^\infty \frac{t^{k-1-\frac{1}{N+1}}}{(1+t)^m}dt \quad \left(\text{Setting } t=r^{2N+2}\right)\\
&= -\frac{1}{2N+2}\int_0^\infty \frac{t^{k-1-\frac{1}{N+1}}}{(m-1)}\frac{d}{dt}\frac{1}{(1+t)^{m-1}}dt\\
&= \frac{1}{(2N+2)}\frac{1}{(m-1)}\left(k-1-\frac{1}{N+1}\right)\int_0^\infty \frac{t^{k-2-\frac{1}{N+1}}}{(1+t)^{m-1}}dt = \cdots\\
&= \frac{\left(k-1-\frac{1}{N+1}\right)\left(k-2-\frac{1}{N+1}\right)\cdots\left(1-\frac{1}{N+1}\right)}{2(N+1)(m-1)(m-2)\cdots(m-k+1)}\int_0^\infty \frac{t^{-\frac{1}{N+1}}}{(1+t)^{m-k+1}}dt.
\end{aligned} \tag{4.2}$$

Now we use the well-known formula from the Mellin transform(see e.g. [7])

$$\int_0^\infty \frac{t^{a-1}}{(1+t)^n}dt = \frac{\pi|(a-1)(a-2)\cdots(a-(n-1))|}{(n-1)!\sin(\pi a)}, \quad a\in(0,1)$$

in order to evaluate

$$\int_0^\infty \frac{t^{-\frac{1}{N+1}}}{(1+t)^{m-k+1}}dt = \int_0^\infty \frac{t^{a-1}}{(1+t)^{m-k+1}}dt = \frac{\pi|(a-1)(a-2)\cdots(a-m+k)|}{(m-k)!\sin(\pi a)}, \tag{4.3}$$

where we set $a=\frac{N}{N+1}$. Substituting (4.3) into (4.2), we obtain (4.1).□

Proof of Lemma 2.1: Let us set $f(r)=\rho_1(r)$. Then, it is found in [1] that the ordinary differential equation(with respect to r), $\Delta w+\rho_1 w = f(r)$ has a solution $w(r)\in Y_\alpha$ given by

$$w(r) = \varphi_0(r)\left\{\int_0^r \frac{\phi_f(s)-\phi_f(1)}{(1-s)^2}ds + \frac{\phi_f(1)r}{1-r}\right\} \tag{4.4}$$

with

$$\phi_f(r) := \left(\frac{1+r^{2M_1+2}}{1-r^{2M_1+2}}\right)^2 \frac{(1-r)^2}{r}\int_0^r \varphi_0(t)tf(t)dt,$$

where $\phi_f(1)$ and $w(1)$ are defined as limits of $\phi_f(r)$ and $w(r)$ as $r\to 1$. From the formula (4.4) we find that

$$w(r) = \varphi_0(r)\int_2^r \left(\frac{1+s^{2M_1+2}}{1-s^{2M_1+2}}\right)^2 \frac{I(s)}{s}ds + (\text{bounded function of } r) \tag{4.5}$$

as $r\to\infty$, where

$$I(s) = \int_0^s \varphi_0(t)t\rho_1(t)dt.$$

Since $\varphi_0(r) \to -1$ as $r \to \infty$, (1.12) follows if we show

$$I = I(\infty) = \int_0^\infty \varphi_0(r) r \rho_1(r) dr = C_0.$$

We evaluate the integral,

$$\begin{aligned}
I &= \int_0^\infty \varphi_0(r) f(r) r dr = \int_0^\infty \varphi_0(r)\rho_1(r)^2 r dr \\
&= 64(M_1+1)^4 \int_0^\infty \frac{(1-r^{2M_1+2})r^{4M_1+1}}{(1+r^{2M_1+2})^5} dr \\
&= 64(M_1+1)^4 \left[\int_0^\infty \frac{r^{4M_1+1}}{(1+r^{2M_1+2})^5} dr - \int_0^\infty \frac{r^{6M_1+3}}{(1+r^{2M_1+2})^5} dr\right] \\
&= 64(M_1+1)^4 \left[\frac{\pi M_1^2(2M_1+1)(3M_1+2)}{2\cdot 4!(M_1+1)^5 \sin\left(\frac{\pi M_1}{M_1+1}\right)} - \frac{\pi M_1^2(2M_1+1)^2}{2\cdot 4!(M_1+1)^5 \sin\left(\frac{\pi M_1}{M_1+1}\right)}\right] \\
&= \frac{4\pi M_1^2(2M_1+1)^6}{15(M_1+1)^5 \sin\left(\frac{\pi M_1}{M_1+1}\right)} = C_0,
\end{aligned}$$

where we used (4.1) with $(k, m) = (2, 5)$ and $(3, 5)$ in the fourth line. This completes the proof of Lemma 2.1. □

Proof of Proposition 2.3: In order to transform the integral we use the formula

$$L\left[\frac{1}{16(1+r^{2M_1+2})^2}\right] = \frac{(M_1+1)^2 r^{4M_1+2}}{(1+r^{2M_1+2})^4},$$

which can be verified by an elementary computation. Using this, we have the following

$$\begin{aligned}
I_\pm &= \int_{\mathbb{R}^2} (\rho_1 w - 2\rho_1^2)\varphi_\pm^2 dx \\
&= \int_0^{2\pi}\int_0^\infty (\rho_1 w - 2\rho_1^2) \frac{r^{2M_1+2}}{(1+r^{2M_1+2})^2} \left\{ \begin{array}{c} \cos^2(M_1+1)\theta \\ \sin^2(M_1+1)\theta \end{array} \right\} r dr d\theta \\
&= \pi \int_0^\infty \left[\frac{8(M_1+1)^2 r^{2M_1}}{(1+r^{2M_1+2})^2} w - 2\rho_1^2\right] \frac{r^{2M_1+2}}{(1+r^{2M_1+2})^2} r dr \\
&= \pi \int_0^\infty \left[\frac{1}{2} L\left\{\frac{1}{(1+r^{2M_1+2})^2}\right\} w - \frac{2\rho_1^2 r^{2M_1+2}}{(1+r^{2M_1+2})^2}\right] r dr \\
&= \pi \int_0^\infty \left[\frac{1}{2} Lw \cdot \frac{1}{(1+r^{2M_1+2})^2} - \frac{2\rho_1^2 r^{2M_1+2}}{(1+r^{2M_1+2})^2}\right] r dr \\
&= \pi \int_0^\infty \left[\frac{\rho_1^2}{2(1+r^{2M_1+2})^2} - \frac{2\rho_1^2 r^{2M_1+2}}{(1+r^{2M_1+2})^2}\right] r dr \\
&= \pi \int_0^\infty \left[\frac{5\rho_1^2}{2(1+r^{2M_1+2})^2} - \frac{2\rho_1^2}{(1+r^{2M_1+2})}\right] r dr
\end{aligned}$$

$$
\begin{aligned}
&= 64\pi(M_1+1)^4 \int_0^\infty \left[\frac{5r^{4M_1+1}}{2(1+r^{2M_1+2})^6} - \frac{2r^{4M_1+1}}{(1+r^{2M_1+2})^5}\right] dr \\
&= 64\pi(M_1+1)^4 \left[\frac{\pi M_1^2(2M_1+1)(3M_1+2)(4M_1+3)}{4\cdot 4!(M_1+1)^6 \sin\left(\frac{\pi M_1}{M_1+1}\right)}\right. \\
&\qquad\qquad \left. - \frac{\pi M_1^2(2M_1+1)(3M_1+2)}{4!(M_1+1)^5 \sin\left(\frac{\pi M_1}{M_1+1}\right)}\right] \\
&= -\frac{2\pi^2 M_1^2(2M_1+1)(3M_1+2)}{3(M_1+1)^2 \sin\left(\frac{\pi M_1}{M_1+1}\right)} < 0,
\end{aligned}
$$

where we used (4.1) with $(k,m)=(2,6)$ and $(2,5)$ respectively in order to evaluate the integrals in the seventh line. This completes the proof of the proposition. □

Acknowledgements

This research is supported partially by the grant no. F01-2002-000-20014 from the KOSEF and by an Interdisciplinary Research Grant of the Seoul National University in 2003.

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