On the coexistence of the strings and antistrings of the nontopological type

Dongho Chae
Department of Mathematics
Sungkyunkwan University
Suwon 440-746, Korea
e-mail: chae@skku.edu

Abstract

We study a semilinear ellptic system modelling the physical system strings and antistrings in cosmology under the boundary condition of the symmetric vacuum(the nontopological type). We construct solutions with the representation having precise informations on the asymptotic behaviors near infinity for arbitray location of strings and antitrings satisfying $1 \leq M - N < \frac{1}{4\pi G}$, where M and N are the total string number and the total antistring number respectively, and G is the gravitational constant. The asymptotic properties, in particular, are completely different to the solutions under the boundary condition of the asymmetric vacuum(the topological type) constructed previously by Y. Yang[19]. We also compute the total magnetic flux, total energy and the total Gaussian curvature of our solutions.

1 Introduction

We consider the following system for (v, η) in $\mathbb{R}^2$:

$$\Delta v = \frac{2e^{\eta}(e^v - 1)}{e^v + 1} - 4\pi \sum_{j=1}^{N} \delta(z - p_j) + 4\pi \sum_{j=1}^{M} \delta(z - q_j), \tag{1.1}$$

$$-\frac{1}{2a}\Delta\eta = \Delta\left[\ln(1 + e^v) - \frac{1}{2}v\right] + 2\pi \sum_{j=1}^{N} \delta(z - p_j) + 2\pi \sum_{j=1}^{M} \delta(z - q_j), \tag{1.2}$$

equipped with the boundary condition

$$e^v \to 0 \quad \text{and} \quad e^{\eta} \to 0 \quad \text{as} \quad |z| \to \infty, \tag{1.3}$$

where we denoted $z = x_1 + ix_2 \in \mathbb{C} = \mathbb{R}^2$, and $a = 8\pi G$ with $G > 0$, the gravitational constant. The system (1.1)-(1.2) represent an equilibrium configuration of cosmic strings and antistrings in cosmology. More precisely, we could start from a lagrangian of the Hilbert-Einstein action coupled with $O(3)$ model(sigma model) as a matter part, which represents cosmic strings in the universe(See e.g. [16].) Then, under the assumption of translational symmetry in the time and one spatial direction, we obtain the Bogomol'nyi type of equations saturating the global minimization of the static energy, which after the standard reduction procedure[8] we obtain (1.1)-(1.2). There are many literature on the related physical models on the cosmic strings(See e.g.[8, 9, 16, 17] and references therein). With the Abelian Higgs model as the matter part mathematically rigorous study of the static cosmic strings was initiated in [7] for the radially symmetric case, and in [18, 21] in the general multistring case. With the $O(3)$ model as a matter part, as we are intersted in this paper, we could have both strings and antistrings simultaneously. In this case, Yang proved existence of general multistrings in [19, 20, 21]. All of the previous mathematical studies are under the boundary condition of the broken vacuum symmetry(topological boundary condition), namely $e^v \to 1$ and $e^{\eta} \to 0$ as $|z| \to \infty$. In this paper, in the cosmic string-sntistring model problem, we study the system under the boundary condition of the unbroken vacuum symmetry(nontopological boundary condition), which is the problem 1.1-1.3. Our boundary condition of the unbroken vacuum symmetry has similar feature to the nontopological boundary condition in the self-dual Chern-Simons theories studied in $\mathbb{R}^2$ by [13, 3, 4, 5, 2], and in the periodic domain by [14, 12, 11]. We use the method developed and refined in [3, 4, 5] to construct solutions of the problem (1.1)-(1.3). We find that the qualititive properties of our solution is completely different to those obtained by Yang in [19, 20, 21]. We note that the study of the Abelian Higgs strings in the broken vacuum symmetry is recently done in [6].

Our main aim in this paper is to prove the following theorem.

Theorem 1.1 *Suppose $Q_1 = \{p_j\}_{j=1}^N$ and $Q_2 = \{q_j\}_{j=1}^M$ are given in $\mathbb{R}^2$ allowing multiplicities, and $Q_1 \cap Q_2 = \varnothing$. Assume*

$$1 \le M - N < \frac{1}{4\pi G}. \tag{1.4}$$

Then, there exists a constant $\varepsilon_1 > 0$ such that for any $\varepsilon \in (0, \varepsilon_1)$ there exists a family of solutions to (P), (u_1, u_2). Moreover, the solutions we constructed have the following representations:

$$\begin{aligned} v(z) &= \ln \rho^I_{\varepsilon,\delta^*_\varepsilon}(z) + \varepsilon^2 w_1(\varepsilon|z|) + \varepsilon^2 u^*_{1,\varepsilon}(\varepsilon z), & (1.5)\\ \eta(z) &= \ln \rho^{II}_{\varepsilon,\delta^*_\varepsilon}(z) + \varepsilon^2 w_2(\varepsilon|z|) + \varepsilon^2 u^*_{2,\varepsilon}(\varepsilon z) & (1.6)\end{aligned}$$

with

$$\begin{aligned} \rho^I_{\delta,\varepsilon}(z) &= \frac{\varepsilon^{2M-2N+2}\prod_{j=1}^M |z-q_j|^2}{\prod_{j=1}^N |z-p_j|^2 (1+|\varepsilon z+\delta|^2)^{\frac{2}{a}}}, & (1.7)\\ \rho^{II}_{\delta,\varepsilon}(z) &= \frac{4\varepsilon^2}{a(1+|\varepsilon z+\delta|^2)^2}, \qquad \delta = \delta_1 + i\delta_2 \in \mathbb{C}, & (1.8)\end{aligned}$$

where and hereafter we denote

$$a = 8\pi G. \tag{1.9}$$

In (1.5) and (1.6), the function $\varepsilon \mapsto \delta^*_\varepsilon$ is a continuous function in a neighborhood of 0, and $|\delta^*_\varepsilon| \to 0$ as $\varepsilon \to 0$. The radial functions w_1, w_2 have the following asymptotic properties.

$$\begin{aligned} w_1(|z|) &= -C_1 \ln|z| + O(1) \qquad \text{as } |z| \to \infty, & (1.10)\\ w_2(|z|) &= -C_2 \ln|z| + O(1) \qquad \text{as } |z| \to \infty & (1.11)\end{aligned}$$

with C_1, C_2 defined by

$$\begin{aligned} C_1 &= \frac{16(a+1)(M-N)![1-a(M-N)]}{a^2 \prod_{k=1+N-M}^2 \left(\frac{2}{a}+k\right)}, & (1.12)\\ C_2 &= \frac{16(a+1)(M-N)![1-a(M-N)]}{a \prod_{k=1+N-M}^2 \left(\frac{2}{a}+k\right)} (= aC_1). & (1.13)\end{aligned}$$

The functions $u^*_{1,\varepsilon}, u^*_{2,\varepsilon}$ satisfy

$$\sup_{z\in\mathbb{R}^2} \frac{|u^*_{1,\varepsilon}(\varepsilon z)| + |u^*_{2,\varepsilon}(\varepsilon z)|}{\ln(e+|z|)} \le o(1) \qquad \text{as } \varepsilon \to 0. \tag{1.14}$$

Given solution (v, η) of the system (1.1)-(1.2) we have the following formulae for the flux density F_{12}, energy density $\mathcal{H}$ and the Gaussian curvature K of the corresponding physical system of cosmic strings and antistrings(vortices and the antivortices) on the conformally flat surface $\mathcal{M}^2 = (\mathbb{R}^2, e^\eta \delta_{jk})$(See Section 11.2 of [18]).

$$
\begin{aligned}
F_{12} &= -\frac{e^\eta(e^v - 1)}{e^v + 1} && (1.15)\\
e^\eta \mathcal{H} &= \Delta \left[\ln(1 + e^v) - \frac{1}{2} v \right] + 2\pi \sum_{j=1}^{N} \delta(z - p_j) + 2\pi \sum_{j=1}^{M} \delta(z - q_j) && (1.16)\\
e^\eta K &= -\frac{1}{2} \Delta \eta. && (1.17)
\end{aligned}
$$

Using these formula we evaluate the total magnetic flux, total energy and the total curvature rather explicitly.

Theorem 1.2 *Let (v, η) be solution constructed in Theorem 1.1. Then, we have the following evaluations of the total magnetic flux, total energy and the total curvature of $\mathcal{M}^2 = (\mathbb{R}^2, e^\eta \delta_{jk})$.*

$$
\begin{aligned}
\int_{\mathbb{R}^2} F_{12} dx &= 4\pi \left[M - N - \frac{1}{a} \right] + \pi C_1 \varepsilon^2 + o(\varepsilon^2), && (1.18)\\
\int_{\mathbb{R}^2} \mathcal{H} e^\eta dx &= \frac{\pi}{a} \left[4 + C_2 \varepsilon^2 + o(\varepsilon^2) \right], && (1.19)\\
\int_{\mathbb{R}^2} K e^\eta dx &= \pi \left[4 + C_2 \varepsilon^2 + o(\varepsilon^2) \right] && (1.20)
\end{aligned}
$$

as $\varepsilon \to 0$, where C_1, C_2 are the constants in (1.10), (1.11) respectively. Moreover, for all sufficiently small positive ε, the conformally flat 2 surface $\mathcal{M}^2$ is complete(incomplete) if and only if $a(M - N) < (>)1$.

2 Proof of the Main Theorems

We first transform our system to more convenient form for our analysis. Using (1.1) we compute,

$$
\begin{aligned}
\Delta \ln(1 + e^v) &= \frac{e^v \Delta v}{e^v + 1} + \frac{|\nabla v|^2 e^v}{e^v + 1} - \frac{|\nabla v|^2 e^{2v}}{(e^v + 1)^2}\\
&= \frac{2e^{\eta+v}(e^v - 1)}{(e^v + 1)^2} + \frac{|\nabla v|^2 e^v}{e^v + 1} - \frac{|\nabla v|^2 e^{2v}}{(e^v + 1)^2} - 4\pi \sum_{j=1}^{M} \delta(z - p_j)
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{2e^{\eta+v}(e^v-1)}{(e^v+1)^2} + \frac{|\nabla v|^2 e^v}{(e^v+1)^2} - 4\pi \sum_{j=1}^{M} \delta(z-p_j) \\
&= \frac{2e^{\eta+v}(e^v-1)}{(e^v+1)^2} + \frac{\Delta e^v - e^v \Delta v}{(e^v+1)^2} - 4\pi \sum_{j=1}^{M} \delta(z-p_j). \qquad (2.1)
\end{aligned}
$$

Substituting (2.1) into (1.2), and then eliminating $\frac{1}{2}\Delta v$ inside $[\cdot]$ using (1.1), we obtain the following system equivalent to (1.1)-(1.2):

$$
\Delta v = \frac{2e^{\eta}(e^v-1)}{e^v+1} - 4\pi \sum_{j=1}^{N} \delta(z-p_j) + 4\pi \sum_{j=1}^{M} \delta(z-q_j), \qquad (2.2)
$$

$$
\Delta \eta = -\frac{2ae^{\eta}(e^v-1)^2}{(e^v+1)^2} - \frac{2a\Delta e^v}{(e^v+1)^2} + \frac{2ae^v \Delta v}{(e^v+1)^2}. \qquad (2.3)
$$

We consider the following 'principal part' of the system, (2.2)-(2.3).

$$
\begin{aligned}
\Delta v_0 &= -2e^{\eta_0} - 4\pi \sum_{j=1}^{N} \delta(z-p_j) + 4\pi \sum_{j=1}^{M} \delta(z-q_j), \qquad (2.4) \\
\Delta \eta_0 &= -2ae^{\eta_0}. \qquad (2.5)
\end{aligned}
$$

As a family of solution (2.5) we have

$$
\eta_0(z) = \ln \rho^{II}_{\delta,\varepsilon}(z) \qquad (2.6)
$$

with $\rho^{II}_{\delta,\varepsilon}(z)$ defined in (1.8). In order to solve (2.4) we rewrite it as

$$
\Delta \left(av_0 + a\sum_{j=1}^{N} \ln|z-p_j|^2 - a\sum_{j=1}^{M} \ln c_0|z-q_j|^2 \right) = -2ae^{\eta_0}, \qquad (2.7)
$$

where c_0 is an arbitrary positive constant. Comparing (2.7) with (2.5), we find that

$$
av_0 + a\sum_{j=1}^{N} \ln|z-p_j|^2 - a\sum_{j=1}^{M} \ln c_0|z-q_j|^2 = \eta_0 + h(z) \qquad (2.8)
$$

for a harmonic function, $h(z)$. We choose $h(z) \equiv 0$. Then, substituting η_0 in (2.6) into (2.8), and solving it for v_0, and choosing the constant c_0 in as

$$
c_0 = \varepsilon^{2M-2N+2} \left(\frac{a}{4\varepsilon^2} \right)^{\frac{1}{a}},
$$

we find that

$$
v_0(z) = \ln \rho^{I}_{\delta,\varepsilon}(z), \qquad (2.9)
$$

with $\rho^I_{\delta,\varepsilon}(z)$ defined in (1.7). We set

$$g^I_{\delta,\varepsilon}(z) = \frac{1}{\varepsilon^2}\rho^I_{\delta,\varepsilon}\left(\frac{z}{\varepsilon}\right), \qquad g^{II}_{\delta,\varepsilon}(z) = \frac{1}{\varepsilon^2}\rho^{II}_{\delta,\varepsilon}\left(\frac{z}{\varepsilon}\right),$$

and define $\rho_1(r), \rho_2)r)$ by

$$\rho_1(r) = \frac{r^{2M-2N}}{(1+r^2)^{\frac{2}{a}}} = \lim_{\varepsilon+|\delta|\to 0} g^I_{\delta,\varepsilon}(z), \quad \rho_2(r) = \frac{4}{a(1+r^2)^2} = \lim_{\varepsilon+|\delta|\to 0} g^{II}_{\delta,\varepsilon}(z).$$

We make transforms from (v,η) to (u_1,u_2) as follows

$$\begin{aligned} v(z) &= \ln\rho^I_{\delta,\varepsilon}(z) + \varepsilon^2 w_1(\varepsilon z) + \varepsilon^2 u_1(\varepsilon z) \\ \eta(z) &= \ln\rho^{II}_{\delta,\varepsilon}(z) + \varepsilon^2 w_2(\varepsilon z) + \varepsilon^2 u_2(\varepsilon z) \end{aligned} \tag{2.10}$$

where w_1, w_2 are the radial functions, $w_j(z) = w_j(|z|)$, $j = 1,2$ to be determined below. Then, (2.2)-(2.3) can be transformed into the functional equation, $P = (P_1, P_2) = 0$, where

$$\begin{aligned} P_1(u_1,u_2,\delta,\varepsilon) &= \Delta u_1 \\ &-\frac{2g^I_{\delta,\varepsilon}(z)g^{II}_{\delta,\varepsilon}(z)e^{\varepsilon^2(u_1+u_2+w_1+w_2)} - \frac{2g^{II}_{\delta,\varepsilon}(z)}{\varepsilon^2}e^{\varepsilon^2(u_2+w_2)}}{\varepsilon^2 g^I_{\delta,\varepsilon}(z)e^{\varepsilon^2(u_1+w_1)}+1} - \frac{2g^{II}_{\delta,\varepsilon}(z)}{\varepsilon^2} + \Delta w_1, \end{aligned}$$

$$\begin{aligned} P_2(u_1,u_2,\delta,\varepsilon) &= \Delta u_2 \\ &+\frac{\frac{2a}{\varepsilon^2}g^{II}_{\delta,\varepsilon}(z)e^{\varepsilon^2(u_2+w_2)}\left[\varepsilon^2 g^I_{\delta,\varepsilon}(z)e^{\varepsilon^2(u_1+w_1)}-1\right]^2}{\left[\varepsilon^2 g^I_{\delta,\varepsilon}(z)e^{\varepsilon^2(u_1+w_1)}+1\right]^2} - \frac{2ag^{II}_{\delta,\varepsilon}(z)}{\varepsilon^2} \\ &+\frac{2a\Delta\left[g^I_{\delta,\varepsilon}(z)e^{\varepsilon^2(u_1+w_1)}\right]}{\left[\varepsilon^2 g^I_{\delta,\varepsilon}(z)e^{\varepsilon^2(u_1+w_1)}+1\right]^2} + \frac{4ag^I_{\delta,\varepsilon}(z)g^{II}_{\delta,\varepsilon}(z)e^{\varepsilon^2(u_1+w_1)}}{\left[\varepsilon^2 g^I_{\delta,\varepsilon}(z)e^{\varepsilon^2(u_1+w_1)}+1\right]^2} \\ &-\frac{2a\varepsilon^2 g^I_{\delta,\varepsilon}(z)e^{\varepsilon^2(u_1+w_1)}\Delta(u_1+w_1)}{\left[\varepsilon^2 g^I_{\delta,\varepsilon}(z)e^{\varepsilon^2(u_1+w_1)}+1\right]^2} + \Delta w_2. \end{aligned}$$

Now we introduce the functions spaces used in [3]. Let us fix $\alpha \in (0, \frac{1}{2})$ throughout this paper. Following [1], we introduce the Banach spaces X_α and Y_α as

$$X_\alpha = \{u \in L^2_{loc}(\mathbb{R}^2) \mid \int_{\mathbb{R}^2}(1+|x|^{2+\alpha})|u(x)|^2dx < \infty\}$$

equipped with the norm $\|u\|^2_{X_\alpha} = \int_{\mathbb{R}^2}(1+|x|^{2+\alpha})|u(x)|^2dx$, and

$$Y_\alpha = \{u \in W^{2,2}_{loc}(\mathbb{R}^2) \mid \|\Delta u\|^2_{X_\alpha} + \left\|\frac{u(x)}{1+|x|^{1+\frac{\alpha}{2}}}\right\|^2_{L^2(\mathbb{R}^2)} < \infty\}$$

equipped with the norm $\|u\|^2_{Y_\alpha} = \|\Delta u\|^2_{X_\alpha} + \left\|\frac{u(x)}{1+|x|^{1+\frac{\alpha}{2}}}\right\|^2_{L^2(\mathbb{R}^2)}$. We recall the following propositions proved in [3].

Proposition 2.1 *Let Y_α be the function space introduced above. Then we have the followings.*

(i) If $v \in Y_\alpha$ is a harmonic function, then $v \equiv constant$.

(ii) There exists a constant $C_1 > 0$ such that for all $v \in Y_\alpha$

$$|v(x)| \le C_1 \|v\|_{Y_\alpha} \ln(e + |x|), \qquad \forall x \in \mathbb{R}^2.$$

Proposition 2.2 *Let $\alpha \in (0, \frac{1}{2})$, and let us set*

$$L = \Delta + \rho : Y_\alpha \to X_\alpha,$$

where $\rho = \frac{8}{(1+r^2)^2}$. We have

$$KerL = Span\{\varphi_+, \varphi_-, \varphi_0\}, \tag{2.11}$$

where we denoted

$$\varphi_+ = \frac{r}{1+r^2}\cos\theta, \qquad \varphi_- = \frac{r}{1+r^2}\sin\theta, \qquad \varphi_0 = \frac{1-r^2}{1+r^2}.$$

Moreover, we have

$$ImL = \{f \in X_\alpha | \int_{\mathbb{R}^2} f\varphi_\pm = 0\}. \tag{2.12}$$

We can check easily that P is a well defined continuous mapping from $B_{\varepsilon_0} \subset (Y_\alpha)^2 \times \mathbb{C} \times \mathbb{R}_+$ into $(X_\alpha)^2$, where $B_{\varepsilon_0} = \{\|u_1\|_{Y_\alpha} + \|u_2\|_{Y_\alpha} + |\delta| \le \varepsilon < \varepsilon_0\}$ for sufficiently small ε_0. In order to have $g^I_{\delta,\varepsilon}(z) \to 0$ as $|z| \to 0$ we impose

$$2a(M - N) < 1,$$

which is equivalent to (1.4).

We now extend continuously $P(0,0,0,\varepsilon)$ to $\varepsilon = 0$ by imposing the condition that $\lim_{\varepsilon\to 0} P(0,0,0,\varepsilon) = 0$. In order to compute the limit $\lim_{\varepsilon\to 0} P(0,0,0,\varepsilon)$ we note the elementary facts,

$$\frac{1}{1+x} - 1 = -x + O(x^2), \qquad \frac{1}{(1+x)^2} - 1 = -2x + O(x^2) \tag{2.13}$$

as $x \to 0$. Using this we compute

$$\lim_{\varepsilon\to 0} P_1(0,0,0,\varepsilon) = -4\rho_1\rho_2 + 2\rho_2 w_2 + \Delta w_1,$$

and

$$\lim_{\varepsilon\to 0} P_2(0,0,0,\varepsilon) = -4a\rho_1\rho_2 + 2a\rho_2 w_2 + 2a\Delta\rho_1 + \Delta w_2.$$

Hence, the condition $\lim_{\varepsilon\to 0} P(0,0,0,\varepsilon) = 0$ implies the following linear system for $w_1(r), w_2(r)$.

$$\Delta w_1 + 2\rho_2 w_2 - 4\rho_1\rho_2 = 0, \tag{2.14}$$

$$\Delta w_2 + 2a\rho_2 w_2 - 4a\rho_1\rho_2 + 2a\Delta\rho_1 = 0. \tag{2.15}$$

We establish the following lemma about asymptotic behaviors of solutions $w_1, w_2 \in Y_\alpha$.

Lemma 2.1 *There exist radial solutions $w_1(|z)), w_2(|z|)$ of (2.14)-(2.15), belonging to Y_α, satisfying the estimate (1.10) and the asymptotic formula in (1.11) respectively.*

Proof: From (2.14)$\times a-$(2.15) we obtain

$$\Delta(aw_1 - w_2 - 2a\rho_1) = 0.$$

We seek w_1, w_2 with $aw_1 - w_2 - 2a\rho_1 \in Y_\alpha$. Then, it follows that $aw_1 - w_2 - a\rho_1 =$constant by Proposition 2.1. We choose this constant$=0$. Then, $\rho_2 w_2 = a\rho_2 w_1 - 2a\rho_1\rho_2$. Substituting this into (2.14) we obtain the following reduced system for w_1, w_2.

$$\Delta w_1 + 2a\rho_2 w_1 = 4(a+1)\rho_1\rho_2, \tag{2.16}$$

$$w_2 = aw_1 - 2a\rho_1. \tag{2.17}$$

Let us set $f(r) = 4(a+1)\rho_1\rho_2$. Then, it is found in [1, 3] that the ordinary differential equation(with respect to r), (2.16) has a solution $w_1(r) \in Y_\alpha$ given by

$$w_1(r) = \varphi_0(r)\left\{\int_0^r \frac{\phi_f(s) - \phi_f(1)}{(1-s)^2}ds + \frac{\phi_f(1)r}{1-r}\right\} \tag{2.18}$$

with

$$\phi_f(r) := \left(\frac{1+r^2}{1-r^2}\right)^2 \frac{(1-r)^2}{r}\int_0^r \varphi_0(t)tf(t)dt,$$

where $\phi_f(1)$ and $w_1(1)$ are defined as limits of $\phi_f(r)$ and $w_1(r)$ as $r \to 1$. From the formula (2.18) we find that

$$w_1(r) = \varphi_0(r)\int_2^r \left(\frac{1+s^2}{1-s^2}\right)^2 \frac{I(s)}{s}ds + (\text{bounded function of } r) \tag{2.19}$$

as $r \to \infty$, where

$$I(s) = \int_0^s \varphi_0(r)f(r)rdr.$$

Since $\varphi_0(r) \to -1$ as $r \to \infty$, (1.10) follows if we show $I = I(\infty) = C_1$. Indeed, substituting $r^2 = t$ in the integrand of I, we evaluate the integral as follows.

$$\begin{aligned} I &= 4(a+1)\int_0^\infty \varphi_0(r)\rho_1(r)\rho_2(r)rdr \\ &= \frac{16(a+1)}{a}\int_0^\infty \frac{(1-r^2)r^{2M-2N}}{(1+r^2)^{3+\frac{2}{a}}}rdr \\ &= \frac{8(a+1)}{a}\int_0^\infty \frac{(1-t)t^{M-N}}{(1+t)^{3+\frac{2}{a}}}dt \end{aligned}$$

$$
\begin{aligned}
&= \frac{8(a+1)}{a}\left[\int_0^\infty \frac{t^{M-N}}{(1+t)^{3+\frac{2}{a}}}dt - \int_0^\infty \frac{t^{M-N+1}}{(1+t)^{3+\frac{2}{a}}}dt\right] \\
&= \frac{8(a+1)}{a}\left[\frac{(M-N)!}{\prod_{k=2+N-M}^{2}\left(\frac{2}{a}+k\right)} - \frac{(M-N+1)!}{\prod_{k=1+N-M}^{2}\left(\frac{2}{a}+k\right)}\right] \\
&= \frac{8(a+1)(M-N)!}{a\prod_{k=1+N-M}^{2}\left(\frac{2}{a}+k\right)}\left[\frac{2}{a}+1+N-M-(M-N+1)\right] \\
&= \frac{16(a+1)(M-N)![1-a(M-N)]}{a^2\prod_{k=1+N-M}^{2}\left(\frac{2}{a}+k\right)} = C_1. \qquad (2.20)
\end{aligned}
$$

This completes the proof of (1.10). The proof follows immediately from (1.10) combined with (2.17). This completes the proof of Lemma 2.1. □

Now we compute the linearized operator of P. By direct computation we have

$$
\lim_{\varepsilon\to 0}\frac{\partial g^I_{\delta,\varepsilon}(z)}{\partial\delta_1}\bigg|_{\delta=0} = -\frac{4}{a}\rho_1\varphi_+, \quad \lim_{\varepsilon\to 0}\frac{\partial g^I_{\delta,\varepsilon}(z)}{\partial\delta_2}\bigg|_{\delta=0} = -\frac{4}{a}\rho_1\varphi_-,
$$

$$
\lim_{\varepsilon\to 0}\frac{\partial g^{II}_{\delta,\varepsilon}(z)}{\partial\delta_1}\bigg|_{\delta=0} = -4\rho_2\varphi_+, \quad \lim_{\varepsilon\to 0}\frac{\partial g^{II}_{\delta,\varepsilon}(z)}{\partial\delta_2}\bigg|_{\delta=0} = -4\rho_2\varphi_-.
$$

Let us set $P'_{u_1,u_2,\delta}(0,0,0,0) = \mathcal{A}$. Then, using the above preliminary computations, we obtain

$$
\begin{aligned}
\mathcal{A}_1[v_1,v_2,\beta] &= \Delta v_1 + 2\rho_2 v_2 \\
&\quad +8\left[2(1+\frac{1}{a})\rho_1\rho_2 - \rho_2 w_2\right](\varphi_+\beta_1+\varphi_-\beta_2),
\end{aligned}
$$

and

$$
\begin{aligned}
\mathcal{A}_2[v_1,v_2,\beta] &= \Delta v_2 + 2a\rho_2 v_2 \\
&\quad +8\left[2(1+a)\rho_1\rho_2 - a\rho_2 w_2\right](\varphi_+\beta_1+\varphi_-\beta_2) \\
&\quad -8\Delta[\rho_1(\varphi_+\beta_1+\varphi_-\beta_2)].
\end{aligned}
$$

For the linearized operator $\mathcal{A}[\cdot]$ we will establish the following key lemma.

Lemma 2.2 *The operator $\mathcal{A} : Y^2_\alpha \times \mathbb{R}^2 \to X^2_\alpha$ defined above is onto. Moreover, kernel of $\mathcal{A}$ is given by*

$$
Ker\mathcal{A} = Span\{(1,0);(\frac{\varphi_\pm}{a},\varphi_\pm);(\frac{\varphi_0}{a},\varphi_0)\}\times\{(0,0)\}. \qquad (2.21)
$$

Thus, if we decompose $Y^2_\alpha\times\mathbb{R}^2 = U_\alpha\oplus Ker\mathcal{A}$, where we set $U_\alpha = (Ker\mathcal{A})^\perp$, then $\mathcal{A}$ is an isomorphism from U_α onto X^2_α.

In order to prove the above lemma we need the following:

Proposition 2.3 *Let $w_2 \in Y_\alpha$ solve (2.14)-(2.15), then*

$$I_\pm = \int_{\mathbb{R}^2} \left[2(1+a)\rho_1\rho_2 - a\rho_2 w_2\right] \varphi_+^2 dx - \int_{\mathbb{R}^2} \Delta(\rho_1\varphi_+)\varphi_\pm dx \neq 0. \tag{2.22}$$

Proof: Integrating by part, we obtain

$$\begin{aligned} I_\pm &= \int_{\mathbb{R}^2} \left\{[2(1+a)\rho_1\rho_2 - aw_2\rho_2]\varphi_\pm^2 - \rho_1\varphi_\pm\Delta\varphi_\pm\right\} dx \\ &= \int_{\mathbb{R}^2} [(4a+2)\rho_1\rho_2 - aw_2\rho_2]\varphi_\pm^2 dx, \end{aligned} \tag{2.23}$$

where we used (2.21 $\Delta\varphi_\pm = -2a\rho_2\varphi_\pm$. (Note that $L = \Delta + 2a\rho_2$.) Below we list useful formulas, which can be checked by elementary computations.

$$\varphi_\pm^2\rho_2 = \frac{1}{16}L_2\rho_2 \left\{ \begin{array}{c} \cos^2\theta \\ \sin^2\theta \end{array} \right\}, \tag{2.24}$$

$$\varphi_\pm^2 = \frac{1}{4}ar^2\rho_2 \left\{ \begin{array}{c} \cos^2\theta \\ \sin^2\theta \end{array} \right\}, \tag{2.25}$$

$$\Delta\rho_2 = 2a(2r^2-1)\rho_2^2, \tag{2.26}$$

Also, from (2.15), we have

$$Lw_2 = 4a\rho_1\rho_2 - 2a\Delta\rho_1. \tag{2.27}$$

Using (2.24)-(2.27), and integrating by parts, we transform the integral successively as follows.

$$\begin{aligned} I_\pm &= \int_{\mathbb{R}^2} (4a+2)\rho_1\rho_2\varphi_\pm^2 dx - \frac{a}{16}\int_0^\infty \int_0^{2\pi} w_2(L_2\rho_2) \left\{ \begin{array}{c} \cos^2\theta \\ \sin^2\theta \end{array} \right\} d\theta r dr \\ &= \int_0^\infty \int_0^{2\pi} \left\{ \frac{a(4a+2)}{4}r^2\rho_1\rho_2^2 - \frac{a}{16}(Lw_2)\rho_2 \right\} \left\{ \begin{array}{c} \cos^2\theta \\ \sin^2\theta \end{array} \right\} d\theta r dr \\ &= \pi\int_0^\infty \left\{ \frac{a(4a+2)}{4}r^2\rho_1\rho_2^2 - \frac{a}{16}(4a\rho_1\rho_2 - 2a\Delta\rho_1)\rho_2 \right\} r dr \\ &= \pi\int_0^\infty \left\{ \frac{a(4a+2)}{4}r^2\rho_1\rho_2^2 - \frac{a^2}{4}\rho_1\rho_2^2 + \frac{a^2}{8}\rho_1\Delta\rho_2 \right\} r dr \\ &= \pi\int_0^\infty \left\{ \frac{a(4a+2)}{4}r^2\rho_1\rho_2^2 - \frac{a^2}{4}\rho_1\rho_2^2 + \frac{a^3}{4}(2r^2-1)\rho_1\rho_2^2 \right\} r dr \\ &= \pi\int_0^\infty \left\{ \frac{a(4a+2)}{4}r^2 - \frac{a^2}{4} + \frac{a^3r^2}{2} - \frac{a^3}{4} \right\} \rho_1\rho_2^2 r dr \end{aligned}$$

$$
\begin{aligned}
&= \frac{a(a+1)\pi}{4}\int_0^\infty [(2a+2)r^2-a]\rho_1\rho_2^2 r dr \\
&= \frac{4(a+1)\pi}{a}\int_0^\infty \frac{[(2a+2)r^2-a]r^{2M-2N+1}}{(1+r^2)^{\frac{2}{a}+4}}dr \quad (\text{Setting } r^2=t) \\
&= \frac{2(a+1)\pi}{a}\int_0^\infty \left[\frac{(2a+2)t^{M-N+1}}{(1+t)^{\frac{2}{a}+4}} - \frac{at^{M-N}}{(1+t)^{\frac{2}{a}+4}}\right]dt \\
&= \frac{2(a+1)\pi}{a}\left[\frac{(2a+2)(M-N+1)!}{\prod_{k=2+N-M}^{3}\left(\frac{2}{a}+k\right)} - \frac{a(M-N)!}{\prod_{k=3+N-M}^{3}\left(\frac{2}{a}+k\right)}\right] \\
&= \frac{2(a+1)\pi(M-N)!}{a\prod_{k=2+N-M}^{3}\left(\frac{2}{a}+k\right)}\left[(2a+2)(M-N+1)-a\left(\frac{2}{a}+2+N-M\right)\right] \\
&= \frac{2(a+1)(3a+2)\pi(M-N)!\cdot(M-N)}{a\prod_{k=2+N-M}^{3}\left(\frac{2}{a}+k\right)} > 0. \qquad (2.28)
\end{aligned}
$$

This completes the proof of Proposition 2.3. □.

We are now ready to prove Lemma 3.1.

Proof of Lemma 2.2: Given $(f_1,f_2)\in X_\alpha^2$, we want first to show that there exists $(v,\eta)\in Y_\alpha^2$, $\beta=(\beta_1,\beta_2)\in\mathbb{R}^2$ such that

$$\mathcal{A}(v_1,v_2,\beta)=(f_1,f_2), \qquad (2.29)$$

which can be rewritten as

$$\Delta v_1 + 2\rho_2 v_2 + 8\left[2(1+\frac{1}{a})\rho_1\rho_2 - \rho_2 w_2\right](\varphi_+\beta_1+\varphi_-\beta_2) = f_1, \qquad (2.30)$$

and

$$\Delta v_2 + 2a\rho_2 v_2 + 8\left[2(1+a)\rho_1\rho_2 - a\rho_2 w_2\right](\varphi_+\beta_1+\varphi_-\beta_2) - 8\Delta[\rho_1(\varphi_+\beta_1+\varphi_-\beta_2)] = f_2. \qquad (2.31)$$

Let us set

$$\beta_1 = \frac{1}{8I_+}\int_{\mathbb{R}^2} f_2\varphi_+ dx, \qquad \beta_2 = \frac{1}{8I_-}\int_{\mathbb{R}^2} f_2\varphi_- dx, \qquad (2.32)$$

where $I_\pm > 0$ is defined in (2.22). We introduce $\tilde{f}$ by

$$\tilde{f}_2 = f_2 - \beta_1\varphi_+ - \beta_2\varphi_-. \qquad (2.33)$$

Using the fact $\int_0^{2\pi}\varphi_+\varphi_- d\theta = 0$, we find easily

$$\int_{\mathbb{R}^2}\tilde{f}_2\varphi_\pm dx = 0. \qquad (2.34)$$

Hence, by (2.12) there exists $v_2 \in Y_\alpha$ such that $\Delta v_2 + 2a\rho_2 v_2 = \tilde{f}_2$. Thus we have found $(v_2, \beta_1, \beta_2) \in Y_\alpha \times \mathbb{R}^2$ satisfying (2.31). Given such (v_2, β_1, β_2), in order to construct $v_1 \in Y_\alpha$ satisfying (2.30), we consider the following equation, obtained by (2.30) $\times\, a -$ (2.31),

$$\Delta(av_1 - v_2 + 8\rho_1\varphi_+\beta_1 + 8\rho_1\varphi_-\beta_2) = af_1 - f_2. \tag{2.35}$$

Obviously, for any harmonic function $\mu(x)$ the function

$$\begin{aligned} v_1(x) &= \frac{1}{2\pi a}\int_{\mathbb{R}^2} \ln(|x-y|)[af_1 - f_2](y)dy \\ &\quad + \frac{1}{a}(v_2 - 8\rho_1\varphi_+\beta_1 - 8\rho_1\varphi_-\beta_2) + \mu(x) \end{aligned} \tag{2.36}$$

satisfies (2.30). The requirement $v_1 \in Y_\alpha$ implies $\mu(x) \equiv$constant, thanks to Proposition 2.1(i). We have just finished the proof that $\mathcal{A} : Y_\alpha^2 \times \mathbb{R}^2 \to X_\alpha^2$ is onto.

Now it is easy to check that the restricted operator(denoted by the same symbol), $\mathcal{A} : U_\alpha \to X_\alpha^2$, where U_α is the space introduced in Lemma 2.2 is one to one. This completes the proof of the lemma.□

We are now ready to prove our main theorem.

Proof of Theorem 1.1: Lemma 2.2 shows that $P'_{(v,\xi,\beta)}(0,0,0,0) : U_\alpha \to X_\alpha \times X_\alpha$ is an isomorphism for $\alpha \in (0, \frac{1}{2})$. Then, the standard implicit function theorem(See e.g. [22]), applied to the functional $P : U_\alpha \times (-\varepsilon_0, \varepsilon_0) \to X_\alpha \times X_\alpha$, implies that there exists a constant $\varepsilon_1 \in (0, \varepsilon_0)$ and a continuous function $\varepsilon \mapsto \psi_\varepsilon^* := (v_{1,\varepsilon}^*, v_{2,\varepsilon}^*, \delta_\varepsilon^*)$ from $(0, \varepsilon_1)$ into a neighborhood of 0 in U_α such that

$$P(u_{1,\varepsilon}^*, u_{2,\varepsilon}^*, \delta_\varepsilon^*, \varepsilon) = (0,0), \quad \text{for all } \varepsilon \in (0, \varepsilon_1).$$

This completes the proof of Theorem 1.1. The representation of solutions u_1, u_2, and the explicit form of $\rho^I_{\varepsilon,\delta_\varepsilon^*}(z)$, $\rho^{II}_{\varepsilon,\delta_\varepsilon^*}(z)$, , together with the asymptotic behaviors of w_1, w_2 described in Lemma 2.1, the fact that $u_{1,\varepsilon}^*, u_{2,\varepsilon}^* \in Y_\alpha$, combined with Proposition 2.1, implies that the solutions satisfy the boundary condition in (P). Now, from Proposition 2.1 we obtain that for each $j = 1, 2$,

$$|u_{j,\varepsilon}^*(x)| \le C\|u_{j,\varepsilon}^*\|_{Y_\alpha}(\ln^+|x| + 1) \le C\|\psi_\varepsilon\|_{U_\alpha}(\ln^+|x| + 1). \tag{2.37}$$

This implies then

$$|u_{j,\varepsilon}^*(\varepsilon x)| \le C\|\psi_\varepsilon\|_{U_\alpha}(\ln^+|\varepsilon x| + 1) \le C\|\psi_\varepsilon\|_{U_\alpha}(\ln^+|x| + 1).$$

From the continuity of the function $\varepsilon \mapsto \psi_\varepsilon$ from $(0, \varepsilon_0)$ into U_α and the fact $\psi_0^* = 0$ we have

$$\|\psi_\varepsilon\|_{U_\alpha} \to 0 \qquad \text{as } \varepsilon \to 0. \tag{2.38}$$

The proof of (1.11) follows from (2.37) combined with (2.38). This completes the proof of Theorem 1.1□

Proof of Theorem 1.2: Combining (1.1) and (1.15), using the Gauss theorem, we deduce

$$\int_{\mathbb{R}^2} F_{12}dx = -\frac{1}{2}\lim_{R\to\infty}\oint_{S_R}\frac{\partial v}{\partial r}ds + 2\pi(M-N), \tag{2.39}$$

where we set $S_R = \{x\in\mathbb{R}^2| \quad |x|=R\}$. For our solution, $v(x)=v_\varepsilon(x)$ given by (2.24), we compute

$$\begin{aligned}\oint_{S_R}\frac{\partial v_\varepsilon}{\partial r}ds &= \oint_{S_R}\frac{\partial}{\partial r}\ln\rho^I_{\varepsilon,\delta^*_\varepsilon}ds + \varepsilon^2\oint_{S_R}\frac{\partial w_1(\varepsilon|z|)}{\partial r}ds + \varepsilon^2\oint_{S_R}\frac{\partial u^*_{1,\varepsilon}(\varepsilon x)}{\partial r}ds \\ &= I_1+\varepsilon^2 I_2+\varepsilon^2 I_3. \end{aligned}\tag{2.40}$$

Following the similar procedure as in [3](pp. 135-138) we easily compute

$$I_1 = -4\pi\left(M-N-\frac{2}{a}\right)+O\left(\frac{1}{R}\right), \tag{2.41}$$

and using (2.16)

$$\begin{aligned} I_2 &= -8\pi(a+1)\int_0^\infty \varphi_0 t\rho_1\rho_2 dt + O\left(\frac{1}{R}\right) \\ &= -2\pi C_1 \end{aligned}\tag{2.42}$$

as $R\to\infty$, where we used the result of the computation in the proof of Lemma 2.1, and finally

$$\begin{aligned}\sup_{R>0}|I_3| &\le \left|\int_{\mathbb{R}^2}\Delta u^*_{1,\varepsilon}dx\right| \\ &\le \left(\int_{\mathbb{R}^2}|\Delta u^*_{1,\varepsilon}|^2(1+|x|^{2+\alpha})dx\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}^2}\frac{dx}{1+|x|^{2+\alpha}}\right)^{\frac{1}{2}} \\ &\le C\|u^*_{1,\varepsilon}\|_{Y_\alpha}\le C\|\psi^*_\varepsilon\|_{U_\alpha}\to 0 \end{aligned}\tag{2.43}$$

as $\varepsilon\to 0$ due to the continuity of $\varepsilon\mapsto\psi^*_\varepsilon$ in U_α on $(-\varepsilon_1,\varepsilon_1)$ as in the proof of Theorem 1.1. Combining (2.41)-(2.43) with (2.40)) we obtain (1.18). Comparing (1.2) with (1.16), and following similar argument we compute

$$\begin{aligned}\int_{\mathbb{R}^2}\mathcal{H}e^\eta dx &= -\frac{1}{2a}\int_{\mathbb{R}^2}\Delta\eta dx = -\frac{1}{2a}\lim_{R\to\infty}\oint_{S_R}\frac{\partial\eta}{\partial r}ds \\ &= -\frac{1}{2a}\lim_{R\to\infty}\oint_{S_R}\frac{\partial}{\partial r}\ln\rho^{II}_{\varepsilon,\alpha^*_\varepsilon}ds \\ &\quad -\frac{\varepsilon^2}{2a}\lim_{R\to\infty}\oint_{S_R}\frac{\partial w_2(\varepsilon|z|)}{\partial r}ds - \frac{\varepsilon^2}{2a}\int_{\mathbb{R}^2}\Delta u^*_{2,\varepsilon}(\varepsilon x)dx \\ &= J_1+\varepsilon^2 J_2+\varepsilon^2 J_3. \end{aligned}\tag{2.44}$$

Similarly to the case of the flux above we easily compute

$$J_1 = \frac{4\pi}{a}, \tag{2.45}$$

$$J_2 = \frac{\pi C_2}{a}. \tag{2.46}$$

Similarly to I_3 above we have

$$|J_3| \leq C\|u^*_{2,\varepsilon}\|_{Y_\alpha} \leq C\|\psi^*_\varepsilon\|_{U_\alpha} \to 0 \tag{2.47}$$

as $\varepsilon \to 0$. We thus obtain (1.19). In order to prove (1.20) we just observe

$$Ke^\eta = -\frac{1}{2}\Delta\eta = a\mathcal{H}e^\eta,$$

and use the result (1.19). In order to obtain the completeness criterion of the metric $g_{jk} = e^\eta\delta_{jk}$ in $\mathbb{R}^2$ we recall the result in Section 10.5 of [21] that $(\mathbb{R}^2, e^\eta\delta_{jk})$ is complete if and only if

$$\int_{\mathbb{R}^2} e^{\frac{1}{2}\eta}dx = \infty.$$

According to the representation formula (1.6), this, in turn, is equivalent to

$$\int_0^\infty (1+r)^{-1-\frac{1}{2}C_2\varepsilon^2+o(\varepsilon^2)}dr = \infty.$$

We note, however, from (1.13) that $C_2 > 0 (< 0)$ if $a(M-N) < (>) 1$. This completes the proof of Theorem 1.2.$\square$

Acknowledgements

The author would like to thank Professor Y. Yang for useful discussion. This work was supported by Korea Research Foundation Grant KRF-2002-015-CS0003.

References

[1] S. Baraket and F. Pacard, *Construction of singular limits for a semilinear elliptic equation in dimension 2*, Calc. Var. and P.D.E.,**6**, (1998), pp.1-38.

[2] H. Chan, C. C. Fu, and C. S. Lin, *Non-topological multi-vortex solutions to the self-dual Chern-Simons-Higgs equation* Comm. Math. Phys. **231** (2002), no. 2, pp. 189-221.

[3] D. Chae and O. Yu Imanuvilov, *The existence of non-topological multi-vortex solutions in the relativistic self-dual Chern-Simons theory,* Comm. Math. Phys. **215**, (2000), pp. 119-142.

[4] D. Chae and O. Yu Imanuvilov, *Non-topological solitons in a generalized self-dual Chern-Simons theory* , Calc. Var. P.D.E., **16**,(2003), pp. 47-61.

[5] D. Chae and O. Yu Imanuvilov, *Non-topological multivortex solutions to the self-dual Maxwell-Chern-Simons-Higgs systems*, J. Funct. Anal., **196**, (2002), pp. 87-118.

[6] D. Chae, *On the multi-string solutions of the self-dual static Einstein-Maxwell-Higgs system,* Calc. Var. P.D.E., **20**, no. 1, (2004), pp. 47-63.

[7] X. Chen, S. Hastings, J.B. McLeod, and Y. Yang, *A nonlinear elliptic equation arising from gauge field theory and cosmology,* Proc. R. Soc. London. A, **446**, (1994), pp. 453-478.

[8] A. Comtet and G. W. Gibbons, *Bogomol'nyi bounds for cosmic strings,* Nucl. Phys. B. **299**, (1988), pp. 719-733.

[9] M. B. Hindmarsh and T. W. B. Kibble, *Cosmic strings,* Rept. Prog. Phys., **58**, (1995), pp.477-562.

[10] K. Kimm, K. Lee, and T. Lee, *Anyonic Bogomol'nyi solitons in a gauged $O(3)$ sigma model,"* Phys. Rev. D **53**, (1996) pp.4436-4440.

[11] M. Nolasco, *Non-topological N-vortex condensates for the self-dual Chern-Simons theory,* preprint.

[12] M. Nolasco and G. Tarantello, *Double vortex condensates in the Chern-Simons-Higgs theory,* Calc. Var. P.D.E., **9**, (1999), issue 1, pp. 31-94.

[13] J. Spruck and Y. Yang, *The existence of nontopological solitons in the self-dual Chern-Simons theory,* Comm. Math. Phys. **149** (1992), no. 2, pp. 361-376.

[14] G. Tarantello, *Multiple condensate solutions for the Chern-Simons-Higgs Theory,* J. Math. Phys. **37**, (1996), pp. 3769-3796.

[15] C. H. Taubes, *Arbitrary N-vortex solutions to the first order Ginzburg-Landau equation,* Comm. Math. Phys. **72**, (1980), 277-292.

[16] A. Vilenkin and E.P.S. Shellard, *"Cosmic strings and other topological defects,"* Cambridge Univ. Press, (1994).

[17] E. Witten, *Superconducting strings,* Nucl. Phys. B **249**,(1985), pp. 557-592.

[18] Y. Yang, *Prescribing topological defects for the coupled Einstein and abelian Higgs equations,* Comm. Math. Phys. **170**, (1995), pp. 541-582.

[19] Y. Yang, *Coexistence of vortices and antivortices in an Abelian gauge theory,* Phys. Rev. Lett. **80**,(1998), pp. 26-29 .

[20] Y. Yang, *Strings of opposite magnetic charges in a gauge field theory,* Proc. Roy. Soc. A. **455**, (1999), pp. 601-629.

[21] Y. Yang, *"Solitons in field theory and nonlinear analysis",* Springer-Verlag, New York, (2001).

[22] E. Zeidler, *"Nonlinear functional analysis and its applications"*, Vol. 1, Springer-Verlag, New York, (1985).