A complete gauge invariant formalism for arbitrary second-order perturbations of a Schwarzschild black hole

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Using recently developed efficient symbolic manipulations tools, we present a general formalism to study arbitrary second-order perturbations of a Schwarzschild black hole. The formalism is both covariant (independent of the background coordinates) and gauge invariant. In particular, we construct the second order Zerilli and Regge-Wheeler equations under the presence of any two first-order modes, reconstruct the perturbed metric in terms of the master scalars, and compute the radiated energy at null infinity.

The results of this paper enable systematic studies of generic second order perturbations of the Schwarzschild spacetime. In particular, studies of mode-mode coupling and non-linear effects in gravitational radiation, the non-linear stability of the Schwarzschild spacetime, or the geometry of the black hole horizon.

PACS numbers:

I. INTRODUCTION

Gravitational waves have two different polarizations. On a Schwarzschild background this can be explicitly seen, since the perturbed Einstein equations can be reduced to a non-constrained wave equation for each of the two degrees of freedom of the gravitational wave. These are the so-called Regge-Wheeler [1] and Zerilli [2] equations (see also [3]).

The first studies of second-order black hole perturbations were carried out in the seventies [6, 7] in order to study the non-linear stability of the Schwarzschild solution.

In the mid-nineties, motivated by the close-limit approximation [8], gauge invariant second order generalizations of the Regge-Wheeler-Zerilli formalisms were developed and applied to a variety of close limit-type initial data for binary black holes; see [9] and references therein. In those references the second order formalism was only developed for the self-coupling of a polar parity axisymmetric $l = 2$ first-order mode. There was also an inclusion of axial modes [10], but also restricted to axisymmetry and to the harmonic $l = 2$ mode both at first and second-order.

The restriction to these particular cases was not conceptual but, in some sense, due to the approach followed. Namely, the metric perturbations were expanded in spherical harmonics, some particular harmonic mode chosen, the perturbed Einstein equations computed for the case under study, the gauge invariant quantities constructed, the second order Regge-Wheeler-Zerilli equations derived, and the radiated energy computed. In order to study a different case within that approach all those steps needed to be re-done, one at a time. It becomes cumbersome and impractical to go beyond a few particular cases.

In contrast, in this paper we present a complete, covariant and gauge invariant Regge-Wheeler-Zerilli like formalism for *arbitrary* first and second order perturbations of a Schwarzschild black hole. In particular, we derive the general first and second order Regge-Wheeler-Zerilli master scalars and the equations that they obey. We also reconstruct the perturbed metric in terms of those scalars, as well as compute the radiated energy at null infinity.

In this way we can study generic, arbitrary first and second order perturbations. This is one of the main differences with previous work on this topic. A second difference is that our formalism is not only gauge invariant but also covariant; that is, independent of the background coordinates. This is analogous to the covariant versions of the Regge-Wheeler and Zerilli formalisms worked out in of references [15, 16] and [4, 5], respectively. This is a necessary step to, for example, test the geometry near the horizon or null infinity, for which Schwarzschild coordinates are inadequate.

There have been several technical advances over the last few years which made the general treatment of this paper possible. One of them is a gauge-invariant formalism to deal with arbitrary second-order perturbations on an arbitrary spherical spacetime [11, 12] . Here we apply that formalism to the case of a Schwarzschild black hole background. The other main advance has been the development of very efficient symbolic manipulation tools for tensor operations and perturbations, `xAct` [13] and `xPert` [14], which use explicit pre-computed formulas for perturbative expansions as well as efficient canonicalization algorithms for tensorial computations.

The organization of this paper is as follows: we start in Section II by reviewing the formalism of [11, 12]. Section III presents the high order Regge-Wheeler and Zerilli equations, Section IV sketches how we compute the radiated energy. Setting up the arena for the second order treatment, Section V rederives in a compact yet complete way a covariant version of the Regge-Wheeler-Zerilli first order formalism. Finally, Section VI presents our general second order Regge-Wheeler and Zerilli equations, their sources, the reconstruction of the metric and the computation of the radiated energy in terms of our second order Regge-Wheeler and Zerilli functions.

II. HIGH-ORDER GERLACH AND SENGUPTA FORMALISM

A. Background spherical spacetime

This section briefly summarizes the formalism introduced in references [11, 12] to deal with high-order perturbations on a spherical spacetime. This formalism can be regarded as the generalization to higher orders of the Gerlach-Sengupta linear formalism [15, 16].

In order to describe the background spacetime, we note that any four-dimensional spherically symmetric spacetime $\mathcal{M}$ can be expressed as the direct product $\mathcal{M} \equiv \mathcal{M}^2 \times S^2$, where $\mathcal{M}^2$ is a two-dimensional Lorentzian manifold and S^2 the two-sphere. We will use Greek letters $(\mu, \nu, \dots)$ for four-dimensional indices, capital Latin letters $(A, B, \dots)$ for indices on $\mathcal{M}^2$ and lowercase Latin letters $(a, b, \dots)$ for indices on the sphere. With this notation, the background metric is given by

$$g_{\mu\nu}(x^D, x^d) dx^\mu dx^\nu = g_{AB}(x^D) dx^A dx^B + r^2(x^D)\, \gamma_{ab}(x^d) dx^a dx^b, \tag{1}$$

where g_{AB} is the metric associated with the manifold $\mathcal{M}^2$ and γ_{ab} is the unit metric on the sphere S^2.

We have made explicit through the notation of its argument that the scalar r is a function on the $\mathcal{M}^2$ manifold.

We define the following notation for the covariant derivatives associated with each metric:

$$g_{AB|C} = 0, \quad \gamma_{ab:c} = 0\,. \tag{2}$$

For future reference, we define the vector $v_A \equiv r_{|A}/r$.

Through this article we will sometimes use Schwarzschild coordinates (t, r) in the background geometry as intermediate tools for computing expressions which in the end will be valid in any asymptotically flat background coordinates.

Hence, we introduce the following shorthand for coordinate derivatives acting on any object ω:

$$\dot{\omega} \equiv \frac{\partial \omega}{\partial t}, \qquad \omega' \equiv \frac{\partial \omega}{\partial r}. \tag{3}$$

B. Non-linear perturbations

In perturbation theory one works with a family of spacetimes $(\mathcal{M}(\varepsilon), g(\varepsilon))$ which depend on a dimensionless parameter ε. The background spacetime is the member of this family for which $\varepsilon = 0$, and it is assumed to be a known solution of the Einstein equations. Performing a Taylor expansion around the background metric $g_{\mu\nu}$,

$$\tilde{g}_{\mu\nu}(\varepsilon) = g_{\mu\nu} + \sum_{n=1}^{\infty} \frac{\varepsilon^n}{n!}\, {}^{\{n\}}h_{\mu\nu}, \tag{4}$$

defines the perturbations ${}^{\{n\}}h_{\mu\nu}$, which are tensors on the background manifold.

Using the basis of tensor harmonics described in appendix A, we decompose the perturbations of the metric in the following way,

$${}^{\{n\}}h_{\mu\nu} \equiv \sum_{l,m} \begin{pmatrix} {}^{\{n\}}H_l^m{}_{AB}\; Z_l^m & {}^{\{n\}}H_l^m{}_A\; Z_l^m{}_b + {}^{\{n\}}h_l^m{}_A\; X_l^m{}_b \\ {}^{\{n\}}H_l^m{}_A\; Z_l^m{}_b + {}^{\{n\}}h_l^m{}_A\; X_l^m{}_b & {}^{\{n\}}K_l^m\; r^2 \gamma_{ab} Z_l^m + {}^{\{n\}}G_l^m\; r^2 Z_l^m{}_{ab} + {}^{\{n\}}h_l^m\; X_l^m{}_{ab} \end{pmatrix}. \tag{5}$$

We define the components of the above tensors in the Schwarzschild coordinates

$$H_{AB} \equiv \begin{pmatrix} H_{tt} & H_{tr} \\ H_{tr} & H_{rr} \end{pmatrix} \tag{6}$$

$$H_A \equiv (H_t \quad H_r) \tag{7}$$

$$h_A \equiv (h_t \quad h_r) \tag{8}$$

Now we proceed to extract the gauge freedom present in the previous decomposition (5). Per each perturbative order n, four of those ten components are free.

The most natural gauge choice in spherical symmetry is the one introduced by Regge and Wheeler (RW) [1] in linear perturbations. Here we use those same conditions at all orders:

$$ {}^{\{n\}}H^m_l{}_A = 0, \quad {}^{\{n\}}G^m_l = 0, \quad {}^{\{n\}}h^m_l = 0. \tag{9}$$

In reference [12] the explicit form of the gauge invariants tied to this gauge was constructed. Essentially, they are obtained by working out the changes in the perturbative quantities when doing a gauge transformation from a generic gauge to the RW one.

Schematically (see [12] for more details) these gauge invariants are given by

$$\begin{aligned} {}^{\{n\}}\mathcal{K}_{AB} &= {}^{\{n\}}H_{AB} + \left(\frac{r^2}{2}\,{}^{\{n\}}G_{|A} - {}^{\{n\}}H_A\right)_{|B} \\ &+ \left(\frac{r^2}{2}\,{}^{\{n\}}G_{|B} - {}^{\{n\}}H_B\right)_{|A} + {}^{\{n\}}R_{AB}, \quad (10) \\ {}^{\{n\}}\mathcal{K} &= {}^{\{n\}}K + 2v^A\left(\frac{r^2}{2}\,{}^{\{n\}}G_{|A} - {}^{\{n\}}H_A\right) \\ &+ \frac{l(l+1)}{2}\,{}^{\{n\}}G + {}^{\{n\}}R, \quad (11) \\ {}^{\{n\}}\kappa_A &= {}^{\{n\}}h_A - \frac{r^2}{2}\left(\frac{{}^{\{n\}}h}{r^2}\right)_{|A} + {}^{\{n\}}R_A, \quad (12) \end{aligned}$$

where ${}^{\{n\}}R_{AB}$, ${}^{\{n\}}R_A$ and ${}^{\{n\}}R$ depend on the harmonic coefficients

$$\{\,{}^{\{k\}}H^m_l{}_A,\ {}^{\{k\}}G^m_l,\ {}^{\{k\}}h^m_l\},\ \text{with}\ k < n. \tag{13}$$

Because of the $k < n$ condition, these coefficients are not present in linearized perturbations ($n = 1$). For second order perturbations ($n = 2$), they are quadratic in the first-order quantities $\{\,{}^{\{1\}}H^m_l{}_A,\ {}^{\{1\}}G^m_l,\ {}^{\{1\}}h^m_l\}$, and so forth for higher n.

All the coefficients in Eq. (13) vanish in the RW gauge.

Because of their gauge invariance, the values of ${}^{\{n\}}\mathcal{K}_{AB}$, ${}^{\{n\}}\mathcal{K}$, ${}^{\{n\}}\kappa_A$ in any gauge coincide with their values in the RW gauge. Because of this we can do calculations in the RW gauge, yet still recover the form of any expression in a generic gauge by making use of the definitions (10-12). We will take advantage of this fact later.

III. HIGH ORDER REGGE-WHEELER AND ZERILLI EQUATIONS

When studying perturbations of the Schwarzschild spacetime, it is possible to reduce all the perturbed Einstein equations to two wave equations for two scalars, one of odd/axial parity and another one even/polar parity. If these equations are obeyed, all the Einstein equations will also be trivially fulfilled. These scalars are called the master scalars because they contain all the physical information of the system; the perturbed metric can be totally reconstructed from them.

A. The even/polar parity sector

We define the nth order Zerilli scalar function as the following combination of polar harmonic coefficients,

$$ {}^{\{n\}}\Psi \equiv \frac{fr^2}{3M+\lambda}(fr^B\,{}^{\{n\}}\mathcal{K}_{AB} - r\,{}^{\{n\}}\mathcal{K}_{|A})r^A + r\,{}^{\{n\}}\mathcal{K}. \tag{14}$$

Here we have defined $\lambda \equiv \frac{1}{2}(l-1)(l+2)$. Notice that the Zerilli scalar is given in terms of the nth order gauge invariants tied to the Regge-Wheeler gauge. If one wants to recover the form of the function in a different gauge, it is sufficient to replace the invariants by their explicit form in terms of a generic gauge (10-12). Then, the Zerilli function takes the following form

$$\begin{aligned} {}^{\{n\}}\Psi &\equiv \frac{(2M-r)}{3M+\lambda r}\left\{(2M-r)\,{}^{\{n\}}H_{rr} + r^2\,{}^{\{n\}}K'\right\} \\ &+ r\,{}^{\{n\}}K + \frac{l(l+1)}{3M+\lambda r}(2M-r)\,{}^{\{n\}}H_r \\ &+ \frac{1}{2}l(l+1)r\,{}^{\{n\}}G + {}^{\{n\}}Q_\Psi\,. \quad (15) \end{aligned}$$

In the previous expression ${}^{\{n\}}Q$ depends on lower order perturbations since it is a function of $\{\,{}^{\{n\}}R_{AB},\ {}^{\{n\}}R_A,\ {}^{\{n\}}R\}$. Notice that in Eq. (15) the last three terms, including ${}^{\{n\}}Q$, are zero when imposing the RW gauge.

The Zerilli scalar satisfies the following wave equation

$$ {}^{\{n\}}\Psi^{|A}{}_A - V_Z\,{}^{\{n\}}\Psi = {}^{\{n\}}\mathcal{S}_\Psi\,. \tag{16}$$

The source term ${}^{\{n\}}\mathcal{S}_\Psi$ depends on the lower order perturbations, while the potential is defined by

$$ V_Z \equiv \frac{l(l+1)}{r^2} - \frac{6M}{r^3}\frac{r^2\lambda(\lambda+2)+3M(r-M)}{(r\lambda+3M)^2}. \tag{17}$$

In particular, when using the tortoise coordinates (t, r^*) [with $r^* = r + 2M\ln(r/(2M)-1)$] the differential operator takes the following simple form

$$ {}^{\{n\}}\Psi^{|A}{}_A \equiv \left(1-\frac{2M}{r}\right)^{-1}\left(-\frac{\partial^2\,{}^{\{n\}}\Psi}{\partial t^2} + \frac{\partial^2\,{}^{\{n\}}\Psi}{\partial r^{*2}}\right). \tag{18}$$

Obviously, the first-order source ${}^{\{1\}}S_\Psi$ is zero. The second order one can be given as

$$
\begin{aligned}
{}^{\{2\}}\mathcal{S}_{\Psi} \equiv & \sum_{\hat{l},\bar{l}}\sum_{\hat{m},\bar{m}} \frac{l(l+1)}{r}\,{}^{(\epsilon)}S^{\hat{m}\,\bar{m}\,m}_{\hat{l}\;\bar{l}\;l} + \frac{fr^3}{3M+\lambda r} r^A \left[{}^{(\epsilon)}S^{\hat{m}\,\bar{m}\,m\,B}_{\hat{l}\;\bar{l}\;l\;\;B|A} - {}^{(\epsilon)}S^{\hat{m}\,\bar{m}\,m}_{\hat{l}\;\bar{l}\;l\;AB}{}^{|B} - \frac{l(l+1)}{r^2}\,{}^{(\epsilon)}S^{\hat{m}\,\bar{m}\,m}_{\hat{l}\;\bar{l}\;l\;A}\right] \qquad (19)\\
& - \frac{r}{4(3M+\lambda r)^2}\left[\left(84M^2 + 12(l^2+l-5)Mr + 2\lambda(l^2+l-4)r^2\right){}^{(\epsilon)}S^{\hat{m}\,\bar{m}\,m}_{\hat{l}\;\bar{l}\;l\;A}{}^{A} + 4rf^2(12M+2\lambda r)r^A r^B\,{}^{(\epsilon)}S^{\hat{m}\,\bar{m}\,m}_{\hat{l}\;\bar{l}\;l\;AB}\right],
\end{aligned}
$$

where $(\hat{l},\hat{m})$ and $(\bar{l},\bar{m})$ are a pair of two first-order modes which contribute to the second-order (l,m) mode. The sources S that appear in the previous expression are explicitly given in reference [11]. The polarity sign ϵ is defined as $\epsilon \equiv (-1)^{(\hat{l}+\bar{l}-l)}$.

At this point we point us something which will be useful afterwards: the lower order terms ${}^{\{n\}}Q$ that appear in the definition of the Zerilli function (15) are arbitrary. We have complete freedom to add terms to ${}^{\{n\}}Q$ and the Zerilli function would obey the same equation. More explicitly, the left-hand side of Eq. (16) would be the same whereas the source ${}^{\{n\}}S$ would change. Here we have chosen ${}^{\{n\}}Q$ as the simplest function that is zero when applying the RW gauge but mantains the Zerilli function being gauge invariant. This convention was also used in, for example, [17].

B. Odd/axial parity sector

The Gerlach and Sengupta (GS) master scalar is defined as the rotational of the axial invariant vector κ_A/r^2,

$$
{}^{\{n\}}\Pi \equiv \epsilon^{AB}\left(\frac{{}^{\{n\}}\kappa_A}{r^2}\right)_{|B}. \qquad (20)
$$

Like the Zerilli function, it is given in terms of some of the RW gauge-invariants (10-12). In a generic gauge it takes the form

$$
{}^{\{n\}}\Pi = \epsilon^{AB}\left(\frac{{}^{\{n\}}h_A}{r^2}\right)_{|B} + {}^{\{n\}}Q_{\Pi}, \qquad (21)
$$

where ${}^{\{n\}}Q$ is a source term that depends on lower-order perturbations, and which is zero in the RW gauge.

It obeys the GS master equation,

$$
-\left[\frac{1}{2r^2}(r^4\,{}^{\{n\}}\Pi)^{|A}\right]_{|A} + \frac{(l-1)(l+2)}{2}\,{}^{\{n\}}\Pi = {}^{\{n\}}\mathcal{S}_{\Pi}. \qquad (22)
$$

The second-order source can be written in terms of the source of the Einstein equations,

$$
{}^{\{2\}}\mathcal{S}_{\Pi} = i\epsilon^{AB}\sum_{\bar{l},\hat{l}}\sum_{\bar{m},\hat{m}} {}^{(-\epsilon)}S^{\bar{m}\,\hat{m}\,m}_{\bar{l}\;\hat{l}\;l\;A|B}. \qquad (23)
$$

As in the even parity case, the sources S that appear in the previous expression are explicitly given in reference [11]. Equation (22) is a wave equation for the scalar ${}^{\{n\}}\Pi$. It contains all the relevant physical information of the axial sector. As we will see, all the metric components can be algebraically reconstructed from this scalar.

We will use the above definition for ${}^{\{n\}}\Pi$ for historical reasons. But, in fact, a better choice [21] is the rescaled ${}^{\{n\}}\tilde{\Pi} \equiv r^3\,{}^{\{n\}}\Pi$, because its evolution equation has no first-order derivatives,

$$
{}^{\{n\}}\tilde{\Pi}_{|A}{}^{A} - V_{\mathrm{RW}}\,{}^{\{n\}}\tilde{\Pi} = -2r\,{}^{\{n\}}\mathcal{S}_{\Pi}. \qquad (24)
$$

This equation is valid in any spherically symmetric background and the potential is given by

$$
V_{\mathrm{RW}} = \frac{l(l+1)}{r^2} + \frac{3}{r^2}(f^2-1). \qquad (25)
$$

When using Schwarzschild coordinates for the background it takes the following form,

$$
V_{\mathrm{RW}} = \frac{l(l+1)}{r^2} - \frac{6M}{r^3}, \qquad (26)
$$

and Eq. (22) becomes the standard RW equation.

One of the main advantages of the GS master scalar is that the perturbation of the metric can be *algebraically* reconstructed (as discussed later in Section V B, the alternative definition about to be introduced does not satisfy this property). But there are some other variables which obey the same RW equation. In particular, one that will be very useful for our purposes is the one introduced by Regge and Wheeler themselves in their seminal paper [1]. Its gauge-invariant generalization to higher orders takes the following form,

$$
{}^{\{n\}}\Phi \equiv v^A\,{}^{\{n\}}\kappa_A. \qquad (27)
$$

When using Schwarzschild coordinates for the background this definition becomes

$$
\Phi = \frac{2M-r}{2r^2}\left[h' - 2h_r - \frac{2}{r}h\right] + {}^{\{n\}}Q_{\Phi}, \qquad (28)
$$

where ${}^{\{n\}}Q_{\Phi}$ is the usual term that depends on lower order perturbations and vanishes when particularizing to the RW gauge. There are practical advantages and disadvantages for each of these definitions for the master scalar in the odd parity sector, as we will see in Section V B.

At linear order, ${}^{\{1\}}\Phi$ obeys the same equation as $\tilde{\Pi}$. But at second order the source term changes,

$$
{}^{\{2\}}\Phi_{|A}{}^{A} - V_{\mathrm{RW}}\,{}^{\{2\}}\Phi = {}^{\{2\}}\mathcal{S}_{\Phi}, \qquad (29)
$$

with

$$ {}^{\{2\}}\mathcal{S}_\Phi \equiv \sum_{\hat{l},\bar{l}} \sum_{\hat{m},\bar{m}} \frac{2i}{r^3}(3M-r)^{(-\epsilon)}S^{\hat{m}\bar{m}\bar{m}m}_{\hat{l}\ \bar{l}\ l} + iv^A \left[{}^{(-\epsilon)}S^{\hat{m}\bar{m}\bar{m}m}_{\hat{l}\ \bar{l}\ l\ |A} - 2^{(-\epsilon)}S^{\hat{m}\bar{m}\bar{m}m}_{\hat{l}\ \bar{l}\ l\ A} \right]. \quad (30) $$

IV. RADIATED POWER

To compute the power radiated to infinity by gravitational waves we will use the Landau and Lifshitz formula [18],

$$ \frac{d\text{Power}}{d\Omega} = \frac{1}{16\pi r^2} \left\{ \frac{1}{\sin^2\theta} \left| \frac{\partial \tilde{g}_{\theta\phi}}{\partial t} \right|^2 + \frac{1}{4} \left| \frac{\partial \tilde{g}_{\theta\theta}}{\partial t} - \frac{1}{\sin^2\theta} \frac{\partial \tilde{g}_{\phi\phi}}{\partial t} \right|^2 \right\}, \quad (31) $$

This expression is only valid in an asymptotically flat (AF) gauge. By which we mean, following [**?**],

$$ {}^{\{n\}}h_{tt},\ {}^{\{n\}}h_{rr},\ {}^{\{n\}}h_{tr} = O(r^{-2}), \quad (32) $$

$$ {}^{\{n\}}h_{t\theta},\ {}^{\{n\}}h_{t\phi},\ {}^{\{n\}}h_{r\theta},\ {}^{\{n\}}h_{r\phi} = O(r^{-1}), \quad (33) $$

$$ {}^{\{n\}}h_{\theta\theta},\ {}^{\{n\}}h_{\phi\phi},\ {}^{\{n\}}h_{\theta\phi} = O(r), \quad (34) $$

$$ \gamma^{ab}\ {}^{\{n\}}h_{ab} = O(r^0) \quad (35) $$

to the desired order n.

Once the Regge-Wheeler and Zerilli functions are known up to a given order, one can easily reconstruct the perturbed metric components in, say, the Regge-Wheeler gauge (we will turn to this below). Once those are known, we could in principle use Eq. (32) to compute the radiated energy.

However, it turns out that the RW gauge is not asymptotically flat. Because of that we will need to (explicitly or implicitly, this will be discussed later) make a gauge transformation from the RW gauge to an asymptotically flat once. Once in that gauge, we will apply Eq. (32) in order to obtain the radiated power.

The Landau and Lifshitz formula (32) is tied to the coordinate spherical coordinate system (θ, ϕ), but it can be covariantly given as discussed next.

We first introduce the trace-free projector to the sphere by (recall that γ_{ab} is the standard, unit metric on the sphere).

$$ P_{ab}{}^{cd} \equiv \gamma_a{}^c \gamma_b{}^d - \frac{1}{2}\gamma_{ab}\gamma^{cd}. \quad (36) $$

This tensor projects any four-dimensional rank-two tensor $A_{\mu\nu}$ into a trace-free one, $P_{ab}{}^{cd}A_{cd}$, on the sphere. In particular, when it acts on a tensor defined on the sphere, it solely takes out its trace.

With this projector at hand we introduce the projected trace-free metric on the sphere,

$$ i_{ab} \equiv P_{ab}{}^{cd}\tilde{g}_{cd}, \quad (37) $$

which allows us to rewrite formula (31) in the following way:

$$ \frac{d\text{Power}}{d\Omega} = \frac{1}{32\pi r^2} \left(\frac{\partial i_{ab}}{\partial t} \right) \gamma^{ac}\gamma^{bd} \left(\frac{\partial i_{cd}}{\partial t} \right)^*, \quad (38) $$

where the star denotes the complex conjugated.

Using the perturbative expansion in spherical harmonics (5), the projected trace-free metric takes the form

$$ i_{ab} = \sum_{n=1}^{\infty} \frac{\varepsilon^n}{n!} \sum_{l,m} \{ r^2\ {}^{\{n\}}G_l^{m\text{AF}} Z_l^m{}_{ab} + {}^{\{n\}}h_l^{m\text{AF}} X_l^m{}_{ab} \}, \quad (39) $$

where the superscript AF stands for any asymptotically flat gauge.

Finally, making use of the normalization shown in appendix A and the fact that the tensor spherical harmonics are trace-free, it is easy to integrate the emitted power over the solid angle to get

$$ \text{Power} = \frac{1}{64\pi r^2} \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} \frac{\varepsilon^{j+k}}{j!k!} \sum_{l,m} \frac{(l+2)!}{(l-2)!} \left\{ r^4 \left(\frac{\partial\ {}^{\{j\}}G_l^{m\text{AF}}}{\partial t} \right) \left(\frac{\partial\ {}^{\{k\}}G_l^{m\text{AF}}}{\partial t} \right)^* + \left(\frac{\partial\ {}^{\{j\}}h_l^{m\text{AF}}}{\partial t} \right) \left(\frac{\partial\ {}^{\{k\}}h_l^{m\text{AF}}}{\partial t} \right)^* \right\}. \quad (40) $$

Therefore, the problem of extracting the radiated power with an order of ε^n reduces to find the value of the time derivative of the harmonic coefficients ${}^{\{k\}}G_l^m$ and ${}^{\{k\}}h_l^m$, for all $k < n$, in an asymptotically flat gauge.

Note that in the last formula there is no coupling between modes with different harmonic labels, which is typical for high-order perturbation theory. This is so because of the integrated character of the total emitted power; in fact, the orthogonality between different spherical harmonics prevents the coupling from happening. This issue has important consequences when one wants to obtain the radiated power up to a given order ε^n consistently, as we will discuss later.

V. FIRST ORDER PERTURBATIONS

In this section we present, for *first order* perturbations, the master scalar equations as well as the reconstruction of the metric components and the computation of the radiated energy in terms of the master scalars. The reason for re-deriving these calculations here is twofold. First, to fix the conventions that we will use in the second order case, in which the first order perturbations will appear as 'source terms'. Second, to sketch in a less complicated case the type of calculations presented in the next section for second order perturbations.

After writing down the first order master scalar equations we reconstruct the metric in the RW gauge. One can explicitly transform the result from the latter to an arbitrary gauge. In fact, in order to use the Landau and Lifshitz formula (31) we must use an asymptotically flat gauge. As we will see, the RW gauge is not asymptotically flat. Because of this, following [9], we will first perform an explicit asymptotic gauge transformation from the RW gauge to an AF one and afterwards apply the LL formula.

In addition, following [17], we use an alternative way of computing the radiated energy, which exploits the gauge invariant form of the master scalars presented in previous section, instead of making an explicit asymptotic gauge transformation.

In this section we will remove all the harmonic as well as the $n=1$ labels since all the objects will be of first-order and correspond to a generic harmonic pair (l,m).

A. Even parity/polar sector

At first order, the Zerilli equation is a wave equation without sources,

$$\Psi^{|A}{}_{A} - V_{\rm Z}\Psi = 0. \tag{41}$$

This master scalar contains all the physical information of the system since it is possible to reconstruct all the components of the perturbation of the metric, in the RW gauge, in terms of it and its derivatives. We explicitly display the results of such reconstruction,

$$\begin{aligned}
H_{tt} &= \frac{2M-r}{4l(l+1)r^3\left(3M+\lambda r\right)^2}\times \qquad (42)\\
&\quad \Big\{\ 2(2M-r)r^2\left(6M+2\lambda r\right)^2\Psi''\\
&+\ 4r\left[\lambda\left(l^2+l-8\right)r^2M-2\lambda^2r^3-18M^3\right]\Psi'\\
&+\ 4\left[18M^3+18\lambda rM^2+6\lambda^2r^2M+l(l+1)\lambda^2r^3\right]\Psi\Big\},\\
H_{rr} &= \frac{r^2}{(2M-r)^2}H_{tt}, \qquad (43)\\
H_{tr} &= \frac{2\left(3M^2+3\lambda rM-\lambda r^2\right)}{l(l+1)\left[6M^2+\left(l^2+l-5\right)rM-\lambda r^2\right]}\dot\Psi\\
&+\ \frac{2r}{l(l+1)}\dot\Psi', \qquad (44)\\
K &= \frac{1}{2l(l+1)r^2\left(3M+\lambda r\right)}\times \qquad (45)\\
&\quad \Big\{\ 2r\left[-12M^2-2\left(l^2+l-5\right)rM+2\lambda r^2\right]\Psi'\\
&+\ \left[24M^2+12\lambda rM+(l-1)l(l+1)(l+2)r^2\right]\Psi\Big\}.
\end{aligned}$$

Introducing the above relations into the linearized Einstein equations, one can show that all of them are trivially satisfied if the Zerilli equation (41) holds.

Next we explicitly display the divergent nature of these quantities (and, as a consequence, of the first order metric perturbations in the RW gauge). For that purpose we temporarily introduce Schwarzschild coordinates (t,r) and the tortoise one r^*. Because of the explicit form of the Zerilli equation in these coordinates, the Zerilli function Ψ can be expanded in inverse powers of r, where the *first* coefficients only depend on the retarded time $u\equiv t-r^*$,

$$\Psi \equiv \Psi_0(u)+\frac{\Psi_1(u)}{r}+\frac{\Psi_2(u)}{r^2}+O\left(\frac{1}{r^3}\right). \tag{46}$$

The coefficients in this expansion are not independent. Introducing this expansion into the Zerilli equation and solving it for each power of r independently we obtain the following relations,

$$\begin{aligned}
\Psi_0(u) &= \frac{2}{l(l+1)}\ddot F(u), \qquad \Psi_1(u)=\dot F(u),\\
\Psi_2(u) &= \frac{\lambda}{2}F(u)-\frac{3M(\lambda+2)}{2\lambda(\lambda+1)}\dot F(u), \qquad (47)
\end{aligned}$$

where $\dot F(u)=dF(u)/du$. The function $F(u)$ can be understood as the free data at null infinity.

In order to see the divergent behaviour of the harmonic coefficients in the RW gauge at null infinity, we replace the expansion (46) in (42-45) to obtain

$$H_{tt} = \frac{4\ddddot{F}}{l^2(l+1)^2}r + \frac{4\lambda\dddot{F}}{l^2(l+1)^2} + O\left(\frac{1}{r}\right), \tag{48}$$

$$H_{rr} = \frac{4\ddddot{F}}{l^2(l+1)^2}r + \frac{16M\ddddot{F} + 4\lambda\dddot{F}}{l^2(l+1)^2} + O\left(\frac{1}{r}\right), \tag{49}$$

$$H_{tr} = -\frac{4\ddddot{F}}{l^2(l+1)^2}r - \frac{8M\ddddot{F} + 4\lambda\dddot{F}}{l^2(l+1)^2} + O\left(\frac{1}{r}\right), \tag{50}$$

$$K = -\frac{4\dddot{F}}{l^2(l+1)^2}r + O\left(\frac{1}{r^2}\right), \tag{51}$$

where the orders r^0 and r^{-1} vanish for the harmonic coefficient K.

In order to apply the LL formula we make an explicit asymptotic (that is, near null infinity) transformation from the RW gauge to an AF one. We will not show the form of the resulting change of coordinates but instead directly show the asymptotic form of the metric coefficients in the new gauge,

$$H_{tt}^{\mathrm{AF}} = 0 + O\left(\frac{1}{r^3}\right), \tag{52}$$

$$H_{rr}^{\mathrm{AF}} = 0 + O\left(\frac{1}{r^3}\right), \tag{53}$$

$$H_{tr}^{\mathrm{AF}} = 0 + O\left(\frac{1}{r^3}\right), \tag{54}$$

$$H_{t}^{\mathrm{AF}} = \left\{\dot{\Phi}_1 - \frac{1}{4l^2(l+1)^2}\left[4M\ddot{F} + \frac{(l+2)!}{(l-2)!}\dot{F}\right]\right\}\frac{1}{r} + O\left(\frac{1}{r^2}\right), \tag{55}$$

$$H_{r}^{\mathrm{AF}} = -\left\{\dot{\Phi}_1 - \frac{1}{2l^2(l+1)^2}\left[2M\ddot{F} + \lambda(l^2+l-8)\dot{F}\right]\right\}\frac{1}{r} + O\left(\frac{1}{r^2}\right), \tag{56}$$

$$G^{\mathrm{AF}} = \frac{4\ddot{F}}{l^2(l+1)^2}\frac{1}{r} + \frac{4\lambda\dot{F}}{l^2(l+1)^2}\frac{1}{r^2} + 2\Phi_1\frac{1}{r^3} + O\left(\frac{1}{r^4}\right), \tag{57}$$

$$K^{\mathrm{AF}} = 0 + O\left(\frac{1}{r^3}\right), \tag{58}$$

where zeros stand to show that in fact one could ask for faster decay rates than the ones defined in (32-35) and $\Phi_1 = \Phi_1(u)$ is a gauge freedom that it is not fixed by the requirement of asymptotic flatness. From the behaviour of the harmonic coefficient G in an asymptotically flat gauge (57) and the asymptotic expansion of the Zerilli function (46), it is easy to obtain

$$G^{\mathrm{AF}} = \frac{2\Psi}{l(l+1)r} + O\left(\frac{1}{r^2}\right). \tag{59}$$

Alternatively, this last relation can be directly obtained from the gauge invariant definition of the Zerilli variable (14-15). Since that definition is valid for any gauge we can suppose that we are in an AF gauge. Imposing the decay rates (32-35) it is straightforward to obtain (59). The advantage of this last method is that we do not have to do an explicit asymptotic gauge transformation.

Either way, using the relation (59) and the LL formula we obtain the radiated power in terms of the Zerilli function,

$$\mathrm{Power} = \frac{\varepsilon^2}{16\pi}\sum_{l,m}\frac{(l-1)(l+2)}{l(l+1)}\left|\frac{\partial\Psi_l^m}{\partial t}\right|^2. \tag{60}$$

Since this expression holds asymptotically, we can now forget that we have used intermediate Schwarzschild-type coordinates to derive it, since it will hold in any AF coordinate system used for the background.

B. Odd parity/axial sector

We proceed as in the even parity/polar sector. We first reconstruct the metric from the RW scalar Π satisfying the RW equation

$$-\left[\frac{1}{2r^2}(r^4\Pi)^{|A}\right]_{|A} + \frac{(l-1)(l+2)}{2}\Pi = 0 \tag{61}$$

in the RW gauge:

$$h_t = \frac{r^2}{2\lambda}(2M-r)(4\Pi + r\Pi'), \tag{62}$$

$$h_r = \frac{r^5}{2\lambda(2M-r)}\dot{\Pi}. \tag{63}$$

Next we expand the master scalar in inverse powers of r near the asymptotic null infinity ($r\to\infty, u = const.$). Since the wave equation is obeyed by the rescaled funtion $\tilde{\Pi} = r^3\Pi$, our master scalar will have the following behavior,

$$\Pi \equiv \frac{\Pi_0(u)}{r^3} + \frac{\Pi_1(u)}{r^4} + \frac{\Pi_2(u)}{r^5} + O\left(\frac{1}{r^6}\right), \tag{64}$$

We can define a function $J(u)$ such that,

$$\Pi_0(u) = \frac{2}{l(l+1)}\ddot{J}(u), \qquad \Pi_1(u) = \dot{J}(u),$$
$$\Pi_2(u) = \frac{\lambda}{2}J(u) - \frac{3M}{l(l+1)}\dot{J}(u). \tag{65}$$

With these expansions at hand, we can obtain the precise divergent behaviour of the RW gauge in terms of the function $J(u)$,

$$h_t = \frac{r}{\lambda l(l+1)}\dddot{J} + \frac{1}{l(l+1)}\ddot{J} + O\left(\frac{1}{r}\right), \tag{66}$$

$$h_r = -\frac{r}{\lambda l(l+1)}\dddot{J} - \frac{2M}{\lambda l(l+1)}\dddot{J} - \frac{1}{2\lambda}\ddot{J} + O\left(\frac{1}{r}\right) \tag{67}$$

As in the even parity/polar sector, we make an explicit asymptotic gauge transformation to an AF gauge. And, as in that sector, we do not show the details of the resulting transformation but instead the final asymptotic behavior of the metric in the new gauge:

$$h_t^{AF} = \left\{\dot{\Xi}_0 + \frac{1}{4}\dot{J} + \frac{(l-2)!}{(l+2)!}M\ddot{J}\right\}\frac{1}{r} + O\left(\frac{1}{r^2}\right), \quad (68)$$

$$h_r^{AF} = -\left\{\dot{\Xi}_0 + \frac{(l-2)!}{(l+2)!}\left[M\ddot{J} + \frac{\lambda}{2}(l^2+l-8)\dot{J}\right]\right\}\frac{1}{r} + O\left(\frac{1}{r^2}\right), \quad (69)$$

$$h^{AF} = -\frac{2r}{\lambda l(l+1)}\ddot{J} - \frac{2}{l(l+1)}\dot{J} + \frac{2}{r}\Xi_0 + O\left(\frac{1}{r^2}\right) \quad (70)$$

where $\Xi_0 = \Xi_0(u)$ is a residual gauge freedom. From there it is easy to obtain that asymptotically,

$$h^{AF} = -\frac{r^4}{\lambda}\Pi + O(r^0). \quad (71)$$

Replacing this result in the LL formula for the emitted power we obtain

$$\text{Power} = \frac{\varepsilon^2 r^6}{16\pi}\sum_{l,m}\frac{l(l+1)}{(l-1)(l+2)}\left|\frac{\partial \Pi_l^m}{\partial t}\right|^2. \quad (72)$$

One can try to obtain relation (71) with a gauge invariant approach, as we have done in the polar case. It is not possible in this sector, however, since the gauge invariant form of the master variable Π does not contain the harmonic coefficient h. Hence, if using that variable one has to go through the explicit gauge transformation. But at this point we note that there is another master variable Φ whose gauge-invariant form (28) does contain the harmonic coefficient h. Making a transformation to outgoing coordinates and assuming that we are in an AF gauge we can easily obtain between Φ and h at null infinity,

$$\dot{h}^{\rm AF} = 2r\Phi + O(r^0). \quad (73)$$

Therefore, the emitted power can also be given in terms of this last variable,

$$\text{Power} = \frac{\varepsilon^2}{16\pi}\sum_{l,m}\frac{(l+2)!}{(l-2)!}\left|\Phi_l^m\right|^2. \quad (74)$$

Because we can apply this gauge-invariant approach to relate the master variable with the harmonic coefficient h at null infinity, at second-order we will use the variable ${}^{\{2\}}\Phi$. But there is one disadvantage of using Φ instead of Π. We have shown above the reconstruction of the perturbations of the metric in the RW gauge in terms of Π (62). These relations are algebraic. If we try to do the same with the variable Φ, we find that the reconstruction of the metric is not algebraic, but differential.

$$h_r = \frac{r^2}{(r-2M)}\Phi, \quad (75)$$

$$\dot{h}_t = \left(1-\frac{2M}{r}\right)^2\left(h_r' + \frac{2M}{r-2M}\frac{h_r}{r}\right). \quad (76)$$

This is why we will use Π at linear order and Φ at second.

VI. SECOND ORDER PERTURBATIONS

In order to solve for the second order perturbations it is *in principle* enough to solve the Zerilli (16) and RW (29) equations with their corresponding sources [given by (19) and (30), respectively]. However, as we will discuss in the next two subsections, *in practice* there are some technical obstacles to overcome first.

A. Sources: even parity sector

As we have defined it, the second-order Zerilli function ${}^{\{2\}}\Psi$, and in consecuence also the source of the equation it obeys, ${}^{\{2\}}S_\Psi$, diverges at large radii. In order to see this, it is sufficient to take its gauge-invariant definition (14-15), to assume an asymptotically flat gauge, and to impose the conditions (52-58) and (68-70). In this way, we find that the quadratic source ${}^{\{2\}}Q_\Psi$ diverges as,

$${}^{\{2\}}Q_\Psi = Q_2 r^2 + Q_1 r + Q_0 + O\left(\frac{1}{r}\right), \quad (77)$$

where Q_0, Q_1 and Q_2 are quadratic functions of $\{\hat{F}, \bar{F}, \hat{J}, \bar{J}\}$. The hat and bar on F and J functions denote the generic harmonic labels $(\hat{l}, \hat{m})$ and $(\bar{l}, \bar{m})$ respectively. For instance, the dominant term is given by

$$Q_2 = \sum_{\hat{l},\hat{m}}\sum_{\bar{l},\bar{m}}\frac{32}{\lambda \bar{l}^2(\bar{l}+1)^2\hat{l}^2(\hat{l}+1)^2}E^{0\bar{l}\bar{m}}_{0\hat{l}\hat{m}l}\dddot{\hat{F}}\dddot{\bar{F}}, \quad (78)$$

where the E-coefficients are Clebsch-Gordan like coefficients defined in Appendix A. In this case, the E coefficient restricts the sums to those harmonic labels $\{\hat{l}, \bar{l}, l\}$ that $\hat{l}+\bar{l}+l$ is an even number. That is, for the cases that $\hat{l}+\bar{l}+l$ is an odd number the term Q_2 cancels out. In contrast, the term Q_1 has a non-vanishing contribution in all the cases.

These divergences are non-physical. They are related to the freedom present when defining a higher (than first) order Zerilli function, as discussed in Section III, and they do not affect quantities such as the radiated energy at infinity.

However, when numerically solving equations with divergent sources, the latter do become a problem in practice due to large roundoff errors. For that reason we will 'regularize' the source in the Zerilli equation by making use of the complete freedom that we have to add any first-order quadratic terms to the definition of the second-order master scalars. In that way they will obey the same equation but with a different, non diverging source term. The following discussion is rather technical, but necessary. The reader not interested in the details, though, might skip it and refer to Eq. (90) [and (95) for the odd parity sector), keeping in mind that we have made use of the freedom in defining the second order master functions in such a way that their associated sources are non-divergent both at the horizon and at infinity.

Our aim is to obtain some quadratic terms on the first-order Zerilli functions and GS master scalars $Q_{reg} = Q_{reg}[\{\hat{\Psi}, \overline{\Psi}, \hat{\Pi}, \overline{\Pi}\}]$ which reproduce the asymptotic divergent behavior of the source Q_Ψ near null infinity. That is, for $r >> M$ with $u = const$,

$$Q_{reg}[\{\hat{\Psi}, \overline{\Psi}, \hat{\Pi}, \overline{\Pi}\}] = Q_2[\{\hat{F}, \bar{F}, \hat{J}, \bar{J}\}]r^2 + Q_1[\{\hat{F}, \bar{F}, \hat{J}, \bar{J}\}]r + Q_0[\{\hat{F}, \bar{F}, \hat{J}, \bar{J}\}] + O\left(\frac{1}{r}\right). \quad (79)$$

In order to construct the function Q_{reg} we make the following replacements in Q_2, Q_1 and Q_0,

$$\ddot{F} \to \frac{1}{2}l(l+1)\Psi, \qquad \ddot{J} \to \frac{r^3}{2}l(l+1)\Pi. \quad (80)$$

These rules include all the cases but the first and zeroth derivatives. There are no F or J terms without derivatives in the divergent terms, but there are some first-order derivatives. Hence, the straightforward definition would be

$$\dot{F} \to -r^2\left(\frac{\partial\Psi}{\partial r}\right)_u. \quad (81)$$

However, these replacements introduce divergences at the horizon $r = 2M$. In order to see this, we choose ingoing Eddington-Finkelstein coordinates, which are smooth at the horizon. They are obtained from the Schwarzschild coordinates (t, r) by following transformation

$$t \to w \equiv t + 2M \ln\left|\frac{r}{2M} - 1\right|. \quad (82)$$

In these coordinates the two-dimensional background metric takes the form

$$g_{AB}dx^A dx^B = -\left(1 - \frac{2M}{r}\right)dw^2 + \frac{4M}{r}dwdr + \left(1 + \frac{2M}{r}\right)dr^2. \quad (83)$$

Therefore, we have the following relation between coordinate derivatives,

$$\left(\frac{\partial\Psi}{\partial r}\right)_u = \left(\frac{\partial\Psi}{\partial r}\right)_\omega + \frac{r+2M}{r-2M}\left(\frac{\partial\Psi}{\partial\omega}\right)_r, \quad (84)$$

which makes explicit the divergence of the radial derivative in outgoing coordinates at the horizon $r = 2M$. Taking into account Eq.(81) this implies that the source diverges there as well.

In order to regularize the source at large radii without introducing divergences at the horizon we proceed in the following way. First, we make a Taylor expansion in inverse powers of r of the right-hand side of Eq. (84). Next, we define a derivative that approaches $(\partial/\partial r)_u$ for large r, but without being divergent at the horizon. Following this method we get

$$\dot{F} = -r^2\left(\frac{\partial\Psi}{\partial r}\right)_\omega - (r^2+4Mr+8M^2)\left(\frac{\partial\Psi}{\partial\omega}\right)_r + O\left(\frac{1}{r}\right), \quad (85)$$

which is finite at the horizon. Converting this last relation into Schwarzschild coordinates gives the following rules to reconstruct the divergent terms,

$$\dot{F} \to -r^2\Psi' + \frac{r^3 - 16M^3}{2M - r}\dot{\Psi}, \quad (86)$$

$$\dot{J} \to -r^2\left(r^3\Pi\right)' + \frac{r^3 - 16M^3}{2M - r}r^3\dot{\Pi}. \quad (87)$$

The replacements (80) and (86) must be done systematically. That is, first take $Q_2 r^2$ and reconstruct the term that will reproduce it,

$$\sum_{\hat{l},\hat{m}}\sum_{\bar{l},\bar{m}} \frac{8r^2}{\lambda\bar{l}(\bar{l}+1)\hat{l}(\hat{l}+1)} E_{0\,\hat{l}\hat{m}l}^{0\bar{l}\bar{m}}\ddot{\hat{\Psi}}\dot{\overline{\Psi}}. \quad (88)$$

When expanding near null infinity, this term will go as $Q_2 r^2 + R_1 r + R_0 + O(r^{-1})$. In order to remove the divergent terms of order $O(r)$, it is not enough to find a term that will reproduce $Q_1 r$, it must reproduce $(Q_1 - R_1)r$, to compensate the new term we have just introduced. Therefore, we take $(Q_1 - R_1)r$ and make the above replacements (80) and (86) again. And so on, until we achieve the desired quadratic function $Q_{reg}[\{\hat{\Psi}, \overline{\Psi}, \hat{\Pi}, \overline{\Pi}\}]$ which asymptotically behaves as in Eq. (79).

In this way, we define the regularized second-order Zerilli funtion as

$${}^{\{2\}}\Psi_{reg} \equiv {}^{\{2\}}\Psi + Q_{reg}. \quad (89)$$

It obeys the following wave equation,

$${}^{\{2\}}\Psi_{reg}{}_l^m{}_{|A}{}^{|A} - V_Z\,{}^{\{2\}}\Psi_{reg}{}_l^m = {}^{\{2\}}\mathcal{S}_\Psi^{reg}, \quad (90)$$

where the regularized source is given by

$${}^{\{2\}}\mathcal{S}_\Psi^{reg} \equiv {}^{\{2\}}\mathcal{S}_\Psi + Q_{reg|A}{}^{|A} - V_Z Q_{reg}. \quad (91)$$

We have implemented this regularization procedure for generic first and second order modes, but the results are quite lengthy. Just to illustrate the point, though, we explicit show the final result for the regularization factor for the particular case $(\hat{l},\hat{m}) = (\bar{l},\bar{m}) = (l,m) = (2,0)$:

$$\begin{aligned} Q_{reg} &= -\frac{1}{252\,(2M-r)}\sqrt{\frac{5}{\pi}}\times \\ &\Big\{ \; 2(2M-r)\left((9M+r)\,{}^{\{1\}}\dot{\Psi}+6\,{}^{\{1\}}\Psi\right)\,{}^{\{1\}}\dot{\Psi} \\ &+ \left(110M^3-21rM^2+14r^2M+4r^3\right)\,{}^{\{1\}}\dot{\Psi}\,{}^{\{1\}}\ddot{\Psi} \\ &- 2(2M-r)\left(4r^2\,{}^{\{1\}}\Psi'-(15M-6r)\,{}^{\{1\}}\Psi\right)\,{}^{\{1\}}\ddot{\Psi}\Big\} \\ &- \frac{3r^6}{224}\sqrt{\frac{5}{\pi}}\left\{16\,{}^{\{1\}}\dot{\Pi}\,{}^{\{1\}}\Pi+(2r-3M)\,{}^{\{1\}}\dot{\Pi}\,{}^{\{1\}}\dot{\Pi}\right\}. \end{aligned} \tag{92}$$

B. Sources: odd parity sector

In the axial case there is no such a divergence. Following the same steps as above, one finds that near null infinity the quadratic part of the RW function ${}^{\{2\}}\Phi$ tends to

$${}^{\{2\}}Q_\Phi = Q_\Phi^{(0)} + O\left(\frac{1}{r}\right). \tag{93}$$

Therefore, in principle there is no need to regularize the second-order RW function. But, as it will be clear in the next subsection, we are still interested in removing the term of order $O(1)$. We do so by applying the same procedure as in the polar case: namely, we obtain a term Q_Φ^{reg} which reproduces $Q_\Phi^{(0)}$ at null infinity.

After that we define the regularized second-order RW variable as

$${}^{\{2\}}\Phi_{reg} \equiv {}^{\{2\}}\Phi + Q_\Phi^{reg}, \tag{94}$$

and its evolution equation

$${}^{\{2\}}\Phi_{reg}{}_l^m{}_{|A}{}^{|A} - V_{\rm RW}\,{}^{\{2\}}\Phi_{reg}{}_l^m = {}^{\{2\}}\mathcal{S}_\Phi^{reg}, \tag{95}$$

where the regularized source is again given by

$${}^{\{2\}}\mathcal{S}_\Phi^{reg} \equiv {}^{\{2\}}\mathcal{S}_\Phi + Q_\Phi^{reg}{}_{|A}{}^{|A} - V_{\rm RW} Q_\Phi^{reg}. \tag{96}$$

As in the even parity case, we shall not present the details of the general procedure but instead simply explicitly show the final result for the regularization factor for the $(\hat{l},\hat{m}) = (\bar{l},\bar{m}) = (l,m) = (2,0)$ case:

$$Q_\Phi^{reg} = -\frac{r^3}{84}\sqrt{\frac{5}{\pi}}\{3\dot{\Pi}\dot{\Psi}+\ddot{\Pi}\Psi+\ddot{\Psi}\Pi\}. \tag{97}$$

The regularized sources for the equations of motion (90) and (95) are one of the main results of this article. We have calculated them for the presence of any first- and second-order axial or polar modes. We do not include their explicit form here because they are quite lengthy and do not contribute to the discussion. They are available from the authors upon request. As an illustration, for the interested reader, in Appendix B we have written down their explicit form for some particular harmonic cases.

C. Radiated power

Once we have solved for the first and second-order master equations we can obtain the radiated power by using Eq. (40). Expanding it explicitly up to order ε^3, it takes the following form

$$\begin{aligned} \text{Power} &= \frac{\varepsilon^2}{64\pi r^2}\sum_{l,m}\frac{(l+2)!}{(l-2)!}\left\{r^4\left|\frac{\partial\,{}^{\{1\}}G_l^{mAF}}{\partial t}\right|^2+\left|\frac{\partial\,{}^{\{1\}}h_l^{mAF}}{\partial t}\right|^2+\varepsilon Re\left[r^4\frac{\partial\,{}^{\{1\}}G_l^{mAF}}{\partial t}\left(\frac{\partial\,{}^{\{2\}}G_l^{mAF}}{\partial t}\right)^*\right.\right. \\ &+ \left.\left.\frac{\partial\,{}^{\{1\}}h_l^{mAF}}{\partial t}\left(\frac{\partial\,{}^{\{2\}}h_l^{mAF}}{\partial t}\right)^*\right]\right\}+O(\varepsilon^4), \end{aligned} \tag{98}$$

where Re means the real part. Again, the problem of finding the radiated power reduces to calculating the harmonic coefficients G_l^m and h_l^m near null infinity, in an asymptotically flat gauge. More precisely, we want to relate them with the regularized master scalars constructed in the previous two subsections.

In those subsections we regularized the second-order master variables so that the quadratic contributions from first-order modes decay as $O(1/r)$ near null infinity. Hence, we can use their gauge-invariant definitions, (15) and (21), and assume an AF gauge (32-35) up to second order. This leads to the very same relations as at first-order; namely,

$$\begin{aligned} {}^{\{2\}}G_l^{m\rm AF} &= \frac{2\,{}^{\{2\}}\Psi_l^m{}_{reg}}{l(l+1)r}+O\left(\frac{1}{r^2}\right), & (99) \\ {}^{\{2\}}\dot{h}_l^{m\rm AF} &= 2r\,{}^{\{2\}}\Phi_l^m{}_{reg}+O(r^0). & (100) \end{aligned}$$

Replacing these expressions in Eq. (98), the radiated power up to order ε^3 is given in terms of the master scalars by

$$\text{Power} = \frac{\varepsilon^2}{64\pi}\sum_{l,m}\frac{(l+2)!}{(l-2)!}\left\{\frac{4}{l^2(l+1)^2}\left|\frac{\partial\, {}^{\{1\}}\Psi_l^m}{\partial t}\right|^2+\frac{r^6}{\lambda^2}\left|\frac{\partial\, {}^{\{1\}}\Pi_l^m}{\partial t}\right|^2+\varepsilon Re\left[\frac{4}{l^2(l+1)^2}\frac{\partial\, {}^{\{1\}}\Psi_l^m}{\partial t}\left(\frac{\partial\, {}^{\{2\}}\Psi^m_{l\ reg}}{\partial t}\right)^*\right.\right.$$
$$\left.\left. - \frac{2r^3}{\lambda}\frac{\partial\, {}^{\{1\}}\Pi_l^m}{\partial t}\, {}^{\{2\}}\Phi^{m*}_{l\ reg}\right]\right\}+O(\varepsilon^4). \tag{101}$$

This last formula, complemented with the evolution equations for the regularized master scalars constructed above, provides a closed set of formulas that permits to obtain the radiated power up to order ε^3 in the most general case in a fully consistent way.

At this point we want to discuss an aspect of second order perturbations of Schwarzschild black holes which seems not have been discussed before in the literature.

Namely, even when we are solving for the metric up to second-order perturbations, we can only obtain the *complete* radiated power up to third-order in ε. In order to obtain the following order ε^4, one should also consider third-order perturbations.

Let us further elaborate on this point. Consider the simplest possible scenario: a unique first-order mode with harmonic labels (l, m) and polarity σ. The polarity σ will take the value 1 for even-parity/polar modes and -1 for odd-parity/axial modes. Because of reality conditions, if the mode (l, m, σ) is present, so is its conjugated $(l, -m, \sigma)$.

The self-coupling of this mode will generate several second-order modes but not the one with labels $(l, \pm m, \sigma)$. In contrast, at third-order the mode with indices $(l, \pm m, \sigma)$ will indeed be generated. This means that the third-order modes will always contribute to the emitted power at order ε^4, coupled to the first-order mode with the same harmonic labels. Therefore, without considering third-order modes, one can only obtain the radiated power consistently up to order ε^3.

In order for the emitted power (40) to have a contribution of that order (ε^3) the self-coupling of the first-order mode must give a second-order mode with the same labels (l, m, σ). It is easy to see that, when only considering a first-order mode $(l, \pm m, \sigma)$, this will happen if and only if $m = 0$ and if, for $\sigma = 1$ ($\sigma = -1$), l is an even (odd) number.

Summarizing, it is in general not true that third-order modes are negligible in comparison with the second-order ones when considering the physically relevant quantity of the emitted power.

In order to make the above discussion more explicit and analyze which problems can be addressed consistently, let us consider the particular case of a first-order even-parity/polar mode with harmonic labels $l = m = 2$ and $l = m = -2$. These modes will generate the second-order $\{l = 4, m = \pm 4, 0\}$, $\{l = 2, m = 0\}$ and $\{l = 0, m = 0\}$ even-parity/polar modes as well as the $\{l = 3, m = 0\}$ odd-parity/axial mode. Particularizing the power formula (40) in terms of the master scalars to this case, we obtain the following contributions from the considered modes,

$$\text{Power} = \frac{\varepsilon^2}{12\pi}\left|\partial_t\, {}^{\{1\}}\Psi_2^2\right|^2+\frac{9\varepsilon^4}{640\pi}\left\{\left|\partial_t\, {}^{\{2\}}\Psi_4^0\right|^2+2\left|\partial_t\, {}^{\{2\}}\Psi_4^4\right|^2\right\}+\frac{15\varepsilon^4}{8\pi}\left|{}^{\{2\}}\Phi_3^0\right|^2+\frac{\varepsilon^4}{96\pi}\left|\partial_t\, {}^{\{2\}}\Psi_2^0\right|^2, \tag{102}$$

where the second-order master scalars are the regularized ones. Here it can be clearly seen that the order ε^3 is not present. The problem with this last formula is that it is not complete since the third-order $\{l = 2, m = \pm 2\}$ polar mode would contribute to the power at order ε^4.

On the other hand, let us consider the first-order mode $l = 2$ with all its possible harmonic labels $m = 0, \pm 1, \pm 2$. By coupling, they will generate the second-order polar modes $l = 0$, $l = 2$ and $l = 4$ with all their possible m. That is, we will have the second-order $\{l = 0, m = 0\}$, $\{l = 2, m = 0, \pm 1, \pm 2\}$ and $\{l = 4, m = 0, \pm 1, \pm 2, \pm 3, \pm 4\}$ polar modes. This particular case will provide a non-vanishing ε^3-order term to the power,

$$\begin{aligned}\text{Power} &= \frac{\varepsilon^2}{24\pi}\left\{2\left|\partial_t\,{}^{\{1\}}\Psi_2^2\right|^2 + 2\left|\partial_t\,{}^{\{1\}}\Psi_2^1\right|^2 + \left|\partial_t\,{}^{\{1\}}\Psi_2^0\right|^2\right\} \\ &+\frac{\varepsilon^3}{24\pi} Re\left[2\partial_t({}^{\{1\}}\Psi_2^2)\partial_t({}^{\{2\}}\Psi_2^2)^* + 2\partial_t({}^{\{1\}}\Psi_2^1)\partial_t({}^{\{2\}}\Psi_2^1)^* + \partial_t({}^{\{1\}}\Psi_2^0)\partial_t({}^{\{2\}}\Psi_2^0)^*\right] \\ &+O(\varepsilon^4),\end{aligned} \tag{103}$$

where, again, the second-order Zerilli function must be understood as regularized. In this last case the formula is exact up to the displayed order, to which the generated second-order axial modes and third-order polar modes do not contribute.

VII. FINAL REMARKS

In this paper we have introduced a complete gauge invariant formalism to study arbitrary perturbations of a Schwarzshild black hole up to second order. In particular, we regularized the resulting equations, making the suitable for a numerical implementation.

This formalism enables a variety of applications and studies. These range from the non-linear stability of the black hole horizon, to non-linear features in gravitational waves and mode-mode coupling. We will report those studies elsewhere.

All the calculations of this paper have been done with –and have been largely possible at all due to– new, efficient symbolic manipulations tools. The resulting expressions in most cases are extremely long and their explicit expressions are not particularly enlighting. For that reason we have refrained from explicitly presenting most of them. They are however, available upon request.

Acknowledgments

We thank Vitor Cardoso and Jorge Pullin for helpful comments on the manuscript.

This research has been supported in part the Spanish MICINN Project FIS2008-06078-C03-03 and by NSF Grant No. 0801213 to the University of Maryland; JMM has been supported by the French A.N.R. Grant *LISA Science* BLAN07-1_201699 and DB by the FPI program of the Regional Government of Madrid.

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[21] In the sense that the differential operator on the left hand side of Eq. (24) is the same one of the 'standard' Regge-Wheeler and Zerilli equations.

Appendix A: Tensor spherical harmonics

Tensor fields of any rank s on the sphere will be decomposed using a basis of tensor spherical harmonics. Such basis can be constructed from the symmetric trace-free (STF) tensors

$$Z_l^m{}_{a_1\ldots a_s} \equiv (Y_l^m{}_{:a_1\ldots a_s})^{\rm STF}, \tag{A1}$$
$$X_l^m{}_{a_1\ldots a_s} \equiv \epsilon_{(a_1}{}^b Z_l^m{}_{ba_2\ldots a_s)}, \tag{A2}$$

together with the metric γ_{ab} and the antisymmetric tensor ϵ_{ab} [11]. For the particular case $s=0$, those objects must be read as $Z_l^m \equiv Y_l^m$ and $X_l^m \equiv 0$. They are normalized in the following way,

$$\int d\Omega Z_l^m{}_{ab}\gamma^{ac}\gamma^{bd}\left(Z_{l'}^{m'}{}_{cd}\right)^* = \frac{1}{2}\frac{(l+2)!}{(l-2)!}\delta_{ll'}\delta_{mm'} \quad \text{(A3)}$$

$$\int d\Omega X_l^m{}_{ab}\gamma^{ac}\gamma^{bd}\left(X_{l'}^{m'}{}_{cd}\right)^* = \frac{1}{2}\frac{(l+2)!}{(l-2)!}\delta_{ll'}\delta_{mm'} \quad \text{(A4)}$$

$$\int d\Omega X_l^m{}_{ab}\gamma^{ac}\gamma^{bd}\left(Z_{l'}^{m'}{}_{cd}\right)^* = 0. \quad \text{(A5)}$$

Going beyond linear perturbation theory, the nonlinear coupling between two first-order modes results in products between two tensor spherical harmonics (l, m, s) and (l', m', s'). Those products can be decomposed into a linear combination of harmonics $(l'', m+m', s+s')$ with an explicit formula involving coefficients

$$E_{s' l' m' l''}^{s\ l\ m} \equiv \frac{k(l', |s'|)k(l, |s|)}{k(l'', |s+s'|)} C_{l'\ \ l\ \ l''}^{m'\ m\ m'+m} C_{l'\ l\ l''}^{s'\ s\ s'+s}, \quad \text{(A6)}$$

where $C_{l'\ l\ l''}^{m'\ m\ m''}$ are the usual Clebsch-Gordan coefficients and k is a normalization factor defined by,

$$k(l, s) = \sqrt{\frac{(2l+1)(l+s)!}{2^{s+2}\,\pi\,(l-s)!}}. \quad \text{(A7)}$$

These E-coefficients encode the geometric selection rules that determine which pairs of modes do actually couple. See [11] for full details.

Appendix B: Regularized sources, some explicit examples

In this appendix we show two particular examples for the regularized sources of the RW (95) and Zerilli equations (90).

Let us first assume that we have the first-order $\{l = 2, m = 1\}$ even-parity/polar and $\{l = 8, m = -4\}$ odd-parity/axial modes. The regularized source generated by them for the equation of motion of the, for example, second-order $\{l = 7, m = 3\}$ even-parity/polar mode is given by

$$\begin{aligned}
S_{\Psi}^{reg} &= -\frac{i\sqrt{\frac{22}{51\pi}}r}{945(U+9)^2(2U-1)(3U+2)^2}\Big\{-60\Pi_{,tr}\left(6U^3+55U^2+7U-18\right)^2\Psi_{,rr}r^3\\
&+ 60\Pi_{,rr}\left(6U^3+55U^2+7U-18\right)^2\Psi_{,tr}r^3\\
&- 20\Pi_{,tr}(U+9)^2\left(18U^4+51U^3-58U^2-26U+20\right)\Psi_{,r}r^2\\
&- 5\Pi_{,t}(3U+2)^2\left(212U^4+3364U^3+11603U^2-13149U+3240\right)\Psi_{,rr}r^2\\
&+ 20\Pi_{,rr}(U+9)^2\left(126U^4+141U^3-82U^2-50U+20\right)\Psi_{,t}r^2\\
&+ 5\Pi_{,r}(3U+2)^2\left(308U^4+4972U^3+17255U^2-22221U+6156\right)\Psi_{,tr}r^2\\
&- 180\Pi_{,tr}(U+9)^2\left(6U^4+9U^3+2U^2+4U-4\right)\Psi r\\
&+ \Pi_{,t}\left(360U^6+2568U^5+57529U^4-14036U^3+375894U^2+88254U\right.\\
&- \left.123444\right)\Psi_{,r}r+\Pi_{,r}\left(14220U^6+253302U^5+1234181U^4+854111U^3\right.\\
&- \left.966354U^2-375354U+220644\right)\Psi_{,t}r+15\Pi(3U+2)^2\left(92U^4+1492U^3\right.\\
&+ \left.8507U^2+14154U-9396\right)\Psi_{,tr}r-\Pi_{,t}\left(6840U^6+112128U^5+422069U^4\right.\\
&+ \left.1424U^3-213006U^2+271944U-48924\right)\Psi\\
&+ \Pi\left(6210U^6+116313U^5+1283789U^4+6929894U^3+6649074U^2\right.\\
&- \left.316926U-1359504\right)\Psi_{,t}\Big\},
\end{aligned} \quad \text{(B1)}$$

where the symbol U stands for the dimensionless mass $U \equiv M/r$.

As a second example, we show the regularized source for the RW equation (95) for the particular case in which the first-order even-parity/polar modes $(\bar{l} = 3, \bar{m} = 0)$ and $(\hat{l} = 4, \hat{m} = -1)$ generate a second-order odd-parity/axial mode with labels $(l = 4, m = -1)$:

$$
\begin{aligned}
S_\Phi^{reg} &= \frac{3i}{8800\sqrt{7\pi}(U+3)^4(2U-1)^2(3U+5)^4}\Big\{-10r\left(3U^2+14U+15\right)^4\hat{\Psi}_{,rrr}\bar{\Psi}_{,rr}(2U-1)^5\\
&+ 26r\left(3U^2+14U+15\right)^4\hat{\Psi}_{,rr}\bar{\Psi}_{,rrr}(2U-1)^5\\
&- \frac{(3U+5)^2}{r^2}\left(3060U^8+49401U^7+332356U^6+1197973U^5+2636572U^4\right.\\
&+ \left.3760905U^3+2764530U^2-467775U-1518750\right)\hat{\Psi}_{,rr}\bar{\Psi}(2U-1)^3\\
&+ \frac{(U+3)^2}{r^2}\left(1620U^8+28161U^7+173844U^6+197637U^5-1511900U^4\right.\\
&- \left.5534775U^3-7023550U^2-3510375U-573750\right)\hat{\Psi}\bar{\Psi}_{,rr}(2U-1)^3\\
&+ 10r\left(3U^2+14U+15\right)^4\hat{\Psi}_{,trr}\bar{\Psi}_{,tr}(2U-1)^3-26r\left(3U^2+14U+15\right)^4\hat{\Psi}_{,tr}\bar{\Psi}_{,trr}(2U-1)^3\\
&- \frac{16(1-2U)^2}{r^4}\left(1701U^{11}+35262U^{10}+320166U^9+1720086U^8+6285736U^7\right.\\
&+ 16821825U^6+34748135U^5+56990175U^4+68601150U^3+42931125U^2\\
&- \left.5703750U-15946875\right)\hat{\Psi}\bar{\Psi}\\
&- \frac{2}{r^3}\left(6U^2+7U-5\right)^2\left(1530U^9+34221U^8+303099U^7+1485635U^6+4592169U^5\right.\\
&+ \left.9179205U^4+10353033U^3+3316365U^2-2994975U-1478250\right)\hat{\Psi}_{,r}\bar{\Psi}\\
&- \frac{10}{r}(U+3)^2\left(6U^2+7U-5\right)^4\left(U^3+9U^2+27U+90\right)\hat{\Psi}_{,rrr}\bar{\Psi}\\
&+ \frac{2}{r^3}\left(2U^2+5U-3\right)^2\left(24138U^9+399357U^8+2535795U^7+8866263U^6\right.\\
&+ \left.20189321U^5+31979265U^4+34936825U^3+20024625U^2-4674375U-9618750\right)\hat{\Psi}\bar{\Psi}_{,r}\\
&+ \frac{8}{r^2}\left(6U^3+25U^2+16U-15\right)^2\left(90U^7+468U^6+1763U^5+5632U^4+22704U^3\right.\\
&- \left.6480U^2-35025U+15750\right)\hat{\Psi}_{,r}\bar{\Psi}_{,r}\\
&+ \frac{2}{r}(U+3)^2\left(6U^2+7U-5\right)^3\left(675U^5+5741U^4+15946U^3+24570U^2+10590U-13950\right)\hat{\Psi}_{,rr}\bar{\Psi}_{,r}\\
&- 5(U+3)^3(11U-9)\left(6U^2+7U-5\right)^4\hat{\Psi}_{,rrr}\bar{\Psi}_{,r}\\
&- \frac{2}{r}(3U+5)^2\left(2U^2+5U-3\right)^3\left(1377U^5+11403U^4+25994U^3+19250U^2-410U-10350\right)\hat{\Psi}_{,r}\bar{\Psi}_{,rr}\\
&- 8(1-2U)^4\left(3U^2+14U+15\right)^3\left(30U^3+91U^2+99U+15\right)\hat{\Psi}_{,rr}\bar{\Psi}_{,rr}\\
&+ \frac{78}{r}(3U+5)^2\left(2U^2+5U-3\right)^4\left(3U^3+15U^2+25U+50\right)\hat{\Psi}\bar{\Psi}_{,rrr}\\
&+ 13(3U+5)^3(17U-15)\left(2U^2+5U-3\right)^4\hat{\Psi}_{,r}\bar{\Psi}_{,rrr}\\
&- \frac{8}{r^2}\left(3U^2+14U+15\right)^2\left(198U^7+1164U^6+2627U^5+17137U^4+45972U^3\right.\\
&+ \left.3140U^2-38100U+11475\right)\hat{\Psi}_{,t}\bar{\Psi}_{,t}\\
&- \frac{2}{r}(U+3)^2(3U+5)^3\left(474U^6+4575U^5+13106U^4+20156U^3+1014U^2-23685U+8100\right)\hat{\Psi}_{,tr}\bar{\Psi}_{,t}\\
&+ 5(1-2U)^2(U+3)^3(3U+5)^4\left(4U^2+23U-9\right)\hat{\Psi}_{,trr}\bar{\Psi}_{,t}\\
&+ \frac{2}{r}(U+3)^3(3U+5)^2\left(2142U^6+13005U^5+21826U^4+13968U^3-4370U^2-21115U+8100\right)\hat{\Psi}_{,t}\bar{\Psi}_{,tr}\\
&+ 24\left(U^2-3U-5\right)\left(6U^3+25U^2+16U-15\right)^3\hat{\Psi}_{,tr}\bar{\Psi}_{,tr}\\
&- 13(1-2U)^2(U+3)^4(3U+5)^3\left(12U^2+37U-15\right)\hat{\Psi}_{,t}\bar{\Psi}_{,trr}\Big\}. \qquad \text{(B2)}
\end{aligned}
$$