GLOBAL EXISTENCE AND REGULARITY OF THE WEAKLY COMPRESSIBLE NAVIER-STOKES SYSTEM

NING JIANG AND C. DAVID LEVERMORE

Abstract. We construct the weakly nonlinear-dissipative approximate system for the general compressible Navier-Stokes system in a periodic domain. It was shown in [11] that because the Navier-Stokes system has an entropy structure, its approximate system will have Leray-like global weak solutions. These solutions decompose into an incompressible part governed by an incompressible Navier-Stokes system, and an acoustic part governed by a nonlocal quadratic equation which couples it to the incompressible part. We obtain regularity results for the acoustic part of the solution via a Littlewood-Paley decomposition that extend to this general setting results found by Masmoudi [18] and Danchin [6] in the γ-law barotropic setting.

1. Introduction

We study the weakly nonlinear-dissipative approximation of the general compressible Navier-Stokes system over a periodic domain. This approximation governs the long-time behavior of small perturbations about a constant state when the dissipation is weak. The theory of such approximations was developed in [11] for general hyperbolic-parabolic systems with an entropy structure. There it was shown that the entropy structure of the original system endows the approximate system with a natural Hilbert space structure within which a Leray-like existence theory of global weak solutions can be established. Here this approximate system is constructed for the general compressible Navier-Stokes system, in which case we call it the *weakly compressible Navier-Stokes system*. Its solutions decompose into an incompressible part that is governed by an incompressible Navier-Stokes system, and an acoustic part that is governed by nonlocal quadratic equations which couples it to the incompressible part. The nonlocal nature of these equations arises because they are derived by time-averaging over the fast acoustic dynamics. We obtain regularity results for the acoustic part of the solution via a Littlewood-Paley decomposition, which extend to this general setting results found by Masmoudi [18] and Danchin [6] in the γ-law barotropic setting. This regularity allows us to show that the uniqueness question for weak solutions reduces to the question of uniqueness for their incompressible parts.

Our starting point is the general compressible Navier-Stokes system for a gas in D-dimensional periodic spatial domain $\mathbb{T}^D$. The system is

$$\begin{aligned} \partial_t\rho + \nabla_x\cdot(\rho u) &= 0\,, \\ \partial_t(\rho u) + \nabla_x\cdot(\rho u\otimes u + pI) &= \nabla_x\cdot S\,, \\ \partial_t\left(\tfrac12\rho|u|^2 + \rho\varepsilon\right) + \nabla_x\cdot\left(\tfrac12\rho|u|^2 u + \rho\varepsilon u + pu\right) &= \nabla_x\cdot(S\cdot u - q)\,. \end{aligned} \tag{1.1}$$

These equations express the local conservation of mass, momentum, and energy respectively. Here I denotes the $D\times D$ identity matrix. We shall take as the basic dependent variables the mass density $\rho(t,x)\geq 0$, the bulk velocity $u(t,x)\in\mathbb{R}^D$, and the specific internal energy

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$\varepsilon(t,x) \geq 0$. The pressure $p = p(\rho,\varepsilon)$ is given by a thermodynamic equation of state. The deviatoric stress tensor S and heat flux q are given by the constitutive relations

$$S = \mu\big[\nabla_x u + (\nabla_x u)^T - \tfrac{2}{D}(\nabla_x \cdot u)I\big] + \lambda(\nabla_x \cdot u)I\,, \qquad q = -\kappa\nabla_x\theta\,, \tag{1.2}$$

where the temperature $\theta = \theta(\rho,\varepsilon) > 0$ is given by a thermodynamic equation of state, while the shear viscosity coefficient $\mu = \mu(\rho,\varepsilon) > 0$, the bulk viscosity coefficient $\lambda = \lambda(\rho,\varepsilon) \geq 0$, and the thermal conductivity coefficient $\kappa = \kappa(\rho,\varepsilon) > 0$ come either from some non-equilibrium (kinetic) theory or from fits to experimental data.

Weakly nonlinear-dissipative approximations of the compressible Navier-Stokes system (1.1) govern regimes in which the gas is near a global equilibrium and the coefficients of viscosity and thermal conductivity are small. Because we have assumed that $\mu > 0$ and $\kappa > 0$, the only global equilibria of (1.1) over a periodic domain are the constant states [11]. By a Galilean transformation, any global equilibrium can then be put into the form $(\rho, u, \varepsilon) = (\rho^o, 0, \varepsilon^o)$ where ρ^o and ε^o are positive constants.

We apply the general theory developed in [11] to derive the averaged equation. For the compressible Navier-Stokes system, the averaged system has fruitful structural features that makes it amenable to analytic study. More specifically, let $\mathcal{A}$ be the acoustic operator, which is the linearization of the Navier-Stokes system (1.1) neglecting dissipation about the constant state $(\rho^o, 0, \varepsilon^o)$,

$$\mathcal{A}\begin{pmatrix}\widetilde{\rho}\\ \widetilde{u}\\ \widetilde{\varepsilon}\end{pmatrix} = \begin{pmatrix}\rho^o\,\nabla_x\cdot\widetilde{u}\\ \dfrac{p^o_\rho}{\rho^o}\,\nabla_x\widetilde{\rho} + \dfrac{p^o_\varepsilon}{\rho^o}\,\nabla_x\widetilde{\varepsilon}\\ \dfrac{p^o}{\rho^o}\,\nabla_x\cdot\widetilde{u}\end{pmatrix}, \tag{1.3}$$

where the coefficients are evaluated at the constant state.

The acoustic operator $\mathcal{A}$ has nontrivial null space $\mathrm{Null}(\mathcal{A})$ which contains the incompressibility and Boussinesq relations

$$\nabla_x\cdot\widetilde{u} = 0\,, \qquad p^o_\rho\widetilde{\rho} + p^o_\varepsilon\widetilde{\varepsilon} = 0\,. \tag{1.4}$$

We call it the *incompressible mode*, while we call its orthogonal complement space $\mathrm{Null}(\mathcal{A})^\perp$ the *acoustic* mode.

The projection of the avaeraged system on $\mathrm{Null}(\mathcal{A})$ is *the incompressible Navier-Stokes equations* with Boussinesq relation, while on $\mathrm{Null}(\mathcal{A})^\perp$ is a *nonlocal quadratic system* which is coupled with the projection on incompressible mode and describes how the fast waves propagate. This is the reason we call the weakly nonlinear-dissipative approximation of the compressible Navier-Stokes system the *weakly compressible Navier-Stokes system.*

In the present paper, there are two key novelties. First, we study the *fully general* gas dynamics. This means we consider the compressible Navier-Stokes system not only includes the energy equation, but also without any unphysical restrictions on the pressure law and entropy. In their studies of incompressible limits of the compressible Navier-Stokes equations, Masmoudi [18] and Danchin [6] also derived the averaged system which describes the propagation of the fast oscillation of the acoustic waves, inspired by the work of Schochet [20]. But they only worked on the γ-law barotropic gases, i.e. $p = a\rho^\gamma$ in (1.1), so that no energy equation is considered. Thus we generalize the results of Masmoudi and Danchin on the averaged system from barotropic gases to general gases.

Second, working in the general setting clarifies the central role played by the entropy, which is used not only to define a natural Hilbert space in which we prove the global existence and regularity of the averaged system, but also to illustrate the structure of the incompressible and acoustic modes. In previous works on the low Mach number limits of γ-law barotropic gas dynamics, or in the work of Hoff-Zumbrun on the diffusion waves for the isentropic compressible Navier-Stokes equations [9, 10] in which they also derived the averaged equations, the entropic structure of the compressible Navier-Stokes was not illustrated.

The outline of this paper is the following: in Section 2, we derive the formal averaged system of the compressible Navier-Stokes-Fourier system, and describe the projections on the incompressible mode $\mathrm{Null}(\mathcal{A})$ and acoustic mode $\mathrm{Null}(\mathcal{A})^\perp$. In Section 4, we use a Littlewood-Paley decomposition to show that in the time interval of the existence of the regular solution to the incompressible Navier-Stokes equations, the averaged system in the acoustic mode has higher regularity.

2. Preliminaries

We begin this section with a review of the framework that was developed in [11] for weakly nonlinear-dissipative approximations of hyperbolic-parabolic systems with a strictly convex entropy. We then characterize when the compressible Navier-Stokes system (1.1) fits into this framework.

2.1. **Weakly Nonlinear-Dissipative Approximations.** In [11] we studied weakly nonlinear-dissipative approximations for a class of hyperbolic-parabolic systems with entropy. These systems have the form

$$\partial_t U + \nabla_x \cdot F(U) = \nabla_x \cdot [D(U) \cdot \nabla_x U] \,, \tag{2.1}$$

where U is a vector of densities, $F(U)$ is a twice continuously differentiable flux, and $D(U)$ is a continuously differentiable diffusion tensor. Moreover, they are assumed to possess a strictly convex, thrice continuously differentiable, real-valued entropy density $H(U)$ such that classical solutions of (2.1) also satisfy

$$\partial_t H(U) + \nabla_x \cdot J(U) = \nabla_x \cdot [H_U(U) D(U) \cdot \nabla_x U] - \nabla_x U \cdot H_{UU}(U) D(U) \cdot \nabla_x U \,, \tag{2.2}$$

where the entropy flux $J(U)$ satisfies $J_U(U) = H_U(U) F_U(U)$ while the tensor $H_{UU}(U)D(U)$ is symmetric and nonnegative definite. Here $H_U(U)$, $J_U(U)$, and $F_U(U)$ denote the derivatives of $H(U)$, $J(U)$, and $F(U)$ with respect to U while $H_{UU}(U)$ denotes the Hessian of $H(U)$ with respect to U.

Weakly nonlinear-dissipative approximations of (2.1) govern regimes in which U is close to a constant state U^o and the dissipation is small. If we express the densities U in terms of any choice of dependent variables V as $U = U(V)$, so that $U^o = U(V^o)$, we then define the matrix R^o by $R^o = U_V(V^o)$. The weakly nonlinear-dissipative approximation of (2.1) governs the perturbation $\widetilde{V}$ of V about V^o by the system

$$\partial_t \widetilde{V} + \mathcal{A}\widetilde{V} + \overline{\mathcal{Q}}\big(\widetilde{V}, \widetilde{V}\big) = \overline{\mathcal{D}}\widetilde{V} \,, \tag{2.3}$$

where the operator $\mathcal{A}$ is formally defined by

$$\mathcal{A}\widetilde{V} = (R^o)^{-1} F_U(U^o) R^o \cdot \nabla_x \widetilde{V} \,, \tag{2.4}$$

while the averaged operators $\overline{\mathcal{Q}}$ and $\overline{\mathcal{D}}$ are formally defined by

$$
\begin{aligned}
\overline{\mathcal{Q}}\big(\widetilde{V},\widetilde{V}\big) &= \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} e^{s\mathcal{A}}\mathcal{Q}\big(e^{-s\mathcal{A}}\widetilde{V},e^{-s\mathcal{A}}\widetilde{V}\big)\,\mathrm{d}s\,,\\
\overline{\mathcal{D}}\widetilde{V} &= \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} e^{s\mathcal{A}}\mathcal{D}e^{-s\mathcal{A}}\widetilde{V}\,\mathrm{d}s\,,
\end{aligned}
\tag{2.5}
$$

with the operators $\mathcal{Q}$ and $\mathcal{D}$ given by

$$
\begin{aligned}
\mathcal{Q}\big(\widetilde{V},\widetilde{V}\big) &= \nabla_x\cdot\Big[\tfrac{1}{2}(R^o)^{-1}F_{UU}(U^o)\big(R^o\widetilde{V},R^o\widetilde{V}\big)\Big]\,,\\
\mathcal{D}\widetilde{V} &= \nabla_x\cdot\Big[(R^o)^{-1}D(U^o)R^o\cdot\nabla_x\widetilde{V}\Big]\,.
\end{aligned}
\tag{2.6}
$$

The first two terms in (2.3) are the linearization of (2.1) with respect to V negelecting the dissipation. The operator $\mathcal{A}$ thereby governs the fast dynamics. Nonlinearity and dissipation will modify the dynamics on longer time scales because the perturbation $\widetilde{V}$ is assumed to be small while the dissipation is assumed to be weak. The operators $\overline{\mathcal{Q}}$ and $\overline{\mathcal{D}}$ are averages of $\mathcal{Q}$ and $\mathcal{D}$ over the fast dynamics that attempt to capture these modifcations.

It was shown in [11] that the entropy structure (2.2) implies that the operator $\mathcal{A}$ is skew-adjoint in the Hilbert space

$$
\mathbb{H} = \left\{\widetilde{V}\in L^2\big(\mathrm{d}x;\mathbb{R}^{D+2}\big)\,:\,\int_{\mathbb{T}^D}\widetilde{V}\,\mathrm{d}x=0\right\}.
\tag{2.7}
$$

equipped with the natural inner product

$$
\Big(\widetilde{V}_1,\widetilde{V}_2\Big)_{\mathbb{H}} = \int_{\mathbb{T}^D}\big(R^o\widetilde{V}_1\big)^T H_{UU}(U^o)R^o\widetilde{V}_2\,\mathrm{d}x\,,
\tag{2.8}
$$

The entropy structure implies moreover that $\overline{\mathcal{Q}}$ formally satisfies the cyclic identity

$$
\Big(\widetilde{V}_1,\overline{\mathcal{Q}}\big(\widetilde{V}_2,\widetilde{V}_3\big)\Big)_{\mathbb{H}} + \Big(\widetilde{V}_2,\overline{\mathcal{Q}}\big(\widetilde{V}_3,\widetilde{V}_1\big)\Big)_{\mathbb{H}} + \Big(\widetilde{V}_3,\overline{\mathcal{Q}}\big(\widetilde{V}_1,\widetilde{V}_2\big)\Big)_{\mathbb{H}} = 0\,,
\tag{2.9}
$$

and that $\overline{\mathcal{D}}$ formally self-adjoint and satisfies the dissipation relation

$$
\Big(\widetilde{V},\overline{\mathcal{D}}\widetilde{V}\Big)_{\mathbb{H}} \leq 0\,.
\tag{2.10}
$$

It was also shown in [11] that over the periodic spatial domain $\mathbb{T}^D$ these formal relations could be made rigorous, and that whenever $\overline{\mathcal{D}}$ is strictly dissipative, the weakly nonlinear-dissipative approximation (2.3) has a Leray-like theory of global weak solutions for initial data in $\mathbb{H}$.

Finally, as was noted in [11], had we made a different choice of dependent variables W, with the densities U expressed as $U = U'(W)$, so that $U^o = U'(W^o)$, then the perturbation $\widetilde{W}$ of W about W^o is governed by the system

$$
\partial_t\widetilde{W} + \mathcal{A}'\widetilde{W} + \overline{\mathcal{Q}}'\big(\widetilde{W},\widetilde{W}\big) = \overline{\mathcal{D}}'\widetilde{W}\,,
$$

where the operators $\mathcal{A}'$, $\overline{\mathcal{Q}}'$, and $\overline{\mathcal{D}}'$, are related to the operators $\mathcal{A}$, $\overline{\mathcal{Q}}$, and $\overline{\mathcal{D}}$ appearing in (2.3) through the tranformation matrix $T = \partial_W\big(U^{-1}(U'(W))\big)\big|_{W=W^o}$ as

$$
\mathcal{A}' = T^{-1}\mathcal{A}T\,,\qquad \overline{\mathcal{Q}}'\big(\widetilde{W},\widetilde{W}\big) = T^{-1}\overline{\mathcal{Q}}\big(T\widetilde{W},T\widetilde{W}\big)\,,\qquad \overline{\mathcal{D}}' = T^{-1}\overline{\mathcal{D}}T\,.
\tag{2.11}
$$

2.2. **Entropy Structure of Gas Dynamics.** In this section we show that the compressible Navier-Stokes system (1.1) is a hyperbolic-parabolic system of the form (2.1) with a strictly convex entropy that satisfies (2.2). Indeed, it is clear that (1.1) has the form (2.1) with

$$U = \begin{pmatrix} \rho \\ \rho u \\ \frac{1}{2}\rho|u|^2 + \rho\varepsilon \end{pmatrix}, \qquad F(U) = \begin{pmatrix} \rho u^T \\ \rho u u^T + pI \\ \frac{1}{2}\rho|u|^2 u^T + \rho\varepsilon u^T + p u^T \end{pmatrix}, \tag{2.12}$$

and some four-tensor $D(U)$ which we will not write down here. We therefore only have to verify that the compressible Navier-Stokes system has a strictly convex entropy structure.

The equations-of-state for pressure and temperature, $p = p(\rho, \varepsilon)$ and $\theta = \theta(\rho, \varepsilon)$, are assumed to be given by twice continuously differentiable functions over $(\rho, \varepsilon) \in \mathbb{R}^2_+$ that satisfy the Maxwell relation [4]

$$\theta\, p_\varepsilon = p\, \theta_\varepsilon + \rho^2 \theta_\rho\,, \qquad \text{for every } \rho > 0 \text{ and } \varepsilon > 0\,. \tag{2.13}$$

In addition, these functions are assumed to satisfy the inequalities

$$p_\rho + \frac{p}{\rho^2}\, p_\epsilon > 0\,, \qquad p_\rho\, \theta_\varepsilon - \theta_\rho\, p_\varepsilon > 0\,, \qquad \text{for every } \rho > 0 \text{ and } \varepsilon > 0\,. \tag{2.14}$$

These inequalites insure the existence of a strictly convex entropy structure for the compressible Navier-Stokes system (1.1). They thereby play a leading role in our analysis.

The Maxwell relation (2.13) implies there exists a function $\sigma(\rho, \varepsilon)$ over $\mathbb{R}^2_+$ such that

$$\sigma_\rho = \frac{p}{\theta\rho^2}\,, \qquad \sigma_\varepsilon = -\frac{1}{\theta}\,. \tag{2.15}$$

This is equivalent to saying σ satisfies the differential relation

$$\mathrm{d}\sigma = \frac{p}{\theta\rho^2}\,\mathrm{d}\rho - \frac{1}{\theta}\,\mathrm{d}\varepsilon\,. \tag{2.16}$$

We can thereby identify $-\sigma$ with the physical specific entropy [4]. The equations of state can then be expressed in terms of σ as

$$p = \rho^2 \frac{\sigma_\rho(\rho, \varepsilon)}{\sigma_\varepsilon(\rho, \varepsilon)}\,, \qquad \theta = -\frac{1}{\sigma_\varepsilon(\rho, \varepsilon)}\,. \tag{2.17}$$

For example, for a polytropic γ-law gas one has

$$\sigma = \log\left(\frac{\rho}{\varepsilon^{\frac{1}{\gamma-1}}}\right), \qquad p = (\gamma - 1)\rho\varepsilon\,, \qquad \theta = (\gamma - 1)\varepsilon\,, \tag{2.18}$$

where the constant $\gamma > 1$ is the adiabatic exponent. Equations (2.15) uniquely determine $\sigma(\rho, \varepsilon)$ up to an additive constant that is usually normalized to a reference state, but whose value does not effect the equations of state (2.17).

Now define $H(U) = \rho\, \sigma(\rho, \varepsilon)$ where U is given by (2.12). It was shown in [8] that $H(U)$ is a strictly convex function of U if and only if the inequalities (2.14) hold. We will sketch the argument here because some of its ingredients will be used later. We first introduce the vector variable V and matrix function $R(V)$ by

$$V = \begin{pmatrix} \rho \\ u \\ \varepsilon \end{pmatrix}, \qquad R(V) = U_V(V) = \begin{pmatrix} 1 & 0 & 0 \\ u & \rho I & 0 \\ \frac{1}{2}|u|^2 + \varepsilon & \rho u^T & \rho \end{pmatrix}. \tag{2.19}$$

A direct calculation [8] then shows that

$$\frac{1}{\rho}\,R(V)^T H_{UU}(U)\,R(V) = \begin{pmatrix} \sigma_{\rho\rho} + \frac{2}{\rho}\,\sigma_\rho & 0 & \sigma_{\rho\varepsilon} \\ 0 & -\sigma_\varepsilon I & 0 \\ \sigma_{\rho\varepsilon} & 0 & \sigma_{\varepsilon\varepsilon} \end{pmatrix}. \tag{2.20}$$

Because $-\sigma_\varepsilon = 1/\theta > 0$, it is clear that $H_{UU}(U)$ is positive definite if and only if the matrix

$$\begin{pmatrix} \sigma_{\rho\rho} + \frac{2}{\rho}\,\sigma_\rho & \sigma_{\rho\varepsilon} \\ \sigma_{\rho\varepsilon} & \sigma_{\varepsilon\varepsilon} \end{pmatrix} = \frac{1}{\rho^2\theta^2}\begin{pmatrix} \theta\,p_\rho - p\,\theta_\rho & \rho^2\theta_\rho \\ \rho^2\theta_\rho & \rho^2\theta_\varepsilon \end{pmatrix} \quad \text{is positive definite}\,.$$

This will be the case if and only if

$$\begin{pmatrix} \rho & \frac{p}{\rho} \end{pmatrix} \begin{pmatrix} \dfrac{\theta\,p_\rho - p\,\theta_\rho}{\rho^2} & \theta_\rho \\ \theta_\rho & \theta_\varepsilon \end{pmatrix} \begin{pmatrix} \rho \\ \dfrac{p}{\rho} \end{pmatrix} = \theta\left(p_\rho + \frac{p}{\rho^2}\,p_\varepsilon\right) > 0\,, \tag{2.21}$$

and

$$\det\begin{pmatrix} \dfrac{\theta\,p_\rho - p\,\theta_\rho}{\rho^2} & \theta_\rho \\ \theta_\rho & \theta_\varepsilon \end{pmatrix} = \frac{\theta}{\rho^2}\left(p_\rho\,\theta_\varepsilon - \theta_\rho\,p_\varepsilon\right) > 0\,, \tag{2.22}$$

where the Maxwell relation (2.13) was used to evaluate the quantities in both (2.21) and (2.22). Finally, it is clear that (2.21) and (2.22) are equivalent to the inequalities (2.14).

We can then show [11] that

$$\partial_t(\rho\sigma) + \nabla_x\cdot\left(\rho u\sigma + \frac{q}{\theta}\right) = -\frac{\mu}{2}\left|\nabla_x u + (\nabla_x u)^T - \tfrac{2}{D} I\,\nabla_x\cdot u\right|^2 - \lambda\,|\nabla_x\cdot u|^2 - \kappa\,|\nabla_x\theta|^2\,. \tag{2.23}$$

Because $\mu > 0$, $\lambda \geq 0$, and $\kappa > 0$ the right-hand side above is nonpositive. Because $H(U)$ is strictly convex, we can see that (2.23) has the form (2.2). The compressible Navier-Stokes system (1.1) is therfore a hyperbolic-parabolic system with a strictly convex entropy given by $H(U)$ to which the theory in [11] applies.

The inequalities (2.14) can be understood better by introducing some notation that is common in thermodynamics, but less so in mathematics [4]. Given any three differntiable quantities $\psi(\rho,\varepsilon)$, $\phi(\rho,\varepsilon)$, and $\eta(\rho,\varepsilon)$, we define

$$\left(\frac{\partial\psi}{\partial\phi}\right)_\eta = \frac{\eta_\varepsilon\psi_\rho - \eta_\rho\psi_\varepsilon}{\eta_\varepsilon\phi_\rho - \eta_\rho\phi_\varepsilon}\,. \tag{2.24}$$

In other words, the subscripted parentheses denote that the enclosed derivative is to be taken while holding the subscript quantity fixed. The inequalities in (2.14) can then be expressed as

$$\left(\frac{\partial p}{\partial\rho}\right)_\sigma > 0\,, \qquad \left(\frac{\partial p}{\partial\rho}\right)_\theta > 0\,. \tag{2.25}$$

Because the first quantity on the left is related to the sound speed c_s by [4, 8]

$$c_\mathrm{s}^2 = \left(\frac{\partial p}{\partial\rho}\right)_\sigma = p_\rho + \frac{p}{\rho^2}\,p_\varepsilon\,, \tag{2.26}$$

the compressible Euler system associated with (1.1) is hyperbolic if and only if the first inequality holds [8]. The second inequality insures that the only stationary solutions of the compressible Navier-Stokes system (1.1) over $\mathbb{T}^D$ are constant states [11].

3. Weakly Compressible Navier-Stokes System

In this section we compute the weakly compressible Navier-Stokes system. More precisely, we will apply the general recipe for weakly nonlinear-dissipative approximations given by (2.3) to the compressible Navier-Stokes system (1.1). We will take as our the constant state $(\rho, u, \varepsilon) = (\rho^o, 0, \varepsilon^o)$ where ρ^o and ε^o are positive. We will take as our dependent variables V given by (2.19). We also see from (2.19) that the matrix $R^o = R(V^o) = U_V(V^o)$ and its inverse are

$$R^o = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \rho^o I & 0 \\ \varepsilon^o & 0 & \rho^o \end{pmatrix}, \qquad (R^o)^{-1} = \frac{1}{\rho^o}\begin{pmatrix} \rho^o & 0 & 0 \\ 0 & I & 0 \\ -\varepsilon^o & 0 & 1 \end{pmatrix}. \tag{3.1}$$

The weakly nonlinear-dissipative approximation to the compressible Navier-Stokes system (1.1) thereby has the form (2.3) where the operator $\mathcal{A}$ defined by (2.4) is given by

$$\mathcal{A}\widetilde{V} = \begin{pmatrix} \rho^o \, \nabla_x \cdot \widetilde{u} \\ \dfrac{p^o_\rho}{\rho^o} \nabla_x \widetilde{\rho} + \dfrac{p^o_\varepsilon}{\rho^o} \nabla_x \widetilde{\varepsilon} \\ \dfrac{p^o}{\rho^o} \nabla_x \cdot \widetilde{u} \end{pmatrix}, \tag{3.2}$$

while the averaged operators $\overline{\mathcal{Q}}$ and $\overline{\mathcal{D}}$ are given by (2.5) with the operators $\mathcal{Q}$ and $\mathcal{D}$ defined by (2.6) given by

$$\begin{aligned} \mathcal{Q}\big(\widetilde{V}, \widetilde{V}\big) &= \begin{pmatrix} \nabla_x \cdot (\tilde{\rho}\, \tilde{u}) \\ \nabla_x \cdot \left(\tilde{u} \otimes \tilde{u} + \left(\dfrac{p^o_{\rho\rho}}{2\rho^o} \tilde{\rho}^2 + \dfrac{p^o_{\rho\varepsilon}}{\rho^o} \tilde{\rho}\, \tilde{\varepsilon} + \dfrac{p^o_{\varepsilon\varepsilon}}{2\rho^o} \tilde{\varepsilon}^2 \right) I \right) \\ \nabla_x \cdot \left(\tilde{\varepsilon}\, \tilde{u} + \left(\dfrac{p^o_\rho}{\rho^o} \tilde{\rho} + \dfrac{p^o_\varepsilon}{\rho^o} \tilde{\varepsilon} \right) \tilde{u} \right) \end{pmatrix}, \\ \mathcal{D}\widetilde{V} &= \begin{pmatrix} 0 \\ \dfrac{\mu^o}{\rho^o} \Delta_x \tilde{u} + \dfrac{\frac{D-2}{D}\mu^o + \lambda^o}{\rho^o} \nabla_x \nabla_x \cdot \tilde{u} \\ \dfrac{\kappa^o \theta^o_\rho}{\rho^o} \Delta_x \tilde{\rho} + \dfrac{\kappa^o \theta^o_\varepsilon}{\rho^o} \Delta_x \tilde{\varepsilon} \end{pmatrix}. \end{aligned} \tag{3.3}$$

All that remains to be done in order to obtain the weakly compressible Navier-Stokes system are the two hardest steps — namely, to compute the averaged operators $\overline{\mathcal{Q}}$ and $\overline{\mathcal{D}}$. That is central task of this section.

3.1. **Structure of the System.**

We first state the formal structure of the averaged system (2.3).

Theorem 1. *A solution V to system* (2.3) *with initial condition $V^{\text{in}} = (\rho^{\text{in}}, u^{\text{in}}, \varepsilon^{\text{in}})^T$ can be decomposed orthogonally to $\mathcal{P}V = (-\frac{p^o_\varepsilon}{p^o_\rho}\vartheta\,, w\,, \vartheta)^T$ and $\mathcal{P}^\perp V = (\frac{1}{c^{o2}}\pi\,, v\,, \frac{p^o}{\rho^{o2} c^{o2}}\pi)^T$, where (ϑ, w) satisfies the* **incompressible Navier-Stokes** *system*

$$\begin{aligned} \nabla_x \cdot w &= 0\,, \\ \partial_t w + w \cdot \nabla_x w + \nabla_x p &= \tfrac{\mu^o}{\rho^o} \Delta_x w\,, \\ \partial_t \vartheta + w \cdot \nabla_x \vartheta &= \tfrac{\kappa^o}{\rho^o c^o_p} \Delta_x \vartheta\,, \end{aligned} \tag{3.4}$$

with initial data $(w,\vartheta)|_{t=0}=(w^{\mathrm{in}},\vartheta^{\mathrm{in}})$*, where* $w^{\mathrm{in}}=\Pi u^{\mathrm{in}}$*, and* $\vartheta^{\mathrm{in}}=-\frac{p^o p^o_\rho}{\rho^{o\,2} c^{o\,2}}\rho^{\mathrm{in}}+\frac{p^o_\rho}{c^{o\,2}}\varepsilon^{\mathrm{in}}$*, while* $\mathcal{P}^\perp V$ *satisfies the* **non-local quadratic** *system*

$$
\begin{aligned}
\partial_t\mathcal{P}^\perp V+\mathcal{A}\mathcal{P}^\perp V+\mathcal{Q}_{2r}(\mathcal{P}V,\mathcal{P}^\perp V)+\mathcal{Q}_{3r}(\mathcal{P}^\perp V,\mathcal{P}^\perp V)&=\bar{\mu}\Delta_x\mathcal{P}^\perp V\,,\\
\mathcal{P}^\perp V\big|_{t=0}&=\mathcal{P}^\perp V^{\mathrm{in}}\,.
\end{aligned}
\tag{3.5}
$$

More explicitly, (π,v) *satisfy the equations*

$$
\begin{aligned}
\partial_t\pi+\rho^o c^{o\,2}\nabla_x\!\cdot v+\left(\tfrac{1}{4}+\tfrac{p^o_\varepsilon}{4\rho^o}\right)w\!\cdot\!\nabla_x\pi+\tfrac{\rho^o c^o c^o_1}{4}\vartheta\nabla_x\!\cdot v&\\
+\,\mathcal{Q}^\pi_{2r}(\mathcal{P}V,\mathcal{P}^\perp V)+\mathcal{Q}^\pi_{3r}(\mathcal{P}^\perp V,\mathcal{P}^\perp V)&=\bar{\mu}\Delta_x\pi\,,
\end{aligned}
\tag{3.6}
$$

and

$$
\partial_t v+\tfrac{1}{\rho^o}\nabla_x\pi+\tfrac{c^o_1}{2\rho^o c^o}\nabla_x(\vartheta\pi)+\mathcal{Q}^v_{2r}(\mathcal{P}V,\mathcal{P}^\perp V)+\mathcal{Q}^v_{3r}(\mathcal{P}^\perp V,\mathcal{P}^\perp V)=\bar{\mu}\Delta_x v\,.
\tag{3.7}
$$

The details of the non-local terms in (3.6) *and* (3.7) *will given later.*

Remark. An important feature of the averaged system (2.3) is that the projection on the slow mode (incompressible Navier-Stokes equations) is completely decoupled from that on the fast motion, so it can be solved separately. One solution is provided by Leray [15]. But the equation on the fast mode is coupled with slow equation. The coefficient of the nonlocal term $\mathcal{Q}_{2r}(\mathcal{P}V,\mathcal{P}^\perp V)$ depends on $\mathcal{P}V$. A similar phenomena appears in many other problems related to the motion of fast oscillating waves [1, 2, 3, 19, 16, 17, 20].

Remark. If we were to use temperature θ rather than specific internal energy ε as the dependent variable then the resulting system is related to the one computed here by (2.11) with T given by

$$
T=\begin{pmatrix}1&0&0\\0&I&0\\\theta^o_\rho&0&\theta^o_\varepsilon\end{pmatrix}^{-1}\,.
$$

In the rest of this section, we give a detailed calculations to show Theorem 1. Our first step will be to give the spectral decomposition of the operator $\mathcal{A}$, which plays a central role in our analysis. We will then compute the averaged dissipation operator $\overline{\mathcal{D}}$, followed by the more complicated averaged quadratic operator $\overline{\mathcal{Q}}$.

3.2. **Acoustic Operator.** Because the operator $\mathcal{A}$ is the linearization of the compressible Euler system about a constant state, the only dynamics associated with it is that of sound waves. It is thereby called the *acoustic operator*. The weakly compressible Navier-Stokes system has a nice structure that it gains because it is an average over the fast acoustic dynamics. Specifically, its solutions can be decomposed into a so-called incompressible component that lies in $\mathrm{Null}(\mathcal{A})$, and an acoustic component that lies in $\mathrm{Range}(\mathcal{A})$. We will see that the incompressible component is governed by an incompressible Navier-Stokes system, and thereby decouples from the acoustic component. On the other hand, we will also see that the acoustic component is governed by a quadratic acoustic system that couples to the incompressible component.

It is clear from (3.2) that the range and null space of $\mathcal{A}$ are given by

$$
\begin{aligned}
\mathrm{Range}(\mathcal{A}) &= \left\{ \begin{pmatrix} \rho^o\beta \\ \nabla_x\phi \\ \frac{p^o}{\rho^o}\,\beta \end{pmatrix} \;:\; \beta\in L^2_0(\mathrm{d}x)\,,\ \phi\in H^1(\mathrm{d}x) \right\}, \\
\mathrm{Null}(\mathcal{A}) &= \left\{ \begin{pmatrix} p^o_\varepsilon\gamma \\ w \\ -p^o_\rho\gamma \end{pmatrix} \;:\; \gamma\in L^2_0(\mathrm{d}x)\,,\ w\in L^2_0(\mathrm{d}x;\mathbb{R}^D)\,,\ \nabla_x\!\cdot w = 0 \right\},
\end{aligned}
\tag{3.8}
$$

where $L^2_0(\mathrm{d}x)$ denotes L^2 functions with mean zero.

Because $\mathcal{A}$ is skew-adjoint in the Hilbert space $\mathbb{H}$, it follows that $\mathrm{Range}(\mathcal{A}) = \mathrm{Null}(\mathcal{A})^\perp$, where $\mathrm{Null}(\mathcal{A})^\perp$ is the orthognal complement of $\mathrm{Null}(\mathcal{A})$ with respect to the the natural inner product on $\mathbb{H}$ given by (2.8). For the weakly compressible Navier-Stokes system we have $H(U) = \rho\,\sigma(\rho,\varepsilon)$ where U is given by (2.12), while the matrix R^o is given by (3.1). Because by (2.20)

$$
R^{o\,T} H_{UU}(U^o) R^o = \begin{pmatrix} \rho^o\sigma^o_{\rho\rho} + 2\sigma^o_\rho & 0 & \rho^o\sigma^o_{\rho\varepsilon} \\ 0 & -\rho^o\sigma^o_\varepsilon I & 0 \\ \rho^o\sigma^o_{\rho\varepsilon} & 0 & \rho^o\sigma^o_{\varepsilon\varepsilon} \end{pmatrix},
\tag{3.9}
$$

the natural inner product (2.8) becomes

$$
\begin{aligned}
\left(\widetilde{V}_1,\widetilde{V}_2\right)_{\mathbb{H}} = \int_{\mathbb{T}^D} \big[&(\rho^o\sigma^o_{\rho\rho} + 2\sigma^o_\rho)\,\tilde\rho_1\tilde\rho_2 + \rho^o\sigma^o_{\rho\varepsilon}\left(\tilde\rho_1\tilde\varepsilon_2 + \tilde\varepsilon_1\tilde\rho_2\right) \\
&+ \rho^o\sigma^o_{\varepsilon\varepsilon}\,\tilde\varepsilon_1\tilde\varepsilon_2 - \rho^o\sigma^o_\varepsilon\,\tilde u_1\cdot\tilde u_2 \big]\,\mathrm{d}x\,.
\end{aligned}
\tag{3.10}
$$

For the notational simplicity, we henceforth denote this inner product by $(\,\cdot\,,\cdot)$.

The spectral decomposition of $\mathcal{A}$ can be characterized in terms of the eigenfunctions of the Laplacian. Let c^o denote the speed of sound, which is given by

$$
c^o = \sqrt{p^o_\rho + \frac{p^o}{\rho^{o\,2}}\,p^o_\varepsilon}\,.
\tag{3.11}
$$

Let ϕ_ν be an eigenfunction of the Laplacian satifying

$$
-\Delta_x\phi_\nu = \nu^2\phi_\nu \qquad \text{over } \mathbb{T}^D \text{ for some } \nu>0\,.
\tag{3.12}
$$

Then a direct calculation shows that

$$
\mathcal{A}\begin{pmatrix} \pm i\rho^o\phi_\nu \\ \frac{c^o}{\nu}\nabla_x\phi_\nu \\ \pm i\frac{p^o}{\rho^o}\phi_\nu \end{pmatrix} = \begin{pmatrix} \frac{\rho^o c^o}{\nu}\phi_\nu \\ \pm i c^{o\,2}\nabla_x\phi_\nu \\ \frac{p^o c^o}{\rho^o\nu}\phi_\nu \end{pmatrix} = \pm i c^o\nu \begin{pmatrix} \pm i\rho^o\phi_\nu \\ \frac{c^o}{\nu}\nabla_x\phi_\nu \\ \pm i\frac{p^o}{\rho^o}\phi_\nu \end{pmatrix}.
\tag{3.13}
$$

We thereby see that if ν^2 is a positive eigenvalue of $-\Delta_x$ with eigenfunction ϕ_ν then $\pm ic^o\nu$ is a conjugate pair of eigenvalues of $\mathcal{A}$ with eigenfunctions given above. Because the real and imaginary parts of these eigenfunctions clearly span $\mathrm{Range}(\mathcal{A})$ as given by (3.8), this must be a complete list of eigenpairs with nonzero eigenvalues.

For periodic domains solutions of (3.12) are given by the Fourier modes. If $\mathbb{T}^D = \mathbb{R}^D/\mathbb{L}$, where $\mathbb{L}\subset\mathbb{R}^D$ is some D-dimensional lattice then the Fourier modes are $e^{ik\cdot x}$ where $k\in\mathbb{L}^*$ and $\mathbb{L}^*$ is the dual lattice of $\mathbb{L}$, which is defined by the property that $k\cdot l = 0 \bmod 2\pi$ for every

$k \in \mathbb{L}^*$ and $l \in \mathbb{L}$. (So that $\mathbb{L}^* = (2\pi\mathbb{Z})^D$ when $\mathbb{L} = \mathbb{Z}^D$.) The Fourier mode $e^{ik\cdot x}$ solves (3.12) with $\nu = |k|$. By (3.13) we can thereby construct an eigenfunction basis for $\mathcal{A}$ in the form

$$H_k^\alpha(x) = \begin{pmatrix} \rho^o \\ \alpha\,\mathrm{sg}(k)\frac{k}{|k|}c^o \\ \frac{p^o}{\rho^o} \end{pmatrix} e^{ik\cdot x}\,,$$

where $k \in \mathbb{L}^*$ and α is $+$ or $-$, and the notation $\mathrm{sg}(k)$ stands for a generalized sign function on $\mathbb{R}^D \setminus \{0\}$: its value is 1 if and only if the first nonzero component of k is positive, -1 elsewhere. One can check that $H_k^\alpha(x)$ is the eigenvector of $\mathcal{A}$ with the eigenvalue $ic^o\alpha\,\mathrm{sg}(k)|k|$, with the norm

$$(H_k^\alpha\,, H_k^\alpha) = -2\rho^o\sigma_\varepsilon^o c^{o\,2} = a^{o\,2}c^{o\,2}\,,$$

where $a^o = \sqrt{-2\rho^o\sigma_\varepsilon^o}$.

Remark. The relation (3.13) between the spectral representation of the acoustic operator $\mathcal{A}$ and the spectral representation of the Laplacian can be applied to more general bounded domains. In that case, zero normal velocity boundary conditions for the acoustic operator corresponds to Neumann boundary conditions for the Laplacian.

The proof of (1) immediately follows the definition of the operator $\mathcal{A}$. The proof of part (2) follows from (1) and the definition of the inner product associated with $\mathbb{H}$. Suppose $\mathcal{A}\widetilde{V}_1 = 0$, i.e. $\widetilde{V}_1 = (\tilde\rho_1, \tilde u_1, -\frac{p_\rho^o}{p_\varepsilon^o}\tilde\rho_1)^T$, while $\nabla_x\cdot\tilde u_1 = 0$. Then $\left(\mathcal{A}\widetilde{V}_1, \widetilde{V}_2\right) = 0$ implies that u_2 is a gradient and $\tilde\varepsilon_2 = d^o\tilde\rho_2$, where

$$d^o = \frac{\rho^o p_\rho^o\sigma_{\rho\varepsilon}^o - \rho^o p_\varepsilon^o\sigma_{\rho\rho}^o - 2p_\varepsilon^o\sigma_\rho^o}{\rho^o p_\varepsilon^o\sigma_{\rho\varepsilon}^o - \rho^o p_\rho^o\sigma_{\varepsilon\varepsilon}^o}\,.$$

Again using the thermodynamical relation of the pressure $p = -\rho^2\frac{\sigma_\rho(\rho,\varepsilon)}{\sigma_\varepsilon(\rho,\varepsilon)}$, we can verify the identity

$$d^o = \frac{p^o}{\rho^{o\,2}}\,.$$

Because the operator $\mathcal{A}$ is skew-adjoint, we have the following decomposition of $\mathbb{H}$:

$$\mathbb{H} = \mathbb{H}_1 \oplus \mathbb{H}_2\,,$$

where

$$\mathbb{H}_1 = \mathrm{Null}(\mathcal{A})\,, \qquad \mathbb{H}_2 = \mathrm{Range}(\mathcal{A}) = \mathrm{Null}(\mathcal{A})^\perp\,.$$

Every $\widetilde{V} \in \mathbb{H}$ has the unique decomposition

$$\widetilde{V} = \mathcal{P}\widetilde{V} + \mathcal{P}^\perp\widetilde{V}\,,$$

where $\mathcal{P}$ and $\mathcal{P}^\perp$ are projections onto $\mathrm{Null}(\mathcal{A})$ and $\mathrm{Null}(\mathcal{A})^\perp$ with

$$\mathcal{P} : \mathbb{H} \longrightarrow \mathbb{H}_1\,, \qquad \mathcal{P}^\perp : \mathbb{H} \longrightarrow \mathbb{H}_2\,,$$

defined by

$$\mathcal{P}\begin{pmatrix}\tilde\rho\\ \tilde u\\ \tilde\varepsilon\end{pmatrix} = \begin{pmatrix} \frac{p^o p_\varepsilon^o}{\rho^{o\,2}c^{o\,2}}\tilde\rho - \frac{p_\varepsilon^o}{c^{o\,2}}\tilde\varepsilon \\ \Pi\tilde u \\ -\frac{p^o p_\rho^o}{\rho^{o\,2}c^{o\,2}}\tilde\rho + \frac{p_\rho^o}{c^{o\,2}}\tilde\varepsilon \end{pmatrix}, \qquad \mathcal{P}^\perp\begin{pmatrix}\tilde\rho\\ \tilde u\\ \tilde\varepsilon\end{pmatrix} = \begin{pmatrix} \frac{p_\rho^o}{c^{o\,2}}\tilde\rho + \frac{p_\varepsilon^o}{c^{o\,2}}\tilde\varepsilon \\ (I-\Pi)\tilde u \\ \frac{p^o p_\rho^o}{\rho^{o\,2}c^{o\,2}}\tilde\rho + \frac{p^o p_\varepsilon^o}{\rho^{o\,2}c^{o\,2}}\tilde\varepsilon \end{pmatrix}, \tag{3.14}$$

where Π is the usual Leray projection onto the space of divergence-free vector fields defined by

$$\Pi = I - \nabla_x\Delta^{-1}\nabla_x\cdot\,.$$

The averaged system (2.3) can be represented in terms of the spectrum of $\mathcal{A}$. Noticing that the averaged operators $\overline{\mathcal{Q}}$ and $\overline{\mathcal{D}}$ are involved the exponential operators $e^{\pm s\mathcal{A}}$, which do not have any effect on the eigenspace associated with the eigenvalue 0, we are only interested in the spectral space associated with the nontrivial eigenvalues.

We define that

$$\vartheta = -\frac{p^o p^o_\rho}{{\rho^o}^2 {c^o}^2}\tilde{\rho} + \frac{p^o_\rho}{{c^o}^2}\tilde{\varepsilon}\,, \quad w = \Pi\tilde{u}\,,$$

and

$$\pi = p^o_\rho \tilde{\rho} + p^o_\varepsilon \tilde{\varepsilon}\,, \quad v = (I - \Pi)\tilde{u}\,.$$

Then we have the following orthogonal decomposition: for every $\widetilde{V} \in \mathbb{H}$,

$$\widetilde{V} = \mathcal{P}\widetilde{V} + \mathcal{P}^\perp \widetilde{V} = \begin{pmatrix} -\frac{p^o_\varepsilon}{p^o_\rho}\vartheta \\ w \\ \vartheta \end{pmatrix} + \begin{pmatrix} \frac{1}{c^{o2}}\pi \\ v \\ \frac{p^o}{\rho^{o2}c^{o2}}\pi \end{pmatrix}.$$

$\mathcal{P}^\perp \widetilde{V}$ can be represented by H^α_k:

$$\mathcal{P}^\perp \widetilde{V} = \sum_{\alpha\,,k} \frac{(\widetilde{V}\,, H^\alpha_k)}{(H^\alpha_k\,, H^\alpha_k)} H^\alpha_k\,.$$

We denote the coefficient of H^α_k in the above representation of $\mathcal{P}^\perp \widetilde{V}$ as

$$V^\alpha_k = \frac{(\widetilde{V}\,, H^\alpha_k)}{(H^\alpha_k\,, H^\alpha_k)} = \frac{1}{2\rho^o (c^o)^2}\widehat{\pi}_k + \frac{1}{2c^o}\alpha\,\mathrm{sg}(k)\frac{k\cdot\widehat{v}_k}{|k|}\,. \tag{3.15}$$

where $\widehat{f}_k$ denotes the Fourier coefficient of function $f(x)$.

3.3. **Averaged Dissipation Operator.** Let η be any eigenvector associated with eigenvalue 0. The exponential operator $e^{s\mathcal{A}}$ does not affect $\mathrm{Null}(\mathcal{A})$. The inner product of $\mathcal{P}\overline{\mathcal{D}}$ with η is:

$$\begin{aligned}\left(\overline{\mathcal{D}}V, \eta\right) &= \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\left(\mathcal{D}e^{-s\mathcal{A}}V, \eta\right)\,\mathrm{d}s \\ &= \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\left(\mathcal{P}\mathcal{D}\mathcal{P}V, \eta\right)\,\mathrm{d}s + \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\left(\mathcal{D}e^{-s\mathcal{A}}\mathcal{P}^\perp V, \eta\right)\,\mathrm{d}s\,.\end{aligned}$$

The first term is the resonant term which is independent of s, so is not affected by time averaging. The second is non-resonant, which is filtered by time averaging. The following Riemann-Lebesgue lemma, the proof of which can be found in [7], guarantees that this second term vanishes. Thus we have

$$\left(\overline{\mathcal{D}}V, \eta\right) = \left(\mathcal{P}\mathcal{D}\mathcal{P}V, \eta\right)\,.$$

Lemma 1. *In the time averaging, the oscillatory integral*

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} e^{isA(k)}\phi(s)\,\mathrm{d}s \tag{3.16}$$

for any integrable function $\phi(t)$ vanishes when $A(k) \neq 0$. The only nonzero contributions that survive the averaging process are the resonance $A(k) = 0$. Here $A(k)$ is any polynomial of k so that (3.16) *is integrable.*

Upon applying the above Lemma, we deduce that the projection of the averaged dissipation operator on $\mathrm{Null}(\mathcal{A})$ is given by

$$\mathcal{P}\overline{\mathcal{D}}V = \mathcal{P}\mathcal{D}\mathcal{P}V = \begin{pmatrix} -\frac{p^o_\varepsilon}{p^o_\rho}\frac{\kappa^o}{\rho^o c^o_p}\Delta_x\vartheta \\ \frac{\mu^o}{\rho^o}\Pi\Delta_x\Pi u \\ \frac{\kappa^o}{\rho^o c^o_p}\Delta_x\vartheta \end{pmatrix}, \tag{3.17}$$

where c^o_p is the specific heat capacity at constant pressure, which is given by

$$c^o_p = -\theta^o\left(\frac{\partial\sigma}{\partial\theta}\right)_p(\rho^o,\varepsilon^o) = -\theta^o\,\frac{p^o_\varepsilon\,\sigma^o_\rho - p^o_\rho\,\sigma^o_\varepsilon}{p^o_\varepsilon\,\theta^o_\rho - p^o_\rho\,\theta^o_\varepsilon} = \frac{c^{o2}}{p^o_\varepsilon\,\theta^o_\rho - p^o_\rho\,\theta^o_\varepsilon},$$

Equilibrium thermodynamics demands that $c^o_p > 0$.

The projection on $\mathrm{Null}(\mathcal{A})^\perp$ is

$$\mathcal{P}^\perp\overline{\mathcal{D}}V = \sum_{\alpha,k}\frac{\left(\overline{\mathcal{D}}V, H^\alpha_k\right)}{(H^\alpha_k, H^\alpha_k)}H^\alpha_k\,,$$

where $\left(\overline{\mathcal{D}}V, H^\alpha_k\right)$ is

$$\begin{aligned}\left(\overline{\mathcal{D}}V, H^\alpha_k\right) &= \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\left(\mathcal{D}e^{-s\mathcal{A}}V, e^{-s\mathcal{A}}H^\alpha_k\right)\,\mathrm{d}s \\ &= \sum_{\beta,l}\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}V^\beta_l e^{-is(\beta\,\mathrm{sg}(l)|l|-\alpha\,\mathrm{sg}(k)|k|)}\left(\mathcal{D}H^\beta_l, H^\alpha_k\right)\,\mathrm{d}s\,.\end{aligned}\tag{3.18}$$

Straightforward calculations imply that $\left(\mathcal{D}H^\beta_l, H^\alpha_k\right)$ is nonzero only when $k = l$. Note that in this case, by Lemma 1, (3.18) is nonzero if and only if $\alpha = \beta$. Simple calculations show that

$$(\mathcal{D}H^\alpha_k, H^\alpha_k) = -\bar{\mu}\,|k|^2\,(H^\alpha_k, H^\alpha_k)\,.$$

Thus

$$\mathcal{P}^\perp\overline{\mathcal{D}}V = -\bar{\mu}\sum_{\alpha,k}|k|^2 V^\alpha_k H^\alpha_k = \bar{\mu}\,\Delta_x\mathcal{P}^\perp V\,, \tag{3.19}$$

where

$$\bar{\mu} = \frac{2\frac{D-1}{D}\mu^o + \lambda^o}{2\rho^o} + \frac{\theta^o(p^o_\varepsilon)^2}{2\rho^{o3}c^{o2}}\kappa\,,$$

is always positive for physically meaningful models. Simple calculation shows that the averaged diffusion term is strictly dissipated, in other words,

$$\begin{aligned}-\left(\overline{\mathcal{D}}V, V\right) &= -\left(\mathcal{P}\overline{\mathcal{D}}V, \mathcal{P}V\right) - \left(\mathcal{P}^\perp\overline{\mathcal{D}}V, \mathcal{P}^\perp V\right) \\ &= \left\|\left(-\tfrac{p^o_\varepsilon}{p^o_\rho}\sqrt{\kappa^o}\nabla_x\vartheta\,,\ \sqrt{\tfrac{\mu^o}{\rho^o}}\nabla_x w\,,\ \sqrt{\kappa^o}\nabla_x\vartheta\right)\right\|^2_{\mathbb{H}} + \bar{\mu}\,\left\|\mathcal{P}^\perp V\right\|^2_{\mathbb{H}} \\ &\geq \delta_o\|V\|_{\mathbb{H}}\,,\end{aligned}$$

for some $\delta_o > 0$. Furthermore, $\left(\overline{\mathcal{D}}V, V\right) = 0$ if and only if $V = 0$.

Remark. The original dissipation operator $\mathcal{D}$ is only *partially* elliptic (dissipative). That is one of the difficulties for the equations of compressible model because the equation of continuity is just a transport equation, does not have dissipation. From our derivation, after taking time averaging, the diffusion term in the averaged system is strictly dissipative. This averaged dissipation operator appeared in the work of Hoff and Zumbrun [9, 10]. They called it an "artificial viscosity term ", applied to the isentropic gas without energy equation. So our averaged system, when we ignore the nonlinearity, is a natural generalization of the Hoff-Zumbrun's so-called "effective artificial viscosity system" [9, 10]. Actually, one of the main motivation of Hoff-Zumbrun's consideration is to modify the dissipative operator so that it has strict parabolicity.

3.4. **Averaged Quadratic Operator.** By a similar approach, we can compute $\overline{\mathcal{Q}}(V\,,V)$. For any $\eta \in \mathrm{Null}(\mathcal{A})$,

$$\begin{aligned}\big(\overline{\mathcal{Q}}(V\,,V),\eta\big) &= \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\big(\mathcal{Q}(e^{-s\mathcal{A}}V\,,e^{-s\mathcal{A}}V),\eta\big)\;\mathrm{d}s\\ &= \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}(I_1+I_2+I_3,\eta)\;\mathrm{d}s\,,\end{aligned}\tag{3.20}$$

where

$$\begin{aligned}I_1 &= \mathcal{Q}(\mathcal{P}V,\mathcal{P}V)\,,\\ I_2 &= \mathcal{Q}(\mathcal{P}V,e^{-s\mathcal{A}}\mathcal{P}^{\perp}V)+\mathcal{Q}(e^{-s\mathcal{A}}\mathcal{P}^{\perp}V,\mathcal{P}V)\,,\\ I_3 &= \mathcal{Q}(e^{-s\mathcal{A}}\mathcal{P}^{\perp}V,e^{-s\mathcal{A}}\mathcal{P}^{\perp}V)\,.\end{aligned}$$

Direct calculations show that

$$\begin{aligned}I_2 &= ic^o V_k^{\alpha}e^{-is\Box_k^{\alpha}}e^{i(k+l)\cdot x}\\ &\times\begin{pmatrix}-\frac{p^o_{\varepsilon}}{p^o_{\rho}}\widehat{\vartheta}_l\alpha\,\mathrm{sg}(k)\frac{(k+l)\cdot k}{|k|}+\frac{\rho^o}{c^o}\widehat{w}_l\cdot(k+l)\\ c_1^o\widehat{\vartheta}_l(k+l)+\alpha\,\mathrm{sg}(k)\frac{[\widehat{w}_l\cdot(k+l)]k+[k\cdot(k+l)]\widehat{w}_l}{|k|}\\ \widehat{\vartheta}_l\alpha\,\mathrm{sg}(k)\frac{k\cdot(k+l)}{|k|}+\left(\frac{p^o}{\rho^o c^o}+c^o\right)\widehat{w}_l\cdot(k+l)\end{pmatrix},\end{aligned}$$

and

$$\begin{aligned}I_3 &= \frac{i}{2}c^{o\,2}V_k^{\alpha}V_l^{\beta}e^{-is\Box_{kl}^{\alpha\beta}}e^{i(k+l)\cdot x}\\ &\times\begin{pmatrix}\frac{\rho^o}{c^o}\left(\alpha\,\mathrm{sg}(k)\frac{k}{|k|}+\beta\,\mathrm{sg}(l)\frac{l}{|l|}\right)\cdot(k+l)\\ \alpha\,\mathrm{sg}(k)\beta\,\mathrm{sg}(l)\frac{[k\cdot(k+l)]l+[l\cdot(k+l)]k}{|k|\,|l|}+c_2^o(k+l)\\ \left(\frac{p^o}{\rho^o c^o}+c^o\right)\left(\alpha\,\mathrm{sg}(k)\frac{k}{|k|}+\beta\,\mathrm{sg}(l)\frac{l}{|l|}\right)\cdot(k+l)\end{pmatrix},\end{aligned}$$

where

$$\begin{aligned}\Box_k^{\alpha} &= c^o\alpha\mathrm{sg}(k)|k|\,,\qquad \Box_{kl}^{\alpha\beta}=c^o\left(\alpha\,\mathrm{sg}(k)|k|+\beta\,\mathrm{sg}(l)|l|\right)\,,\\ c_1^o &= \frac{1}{c^o}\left(-\frac{p^o_{\varepsilon}}{p^o_{\rho}}\big(p^o_{\rho\rho}+p^o_{\rho\varepsilon}\tfrac{p^o}{\rho^{o\,2}}\big)+\big(p^o_{\rho\varepsilon}+p^o_{\varepsilon\varepsilon}\tfrac{p^o}{\rho^{o\,2}}\big)\right)\,,\\ c_2^o &= \frac{1}{c^{o2}}\left(\rho^o p^o_{\rho\rho}+\frac{2p^o p^o_{\rho\varepsilon}}{\rho^o}+\frac{p^{o\,2}p^o_{\varepsilon\varepsilon}}{\rho^{o3}}\right)\,,\end{aligned}$$

and

$$w=\sum_l \widehat{w}_l e^{il\cdot x}\,,\qquad \vartheta=\sum_l\widehat{\vartheta}_l e^{il\cdot x}\,,$$

with $l\cdot\widehat{u}(l)=0$.

Using Lemma 1, in (3.20),

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}(I_2,\eta)\;\mathrm{d}s=0\,.$$

Now we claim that we also have

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}(I_3,\eta)\,\mathrm{d}s=0\,.$$

By Lemma 1, the only non-trivial contribution is on the resonant set, $\Box_{kl}^{\alpha\beta}=0$, i.e.

$$|k|=|l|\,,\qquad \alpha\,\mathrm{sg}(k)|k|+\beta\,\mathrm{sg}(l)|l|=0\,.$$

On this resonant set, it is easy to see that the first and the last component in I_3 can factor out the term $\alpha\,\mathrm{sg}(k)|k|+\beta\,\mathrm{sg}(l)|l|$, which is zero, and the second component of I_3 can factor out the term $e^{i(k+l)\cdot x}i(k+l)$ which is a gradient form. So after time averaging, I_3 is in $\mathrm{Null}(\mathcal{A})$. So its inner product with η is zero. Thus $\mathcal{P}\overline{\mathcal{Q}}(V\,,V)=\mathcal{P}\mathcal{Q}(\mathcal{P}V,\mathcal{P}V)$. A direct calculation yields

$$\mathcal{P}\mathcal{Q}(\mathcal{P}V,\mathcal{P}V)=\begin{pmatrix}-\frac{p_\varepsilon^o}{p_\rho^o}w\cdot\nabla_x\vartheta\\ w\cdot\nabla_x w\\ w\cdot\nabla_x\vartheta\end{pmatrix}\,. \tag{3.21}$$

The projection on the acoustic mode $\mathcal{P}^\perp\overline{\mathcal{Q}}(V\,,V)$ is

$$\begin{aligned}\frac{1}{a^{o\,2}c^{o\,2}}\sum_{\delta,m}\big(\overline{\mathcal{Q}}(V\,,V),H_m^\delta\big)\,H_m^\delta&=\sum_{\delta,m}\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\big(I_2+I_3,e^{-isc^o\delta\mathrm{sg}(m)|m|}H_m^\delta\big)\,H_m^\delta\,\mathrm{d}t\\&=\mathcal{Q}_{2r}(\mathcal{P}V,\mathcal{P}^\perp V)+\mathcal{Q}_{3r}(\mathcal{P}^\perp V,\mathcal{P}^\perp V)\,,\end{aligned}$$

where $\mathcal{Q}_{2r}$ and $\mathcal{Q}_{3r}$ denote the averaged quadratic operator over the two-wave and three-wave resonant sets respectively. Note that $\mathcal{Q}_{2r}$ depends on both the incompressible and acoustic modes, while $\mathcal{Q}_{3r}$ depends only on the acoustic modes.

3.4.1. *Two-wave resonant term.* Applying Lemma 1 again, we derive

$$\mathcal{Q}_{2r}\big(\mathcal{P}V,\mathcal{P}^\perp V\big)=\sum_{\delta,m}\lambda_m^\delta\cdot imH_m^\delta(x)\,, \tag{3.22}$$

where

$$\lambda_m^\delta=\frac{1}{a^{o2}}\sum_{\substack{\alpha\,\mathrm{sg}(k)=\delta\,\mathrm{sg}(m)\\ k+l=m\\ |k|=|m|}}V_k^\alpha\left[c_3^o\widehat{w}_l-\rho^o\sigma_\varepsilon^o\frac{2k\cdot m}{|k||m|}\widehat{w}_l+c_4^o\widehat{\vartheta}_l\delta\,\mathrm{sg}(m)\frac{k}{|k|}-c_1^o\rho^o\sigma_\varepsilon^o\widehat{\vartheta}_l\delta\,\mathrm{sg}(m)\frac{m}{|m|}\right],$$

where

$$\begin{aligned}c_3^o&=\frac{1}{c^{o\,2}}\big(\rho^{o\,3}\sigma_{\rho\rho}^o+2\rho^o p^o\sigma_{\rho\varepsilon}^o+\tfrac{p^{o2}}{\rho^o}\sigma_{\varepsilon\varepsilon}^o+2\rho^{o\,2}\sigma_\rho^o\big)+\big(p^o\sigma_{\varepsilon\varepsilon}^o+\rho^{o\,2}\sigma_{\rho\varepsilon}^o\big)\,,\\ c_4^o&=\frac{1}{c^o}\big(-\tfrac{p_\varepsilon^o}{p_\rho^o}(\rho^{o\,2}\sigma_{\rho\rho}^o+2\rho^o\sigma_\rho^o+p^o\sigma_{\rho\varepsilon}^o)+\rho^{o\,2}\sigma_{\rho\varepsilon}^o+p^o\sigma_{\varepsilon\varepsilon}^o\big)\,.\end{aligned}$$

Using the Maxwell relation, we can simplify c_3^o and c_4^o as

$$c_3^o=\frac{1}{\theta^o}\left(\rho^o+p_\varepsilon^o\right)\,,\qquad c_4^o=0\,.$$

Thus,

$$\frac{c_3^o}{a^{o2}}=\frac{1}{2}+\frac{p_\varepsilon^o}{2\rho^o}\,.$$

Then

$$\lambda_m^\delta = \sum_{\substack{\alpha\,\mathrm{sg}(k)=\delta\,\mathrm{sg}(m)\\ k+l=m\\ |k|=|m|}} V_k^\alpha \left[\left(\frac{1}{2} + \frac{p_\varepsilon^o}{2\rho^o} \right) \widehat{w}_l + \frac{k\cdot m}{|k|\,|m|} \widehat{w}_l + \frac{c_1^o}{2} \widehat{\vartheta}_l \delta\,\mathrm{sg}(m) \frac{m}{|m|} \right] . \tag{3.23}$$

To express the components in (3.22) explicitly, we plug (3.15) into (3.23). In the component of density, the following two terms can factor out local terms:

$$\begin{aligned} &\sum_{\delta,m} \sum_{\substack{\alpha\,\mathrm{sg}(k)=\delta\,\mathrm{sg}(m)\\ k+l=m\\ |k|=|m|}} \tfrac{1}{2\rho^o (c^o)^2} \widehat{\pi}_k \left(\tfrac{1}{2} + \tfrac{p_\varepsilon^o}{2\rho^o} \right) \widehat{w}_l \cdot im\rho^o e^{im\cdot x} \\ =&\tfrac{1}{c^{o\,2}} \left(\tfrac{1}{4} + \tfrac{p_\varepsilon^o}{4\rho^o} \right) w\cdot\nabla_x \pi - \sum_{\delta,m} \sum_{\substack{\alpha\,\mathrm{sg}(k)=\delta\,\mathrm{sg}(m)\\ k+l=m\\ |k|\neq|m|}} \tfrac{1}{2\rho^o (c^o)^2} \widehat{\pi}_k \left(\tfrac{1}{2} + \tfrac{p_\varepsilon^o}{2\rho^o} \right) \widehat{w}_l \cdot im\rho^o e^{im\cdot x} , \end{aligned}$$

and

$$\begin{aligned} &\sum_{\delta,m} \sum_{\substack{\alpha\,\mathrm{sg}(k)=\delta\,\mathrm{sg}(m)\\ k+l=m\\ |k|=|m|}} \tfrac{1}{2c^o} \alpha\,\mathrm{sg}(k) \tfrac{k\cdot\widehat{v}_k}{|k|} \tfrac{c_1^o}{2} \widehat{\vartheta}_l \delta\,\mathrm{sg}(m) \tfrac{m}{|m|} \cdot im\rho^o e^{im\cdot x} \\ =&\tfrac{\rho^o c_1^o}{4c^o} \vartheta \nabla_x\!\cdot v - \sum_{\delta,m} \sum_{\substack{\alpha\,\mathrm{sg}(k)=\delta\,\mathrm{sg}(m)\\ k+l=m\\ |k|\neq|m|}} \tfrac{1}{2c^o} \alpha\,\mathrm{sg}(k) \tfrac{k\cdot\widehat{v}_k}{|k|} \tfrac{c_1^o}{2} \widehat{\vartheta}_l \delta\,\mathrm{sg}(m) \tfrac{m}{|m|} \cdot im\rho^o e^{im\cdot x} . \end{aligned}$$

In the component of velocity, the following term can factor out a local term:

$$\begin{aligned} &\sum_{\delta,m} \sum_{\substack{\alpha\,\mathrm{sg}(k)=\delta\,\mathrm{sg}(m)\\ k+l=m\\ |k|=|m|}} \tfrac{1}{2\rho^o (c^o)^2} \widehat{\pi}_k \tfrac{c_1^o}{2} \widehat{\vartheta}_l \delta\,\mathrm{sg}(m) \tfrac{m}{|m|} \cdot im\delta\,\mathrm{sg}(m) \tfrac{m}{|m|} c^o e^{im\cdot x} \\ =&\tfrac{c_1^o}{4\rho^o c^o} \nabla_x(\vartheta\pi) - \sum_{\delta,m} \sum_{\substack{\alpha\,\mathrm{sg}(k)=\delta\,\mathrm{sg}(m)\\ k+l=m\\ |k|\neq|m|}} \tfrac{1}{2\rho^o (c^o)^2} \widehat{\pi}_k \tfrac{c_1^o}{2} \widehat{\vartheta}_l \delta\,\mathrm{sg}(m) \tfrac{m}{|m|} \cdot im\delta\,\mathrm{sg}(m) \tfrac{m}{|m|} c^o e^{im\cdot x} \end{aligned}$$

Thus, the density component of $\mathcal{Q}_{2r}\big(\mathcal{P}V, \mathcal{P}^\perp V\big)$ can be written as

$$\tfrac{1}{c^{o\,2}} \left(\tfrac{1}{4} + \tfrac{p_\varepsilon^o}{4\rho^o} \right) w\cdot\nabla_x\pi + \tfrac{\rho^o c_1^o}{4c^o} \vartheta\nabla_x\!\cdot v + \tfrac{1}{c^{o\,2}} \mathcal{Q}_{2r}^\pi\big(\mathcal{P}V, \mathcal{P}^\perp V\big) , \tag{3.24}$$

where $\tfrac{1}{c^{o\,2}} \mathcal{Q}_{2r}^\pi\big(\mathcal{P}V, \mathcal{P}^\perp V\big)$ inludes all the other non-local terms in the density component of $\mathcal{Q}_{2r}\big(\mathcal{P}V, \mathcal{P}^\perp V\big)$. Similarly, the velocity component of $\mathcal{Q}_{2r}\big(\mathcal{P}V, \mathcal{P}^\perp V\big)$ can be written as

$$\tfrac{c_1^o}{4\rho^o c^o} \nabla_x(\vartheta\pi) + \mathcal{Q}_{2r}^v\big(\mathcal{P}V, \mathcal{P}^\perp V\big) , \tag{3.25}$$

where $\mathcal{Q}_{2r}^v\big(\mathcal{P}V, \mathcal{P}^\perp V\big)$ includes all the non-local terms in the velocity component of $\mathcal{Q}_{2r}\big(\mathcal{P}V, \mathcal{P}^\perp V\big)$.

3.4.2. *Three-wave resonant term.* The 3-wave interaction term $\mathcal{Q}_{3r}\big(\mathcal{P}^\perp V, \mathcal{P}^\perp V\big)$ can be written as

$$\mathcal{Q}_{3r}\big(\mathcal{P}^\perp V, \mathcal{P}^\perp V\big) = i\frac{c^o}{2a^{o\,2}} \sum_{\delta,m} \sum_{\substack{k+l=m \\ \alpha\,\mathrm{sg}(k)|k|+\beta\,\mathrm{sg}(l)|l|=\delta\,\mathrm{sg}(m)|m|}} V_k^\alpha\, V_l^\beta \bigg[- c_2^o\rho^o\sigma_\varepsilon^o \delta\,\mathrm{sg}(m)|m| + c_3^o \left(\alpha\,\mathrm{sg}(k)\frac{k\cdot m}{|k|} + \beta\,\mathrm{sg}(l)\frac{l\cdot m}{|l|} \right) - \rho^o\sigma_\varepsilon^o \alpha\,\mathrm{sg}(k)\,\beta\,\mathrm{sg}(l)\delta\,\mathrm{sg}(m)\frac{2(k\cdot m)(l\cdot m)}{|k||l||m|} \bigg] H_m^\delta(x)\,.$$

We apply basically Masmoudi's arguments [18] to analyze the structure of the resonant set. The resonance condition between (k,α), (l,β), (m,δ), namely $(H_k^\alpha, H_l^\beta, H_m^\delta)$ is

$$k+l=m\,, \qquad \alpha\,\mathrm{sg}(k)|k| + \beta\,\mathrm{sg}(l)|l| = \delta\,\mathrm{sg}(m)|m|\,. \tag{3.26}$$

Hence, $2k\cdot l = 2\alpha\,\mathrm{sg}(k)\,\beta\,\mathrm{sg}(l)|k||l|$, which means that k is parallel to l, so is parallel to m, i.e., all the vectors in a 3-waves resonant set are parallel to each other. Rewriting this product again and using that k is parallel to l, we deduce that $k\cdot l = \mathrm{sg}(k)\mathrm{sg}(l)|k||l|$. This yields that we have $\alpha=\beta$ and then we can see easily that (3.26) is equivalent to

$$k+l=m\,, \qquad \mathrm{sg}(k)|k| + \mathrm{sg}(l)|l| = \mathrm{sg}(m)|m|\,, \qquad \alpha=\beta=\delta\,.$$

The above relations mean that we can only get resonances between the triplet (H_k^+, H_l^+, H_m^+) and (H_k^-, H_l^-, H_m^-) separately. As Masmoudi mentioned in [18], that is the reason why we have introduced the notation $\mathrm{sg}(k)$. Applying the above analysis of the resonant sets, we can rewrite $\mathcal{Q}_{3r}(\mathcal{P}^\perp V, \mathcal{P}^\perp V)$ as

$$\mathcal{Q}_{3r}(\mathcal{P}^\perp V, \mathcal{P}^\perp V) = \sum_{\alpha,m} \chi_m^\alpha \cdot im H_m^\alpha(x)\,, \tag{3.27}$$

where

$$\chi_m^\alpha = C^o \sum_{\substack{\alpha,k,l \\ k+l=m \\ \mathrm{sg}(k)|k|+\mathrm{sg}(l)|l|=\mathrm{sg}(m)|m|}} V_k^\alpha V_l^\alpha \alpha \mathrm{sg}(m)\frac{m}{|m|}\,,$$

where $C^o = 2c_3^o - c_2^o\rho^o\sigma_\varepsilon^o - 2\rho^o\sigma_\varepsilon^o$. This is a very simple form.

Different with the two-wave resonant term, we can not factor out local terms from three-wave resonant terms. All terms in $\mathcal{Q}_{3r}(\mathcal{P}^\perp V, \mathcal{P}^\perp V)$ are non-local. we denote its density component by $\frac{1}{c^{o\,2}}\mathcal{Q}_{3r}^\pi(\mathcal{P}^\perp V, \mathcal{P}^\perp V)$, while the velocity component is denoted by $\mathcal{Q}_{3r}^v(\mathcal{P}^\perp V, \mathcal{P}^\perp V)$.

4. Existence and Regularity for the Averaged System

4.1. **Global Weak Solutions.** The existence of global weak solutions theory to the averaged system (2.3) is a direct consequence of that of general hyperbolic-parabolic system developed in [11]. Because of the structure of (3.5), it is natural to define weak solutions in the sense of (3.4) in [11]. The quadratic terms $\mathcal{Q}_{2r}$ and $\mathcal{Q}_{3r}$ do not contribute in energy estimate, see [11]. The dissipative term in (3.5) is $\bar{\mu}\Delta_x$, automatically satisfies the structure assumption in Leray-type global weak solutions to the averaged system of the general hyperbolic-parabolic system with entropy. As a corollary of Theorem 1 of [11], we state global existence of weak solutions to the averaged system (3.5).

Theorem 2. *For any given initial data* $U^{in} \in L^2$*, first we solve the incompressible Navier-Stokes equation* (3.4)*. Using a Leray solution to* (3.4)*, we can find a global weak solution to* (3.5)*, and the energy inequality holds:*

$$\tfrac{1}{2}\|\mathcal{P}^\perp V(t)\|_{\mathbb{H}}^2 + \bar{\mu}\int_0^t \|\nabla_x \mathcal{P}^\perp V(t')\|_{\mathbb{H}}^2 \,\mathrm{d}t' \le \tfrac{1}{2}\|\mathcal{P}^\perp V^{in}\|_{\mathbb{H}}^2 \,.$$

Remark. In their consideration of compressible incompressible limit of Navier-Stokes equations for *barotropic* fluid, Masmoudi [18] and Danchin [6] derived the averaged equations of fast waves, which are very similar to (3.4) (without equations for temperature) and (3.5). Next, we state the higher regularity of (3.5) with generalization to the full Navier-Stokes equations.

In this section, we investigate the global well-posedness of the averaged system (3.5) in the general Sobolev spaces $H^s(\mathbb{T}^D)$ and the Besov spaces $B^s(\mathbb{T}^D)$. We introduce the Littlewood-Paley decomposition to characterize these spaces.

4.2. **Littlewood-Paley Decomposition.** First, we introduce a couple of smooth radial functions (χ, φ) such that χ is supported in the ball $\{k \in \mathbb{R}^D, |k| \le 6/5\}$ supported in $\{k \in \mathbb{R}^D, 5/6 \le |k| \le 12/5\}$ and such that

$$\chi(k) + \sum_{q\in\mathbb{N}} \varphi(2^{-q}k) = 1 \quad \text{for every } k \in \mathbb{R}^D \,.$$

Denoting

$$h_q(x) = \sum_{k\in\mathbb{Z}^N} \varphi(2^{-q}k) e^{ik\cdot x} \,, \tag{4.1}$$

one can then define the periodic dyadic blocks as

$$\Delta_q u \triangleq \sum_{k\in\mathbb{Z}^N} \varphi(2^{-q}k)\, \widehat{u}_k \, e^{ik\cdot x} = \frac{1}{|\mathbb{T}^D|}\int_{\mathbb{T}^D} h_q(y)\, u(x-y)\,\mathrm{d}y \,,$$

and the low-frequency cut-off by

$$S_q u \triangleq \sum_{k\in\mathbb{Z}^N} \chi(2^{-q}k)\, \widehat{u}_k \, e^{ik\cdot x} \,.$$

Obviously, $\Delta_p u = 0$ for negative enough p (depending on the periodic box $\mathbb{T}^N$) and $u = \widehat{u}_0 + \sum_q \Delta_q u$ in $\mathcal{S}'(\mathbb{T}^N)$. The dyadic blocks $\Delta_q u$ are no longer orthogonal in $L^2(\mathbb{T}^N)$ but they still have some properties of quasi-orthogonality: with our choice of φ, we have

$$\Delta_k \Delta_q u \equiv 0 \quad \text{if} \quad |k-q| \ge 2 \quad \text{and} \quad \Delta_k(S_{q-1} u \Delta_q u) \equiv 0 \quad \text{if} \quad |k-q| \ge 4 \,.$$

The Sobolev spaces and Besov spaces can be characterized by means of Littlewood-Paley decomposition:

$$H^s = \left\{ u \in \mathcal{S}'(\mathbb{T}^N) \,:\, \|u\|_{H^s} \triangleq \left(|\widehat{u}_0|^2 + \sum_{q\in\mathbb{Z}} 2^{2sq}\|\Delta_q u\|_{L^2}^2 \right)^{\frac{1}{2}} < +\infty \right\} ,$$

$$B^s_{2,1} = \left\{ u \in \mathcal{S}'(\mathbb{T}^N) \,:\, \|u\|_{B^s_{2,1}} \triangleq |\widehat{u}_0| + \sum_{q\in\mathbb{Z}} 2^{sq}\|\Delta_q u\|_{L^p} < +\infty \right\} .$$

In the rest of this paper, for notational simplicity, we use B^s denote to $B^s_{2,1}$.

We will use the energy methods to the averaged equation which is localized in dyadic Fourier variables according to Littlewood-Paley decomposition. When taking an integrating in time, we get estimates in space $L^r(0,T\,;L^2)$ for *each dyadic block*. This leads to the definition of the following spaces first introduced by Chemin and Lerner in [5].

$$\widetilde{L}^r_T(H^s)=\left\{u\in\mathcal{S}'([0,T]\times\mathbb{T}^N)\,:\,\|u\|_{H^s}\triangleq\left(\|\widehat{u}_0\|^2_{L^r_T(L^2)}+\sum_{q\in\mathbb{Z}}2^{2sq}\|\Delta_q u\|^2_{L^r_T(L^2)}\right)^{\frac{1}{2}}<+\infty\right\}.$$

Note that $\widetilde{L}^r_T(H^s)$ coincides with $L^r(0,T\,;H^s)$ if $r=2$. We further denote $\widetilde{C}(X)=\widetilde{L}^\infty_T(X)\cap C([0,T]\,;X)$, and the spaces where solutions of the averaged system belong to

$$F^s_T=\left\{V\in\widetilde{C}_T(H^{s-1})\,:\,V\in\widetilde{L}^1_T(H^{s+1})\right\}.$$

4.3. **Higher Regularity.** The results of this section closely follow the work of Masmoudi [18] and Danchin [6]. In the last section, we proved global existence in the sense of Leray to the averaged system

$$\partial_t\mathcal{P}^\perp V+\mathcal{Q}_{2r}(\mathcal{P}V\,,\mathcal{P}^\perp V)+\mathcal{Q}_{3r}(\mathcal{P}^\perp V\,,\mathcal{P}^\perp V)=\bar{\mu}\Delta_x\mathcal{P}^\perp V\,,\tag{4.2}$$

for a fixed $\mathcal{P}V=(-\frac{p^o_\varepsilon}{p^o_\rho}\vartheta,w,\vartheta)$ and (w,ϑ) is a global weak solution to the incompressible Navier-Stokes system (3.4).

To prove the global well-posedness of solutions to the averaged system , we need the following a priori estimates.

Lemma 2. *For all $s\geq 0$, we have the identities*

$$\big(\mathcal{Q}_{2r}(\mathcal{P}V\,,\mathcal{P}^\perp V)\,,\mathcal{P}^\perp V\big)_{H^s}=0\,.\tag{4.3}$$

and

$$\big(\Delta_q\mathcal{Q}_{2r}(\mathcal{P}V\,,\mathcal{P}^\perp V),\Delta_q\mathcal{P}^\perp V\big)_{L^2}=0\,.\tag{4.4}$$

Proof. The proof employ the symmetry of $\mathcal{Q}_{2r}(\mathcal{P}V\,,\mathcal{P}^\perp V)$. Noticing that $H^\alpha_k=\overline{H}^\alpha_{-k}$ and that $V^\alpha_k=\overline{V}^\alpha_{-k}$

$$\begin{aligned}&\big(\mathcal{Q}_{2r}(\mathcal{P}V\,,\mathcal{P}^\perp V)\,,\mathcal{P}^\perp V\big)_{H^s}\\ =&i\sum_{\delta,m}\sum_{\substack{k+l=m\\ \alpha\mathrm{sg}(k)=\delta\mathrm{sg}(m)\\ |k|=|m|}}V^\alpha_kV^\delta_{-m}|m|^{2s}\left[\frac{(\widehat{w}_l\cdot m)(k\cdot m)}{|k||m|}+\left(\frac{1}{2}+\frac{1}{2}\frac{p^o_\varepsilon}{\rho^o}\right)\widehat{w}_l\cdot m\right.\\ &\left.-c^o_1\rho^o\sigma^o_\varepsilon\widehat{\theta}_l\delta\,\mathrm{sg}(m)|m|\right].\end{aligned}$$

In above summation, exchange α and δ, and change k to $-m$ and change m to $-k$. Notice that under this changing index, the relation $l = m - k$ is invariant, so

$$\begin{aligned}&\big(\mathcal{Q}_{2r}(\mathcal{P}V\,,\mathcal{P}^\perp V)\,,\mathcal{P}^\perp V\big)_{H^s}\\ =&i\sum_{\delta,m}\sum_{\substack{k+l=m\\ \alpha\mathrm{sg}(k)=\delta\mathrm{sg}(m)\\ |k|=|m|}} V_k^\alpha V_{-m}^\delta |m|^{2s}\Big[-\frac{(\widehat{w}_l\cdot m)(k\cdot m)}{|k||m|}-\Big(\frac12+\frac12\frac{p_\varepsilon^o}{\rho^o}\Big)\widehat{w}_l\cdot k\\ &+c_1^o\rho^o\sigma_\varepsilon^o\widehat{\theta}_l\alpha\,\mathrm{sg}(k)|k|\Big]\,.\end{aligned}$$

Notice that u is divergence-free, so $\widehat{w}_l\cdot m = \widehat{w}_l\cdot k$, then

$$\big(\mathcal{Q}_{2r}(\mathcal{P}V\,\mathcal{P}^\perp V)\,,\mathcal{P}^\perp V\big)_{H^s} = -\big(\mathcal{Q}_{2r}(\mathcal{P}V\,,\mathcal{P}^\perp V)\,,\mathcal{P}^\perp V\big)_{H^s}\,.$$

Then $\big(\mathcal{Q}_{2r}(\mathcal{P}V\,,\mathcal{P}^\perp V)\,,\mathcal{P}^\perp V\big)_{H^s} = 0$. The proof of the identity (4.4) is similar, so we omit it here. □

We already know that $\big(\mathcal{Q}_{3r}(\mathcal{P}^\perp V\,,\mathcal{P}^\perp V)\,,\mathcal{P}^\perp V\big) = 0$, but for general $s > 0$,

$$\big(\mathcal{Q}_{3r}(\mathcal{P}^\perp V\,,\mathcal{P}^\perp V)\,,\mathcal{P}^\perp V\big)_{H^s} \neq 0\,.$$

However, we have the following key estimate for $\mathcal{Q}_{3r}$.

Lemma 3. *For every $V, W \in \mathrm{Null}(\mathcal{A})^\perp$,*

$$(\mathcal{Q}_{3r}(V,W),W) \lesssim \|W\|_{L^2}\|W\|_{B^{\frac12}}\|V\|_{H^1}\,, \tag{4.5}$$

and

$$\|\mathcal{Q}_{3r}(V,W)\|_{H^s} \lesssim \|V\|_{B^{\frac12}}\|W\|_{H^{s+1}} + \|W\|_{B^{\frac12}}\|V\|_{H^{s+1}}\,. \tag{4.6}$$

The proof of this technical lemma is in [6], inspired by [18]. Based on these *a priori* estimates, we can prove the following global well-posedness (in the sense that solution $\mathcal{P}V$ to the incompressible Navier-Stokes equation is fixed.)

Theorem 3. *Let $s \geq 1$, $T \in (0, +\infty]$, $V_0 \in H^{s-1}\cap\mathrm{Null}(\mathcal{A})^\perp$ and $(w,\vartheta) \in F_T^s$ is a fixed solution to the incompressible Navier-Stokes system. Then the averaged system* (4.2) *has a solution $V \in F_T^s$ which remains in $\mathrm{Null}(\mathcal{A})^\perp$ for all time, and uniqueness holds in $C([0,T];L^2)\cap L^2(0,T;H^1)$. The solution V satisfies the energy estimates*

$$\tfrac12\|V(t)\|_{L^2}^2 + \bar\mu\int_0^t \|\nabla_x V(\tau)\|_{L^2}^2\,\mathrm{d}\tau \leq \tfrac12\|V_0\|_{L^2}^2\,, \tag{4.7}$$

and

$$\|V\|^2_{\widetilde{L}^\infty_T(H^{s-1})} + c\bar\mu\|V\|^2_{\widetilde{L}^1_T(H^{s+1})} \leq \|V_0\|^2_{H^{s-1}}\exp\big(C\tfrac{\|V_0\|_{L^2}^2}{\bar\mu^2}\big)\,. \tag{4.8}$$

Proof. Given the above technical lemma, the proof of the theorem is standard. We first prove the a priori estimates, i.e, any solutions $V \in F_T^s$ satisfies the energy estimates (4.7) and (4.8). Take the H^{s-1} inner product of the averaged system with V. Since, according to (4.3), $(\mathcal{Q}_{2r}(\mathcal{P}U,V),V)_{H^{s-1}} = 0$, we obtain

$$\tfrac12\frac{\mathrm{d}}{\mathrm{d}t}\|V\|_{H^{s-1}}^2 + \bar\mu\|\nabla_x V\|_{H^{s-1}}^2 + (\mathcal{Q}_{3r}(V,V),V)_{H^{s-1}} = 0\,. \tag{4.9}$$

If $s = 1$, the last term vanishes, so one time integration yields (4.7).

When $s > 1$, the inequality (4.6) and Young inequality and embedding $H^1 \hookrightarrow B^{\frac{1}{2}}$ yield

$$|\,(\mathcal{Q}_{3r}(V,V)\,,V)_{H^{s-1}}\,| \leq \tfrac{\bar{\mu}}{2}\|V\|_{H^s}^2 + \tfrac{C}{\bar{\mu}}\|V\|_{H^1}^2\|V\|_{H^{s-1}}^2\,.$$

Plug this above inequality into (4.9) and take integration in time, we get

$$\tfrac{1}{2}\|V(t)\|_{H^{s-1}}^2 + \frac{\bar{\mu}}{2}\int_0^T \|\nabla_x V(\tau)\|_{H^s}^2\,\mathrm{d}\tau \leq \tfrac{1}{2}\|V_0\|_{H^{s-1}}^2 + \frac{C}{\bar{\mu}}\int_0^T \|V(\tau)\|_{H^1}^2\|V(\tau)\|_{H^s}^2\,\mathrm{d}\tau\,.$$

The Gronwell inequality yields

$$\tfrac{1}{2}\|V(t)\|_{H^{s-1}}^2 + \frac{\bar{\mu}}{2}\int_0^T \|\nabla_x V(\tau)\|_{H^s}^2\,\mathrm{d}\tau \leq \tfrac{1}{2}\|V_0\|_{H^{s-1}}^2 e^{\frac{C}{\bar{\mu}}\int_0^T \|V(\tau)\|_{H^1}^2\,\mathrm{d}\tau}\,.$$

Once the a priori estimates (4.7) and (4.8) have been proved, we can use the classical type of regularization, for instance, one can use a Galerkin approximation method as we did in the last section when we proved the global Leray-type solution, to get the existence of a solution to the averaged system (4.2) in $C([0,T];H^{s-1}) \cap L^2(0,T;H^s)$.

Next, we show that any solution $V \in C([0,T];H^{s-1}) \cap L^2(0,T;H^s)$ satisfies the estimate (4.8) which includes the bounds for the L^1 norm in time. Applying the Δ_q to (4.2), taking the L^2 product with $\Delta_q V$ yields

$$\tfrac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\Delta_q V\|_{L^2}^2 + \frac{\bar{\mu}}{2}\|\nabla_x\Delta_q V\|_{L^2}^2 + (\Delta_q\mathcal{Q}_{3r}(V,V),\Delta_q V)_{L^2} = 0\,.$$

Taking integral in time, then multiplying by $2^{q(s-1)}$ yields

$$2^{q(s-1)}\|\Delta_q V\|_{L_t^\infty(L^2)} + c\bar{\mu}2^{q(s+1)}\|\Delta_q V\|_{L_t^1(L^2)} \leq 2^{q(s-1)}\|\Delta V_0\|_{L^2} + 2^{q(s-1)}\|\Delta_q\mathcal{Q}_{3r}(V,V)\|_{L_t^1(L^2)}\,,$$

Taking summation in q, we have

$$\|V\|_{\widetilde{L}_T^\infty(H^{s-1})}^2 + c^2\bar{\mu}^2\|V\|_{\widetilde{L}_T^1(H^{s+1})}^2 \leq 2\left(\|V_0\|_{H^{s-1}}^2 + \|\mathcal{Q}_{3r}(V,V)\|_{\widetilde{L}_T^1(H^{s-1})}^2\right)\,.$$

From (4.6), and noticing the embedding $L_t^1(H^{s-1}) \hookrightarrow \widetilde{L}_t^1(H^{s-1})$ and $H^1 \hookrightarrow B^{1/2}$, we have

$$\|\mathcal{Q}_{3r}(V,V)\|_{\widetilde{L}_T^1(H^{s-1})} \leq \|V\|_{L_t^2(H^1)}\|V\|_{L_t^2(H^s)}\,.$$

Thus we proved (4.8). □

4.4. **Uniqueness of Weak Solutions.** Let us now consider the uniqueness of the weak solutions to averaged system in $C([0,T];L^2) \cap L^2(0,T;H^1)$. The property is not known for the incompressible Navier-Stokes equations for dimension $\mathrm{D} \geq 3$. This means weak solutions to the averaged system in $\mathrm{Null}(\mathcal{A})^\perp$ have better properties.

Proof. Let V_1 and V_2 be two weak solutions of the averaged system (4.2) in $C([0,T];L^2) \cap L^2(0,T;H^1)$. Then $\delta V = V_1 - V_2$ satisfies

$$\partial_t\delta V - \bar{\mu}\Delta\delta V + \mathcal{Q}_{2r}(\mathcal{P}V,\delta V) = -\mathcal{Q}_{3r}(V_1+V_2,\delta V)\,.$$

Take inner product for the above equation with δV, and notice the identity for $\mathcal{Q}_{2r}$, we obtain

$$\frac{1}{2}\frac{d}{dt}\|\delta V\|_{L^2}^2 + \bar{\mu}\|\nabla_x\delta V\|_{L^2}^2 = (\mathcal{Q}_{3r}(V_1+V_2,\delta V),\delta V)\,.$$

Now apply the inequality (4.5) and Young inequality, we have

$$|\,(\mathcal{Q}_{3r}(V_1+V_2,\delta V),\delta V)\,| \leq \tfrac{C}{\bar{\mu}}\,\|V_1+V_2\|_{H^1}^2\|\delta V\|_{L^2}^2 + \bar{\mu}\|\nabla_x\delta V\|_{L^2}^2\,.$$

Then the Gronwell inequality ensures that $\delta V \equiv 0$. Thus we prove the uniqueness. □

We state the well-posedness of the averaged system in Besov space B^s. The proof is similar as for the Sobolev space, thus we omit the proof here.

Theorem 4. *Let $s \geq 1$, $T \in (0, +\infty]$, $V_0 \in B^{s-1} \cap \mathrm{Null}(\mathcal{A})^\perp$ and $(u, \vartheta) \in L^1([0,T]\,; B^{s+1}) \cap C([0,T]\,; B^{s-1})$ is a fixed solution to the incompressible Navier-Stokes equation. Then the averaged system (4.2) has a unique solution $V \in L^1([0,T]\,; B^{s+1}) \cap C([0,T]\,; B^{s-1})$ which remains in $\mathrm{Null}(\mathcal{A})^\perp$ for all time. The solution V satisfies the energy estimate*

$$\|V(t)\|^2_{B^{s-1}} + c\bar{\mu}\|V\|^2_{L^1_T(B^{s+1})} \leq \|V_0\|^2_{B^{s-1}} \exp\big(C\frac{\|V_0\|^2_{L^2}}{\bar{\mu}^2}\big)\,.$$

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(N. Jiang) Department of Mathematics & Center for Scientific Computation and Mathematical Modeling (CSCAMM), University of Maryland, College Park, MD 20742
Current address: Courant Institute of Mathematical Sciences, 251 Mercer Street, New York, NY 10012
E-mail address: `njiang@cims.nyu.edu`

(C.D. Levermore) Department of Mathematics & Institute for Physical Science and Technology (IPST), University of Maryland, College Park, MD 20742
E-mail address: `lvrmr@math.umd.edu`