Variational formulation of entropy solutions for nonlinear conservation laws

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SIAM Invited Address

Outline

1 Entropy and dissipation
   • Equations of mass, momentum, energy
   • Conservation laws of mass momentum and energy
   • Dissipation of entropy

2 Hyperbolic conservation laws
   • Entropic solutions

3 Entropy and symmetry
   • Entropy and stability

4 Entropy at the kinetic level
   • H-Theorem
   • Maximum entropy principle

5 A new variational formulation
Equations of mass and momentum

Hydrodynamics: Euler (1755-1757)

\[ \rho_t + \nabla_x \cdot (\rho \mathbf{v}) = 0 \]
\[ \mathbf{v}_t + \mathbf{v} \cdot \nabla_x \mathbf{v} = -\frac{1}{\rho} \nabla_x \rho \]
Equations of mass and momentum

- Hydrodynamics: Euler (1755-1757)

\[
\rho_t + \nabla \cdot (\rho \mathbf{v}) = 0
\]
\[
\mathbf{v}_t + \mathbf{v} \cdot \nabla \mathbf{v} = -\frac{1}{\rho} \nabla p
\]

- Thermodynamics:
  - density \( \rho \), pressure \( p \), temperature \( T \), internal energy \( e \), ...
  - First law of thermodynamics: work is converted into gained heat \( \rightsquigarrow e \)
  - Single-phase systems: equation of state link any three \( p = f(\rho, e) \rho e \)

Émilie du Châtelet (1749) Helmholtz (1847)

\[
e_t + \mathbf{v} \cdot \nabla e = -\frac{p}{\rho} \nabla \cdot \mathbf{v}
\]

1 von Mayer (1841)
Conservation of mass, momentum, energy

\[
\rho_t + \nabla \cdot (\rho \mathbf{v}) = 0
\]
\[
(\rho \mathbf{v})_t + \nabla \cdot (\mathbf{v} \otimes (\rho \mathbf{v})) + \nabla p = 0
\]
\[
E_t + \nabla \cdot (\mathbf{v}(E + p)) = 0
\]

No 100% energy efficiency

Viscosity \( \Sigma = \mu \left( \nabla \mathbf{v} + \nabla \mathbf{v}^T \right) + \lambda \nabla \cdot \mathbf{v} \mathbb{I} \) and heat conduction \( \nabla T \)

Entropy \( S = S(\rho, p) = \rho \ln(p \rho^{-\gamma}) \) increases

\[
S_t + \nabla \cdot (\rho S - \kappa \nabla \ln T) = \kappa \frac{\left| \nabla T \right|^2}{T^2} + \frac{2\mu \mathrm{tr}(D^2(\mathbf{v})) + \lambda (\nabla \cdot \mathbf{v})^2}{T} \geq 0
\]
Dissipation of entropy

• Second law of thermodynamics — irreversibility

\[
\begin{align*}
\text{"It is impossible to construct a device that operates in a cycle and produces no effect other than the transfer of heat from a cooler body to a hotter body"}
\end{align*}
\]

\[\sim \text{ Complete conservation of heat into work is impossible} \]

Does lack of conservation vanish with vanishing dissipation \( \kappa, \mu, \lambda \to 0? \)

\[
S_t + \nabla_x \cdot (vS) = \frac{2\mu \text{tr}(D^2(v)) + \lambda(\nabla_x \cdot v)^2}{T} \geq 0, \quad D(v) = \frac{\nabla_x v + \nabla_x v^T}{2}
\]

• Traveling viscous wave: \( v^e(x, t) \simeq w \left( \frac{x \cdot n - st}{\epsilon}, \xi_-, t \right) \) with speed \( s: \)

\[
-sS_{\xi_1} + v \cdot n S_{\xi_1} \simeq \frac{|w_{\xi_1}|^2}{T} \text{ independent of } \epsilon = 2\mu + \lambda \downarrow 0
\]

• Rate of increase of entropy does not go to zero

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   • Entropic solutions

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   • H-Theorem
   • Maximum entropy principle

5. A new variational formulation
Euler equations as hyperbolic system of conservation laws

- Conservation laws: \( \mathbf{u}_t + \nabla_x \cdot \mathbf{f}(\mathbf{u}) = 0 \):
  \[
  \text{Conservation: } \int_{\Omega} \mathbf{u}(\cdot, t) \, d\mathbf{x} = - \int_{\partial \Omega} \mathbf{f} \cdot \mathbf{n} \, d\omega, \quad \text{flux } \mathbf{f} = (f^1, \ldots, f^d)
  \]
  \[
  \text{Hyperbolicity: } \mathbf{u}_t + \sum_j A_j(\mathbf{u}) \mathbf{u}_{x_j} = 0, \quad A_j(\mathbf{u}) = \frac{\partial f_j(\mathbf{u})}{\partial \mathbf{u}} \text{ symmetrizable}
  \]

- Symmetric hyperbolic systems: \( \mathbf{u}(x, t): \mathbb{R}^d_x \times \mathbb{R}_t \mapsto \mathbb{R}^m \)
  \[
  \mathbf{u}_t + \sum_{j=1}^d A_j(\mathbf{u}) \mathbf{u}_x^j = 0 \quad \text{such that the } A_j \text{'s are symmetrizable: } \exists H > 0 : \quad HA_j = (HA_j)^T
  \]

- Primary example — Euler eqs for the conserved variables \( \mathbf{u} := \begin{bmatrix} \rho \\ \rho \mathbf{v} \\ E \end{bmatrix} \):
  \[
  f^j(\mathbf{u}) = \begin{bmatrix} \rho v_j \\ \rho v_j \mathbf{v} + p \delta_{ij} \\ v_j(E + p) \end{bmatrix} \quad A_j(\mathbf{u}) \sim \mathbf{v}_j I_{5 \times 5} + c \begin{bmatrix} 0 & \mathbf{e}_j^\top & 0 \\ \mathbf{e}_j & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad c = \sqrt{\frac{\gamma p}{\rho}}
  \]

Basic facts on nonlinear conservation laws

- Spontaneous formation of shock discontinuities \( \rightsquigarrow \)
  \[
  \text{weak solutions: } \int_{\Omega \times \mathbb{R}_t} \langle \psi, \mathbf{u}_t + \nabla_x \cdot \mathbf{f}(\mathbf{u}) \rangle \, d\mathbf{x} dt = 0, \quad \psi \in C_0^\infty
  \]

- Among the many weak solutions: solution realized as \( \rightsquigarrow \)
  \[
  \text{vanishing viscosity limit: } \mathbf{u} = \lim \mathbf{u}^\epsilon, \quad \mathbf{u}^\epsilon_t + \nabla_x \cdot \mathbf{f}(\mathbf{u}^\epsilon) = \epsilon \Delta_x \mathbf{u}^\epsilon
  \]

- \( d = 1 \): Bianchini & Bressan (2005)\(^1\)

- \( d > 1 \): P. D. Lax (2007)
  "There is no theory for the initial value problem for compressible flows in two space dimensions once shocks show up, much less in three space dimensions. This is a scientific scandal and a challenge."\(^2\)

- Vanishing viscosity solutions are entropic solutions \( \rightsquigarrow \)

\(^1\)For initial data \( \|\mathbf{u}_0\|_{BV} \ll 1; \quad ^2\)SIAM mini-symposium on entropic solutions...1pm
Entropy in nonlinear conservation laws

- $u_t + \nabla x \cdot f(u) = 0$ admits an extension: $\eta(u)_t + \nabla x \cdot q(u) = 0$

- An entropy pair $(\eta, q)$:
  $$\begin{align*}
  (\eta'(u))^T A_j(u) &= (q'_j(u))^T, \\
  q &= (q_1, \ldots, q_d)
  \end{align*}$$

  \text{NOTATION: } (\cdot)' = \frac{\partial(\cdot)}{\partial u}, \text{ e.g., } (\eta'(u))^T = (\eta u_1, \ldots, \eta u_d)$

- Lax entropy condition (1971)
  \text{for all convex } \eta \text{'s: } \eta(u)_t + \nabla x \cdot q(u) \leq 0$

- Vanishing viscosity limits: $u^\epsilon_t + \nabla x \cdot f(u^\epsilon) = \epsilon \Delta x u^\epsilon$

  \text{For convex } \eta: \langle \eta'(u^\epsilon), \Delta x u^\epsilon \rangle \equiv \Delta x \eta(u^\epsilon) - \langle \nabla x u^\epsilon, \eta''(u^\epsilon) \nabla x u^\epsilon \rangle \leq \Delta x \eta(u^\epsilon)$

  $$0 = \langle \eta'(u^\epsilon), u^\epsilon_t + \nabla x \cdot f(u^\epsilon) - \epsilon \Delta u^\epsilon \rangle$$

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Variational formulation of entropy solutions for nonlinear conservation laws
Entropy in nonlinear conservation laws

- \( \mathbf{u}_t + \nabla_x \cdot f(\mathbf{u}) = 0 \) admits an extension: \( \eta(\mathbf{u})_t + \nabla_x \cdot \mathbf{q}(\mathbf{u}) = 0 \)

- An entropy pair \((\eta, \mathbf{q})\):
  \[
  (\eta'(\mathbf{u}))^\top A_j(\mathbf{u}) = (q'_j(\mathbf{u}))^\top, \quad \mathbf{q} = (q_1, \ldots, q_d)
  \]

- \( \langle \eta'(\mathbf{u}), \mathbf{u}_t + \nabla_x \cdot f(\mathbf{u}) \rangle = \eta(\mathbf{u})_t + \sum_j \eta'(\mathbf{u}) A_j(\mathbf{u}) u_{xj} = \eta(\mathbf{u})_t + \nabla_x \cdot \mathbf{q}(\mathbf{u}) \)

- Lax entropy condition (1971)
  for all convex \(\eta\)'s: \( \eta(\mathbf{u})_t + \nabla_x \cdot \mathbf{q}(\mathbf{u}) \leq 0 \)

- Vanishing viscosity limits: \( \mathbf{u}^\epsilon_t + \nabla_x \cdot f(\mathbf{u}^\epsilon) = \epsilon \Delta_x \mathbf{u}^\epsilon \)

  For convex \(\eta\):
  \[
  \langle \eta'(\mathbf{u}^\epsilon), \Delta_x \mathbf{u}^\epsilon \rangle \equiv \Delta_x \eta(\mathbf{u}^\epsilon) - \langle \nabla_x \mathbf{u}^\epsilon, \eta''(\mathbf{u}^\epsilon) \nabla_x \mathbf{u}^\epsilon \rangle \leq \Delta_x \eta(\mathbf{u}^\epsilon)
  \]

  \[
  0 = \langle \eta'(\mathbf{u}^\epsilon), \mathbf{u}^\epsilon_t + \nabla_x \cdot f(\mathbf{u}^\epsilon) \rangle - \epsilon \Delta \mathbf{u}^\epsilon = \eta(\mathbf{u}^\epsilon)_t + \nabla_x \cdot \mathbf{q}(\mathbf{u}^\epsilon)
  \]
Entropy in nonlinear conservation laws

- \( \mathbf{u}_t + \nabla \cdot \mathbf{f}(\mathbf{u}) = 0 \) admits an extension: \( \eta(\mathbf{u})_t + \nabla \cdot \mathbf{q}(\mathbf{u}) = 0 \)

- An entropy pair \((\eta, \mathbf{q})\):

\[
\langle \eta'(\mathbf{u}), \mathbf{u}_t + \nabla \cdot \mathbf{f}(\mathbf{u}) \rangle = \eta(\mathbf{u})_t + \sum_j \eta'(\mathbf{u}) A_j(\mathbf{u}) u_{xj} = \eta(\mathbf{u})_t + \nabla \cdot \mathbf{q}(\mathbf{u})
\]

- Lax entropy condition (1971)

  for all convex \(\eta\)'s: \(\eta(\mathbf{u})_t + \nabla \cdot \mathbf{q}(\mathbf{u}) \leq 0\)

- Vanishing viscosity limits: \(\mathbf{u}_t^\varepsilon + \nabla \cdot \mathbf{f}(\mathbf{u}^\varepsilon) = \varepsilon \Delta \mathbf{u}^\varepsilon\)

For convex \(\eta\):

\[
0 = \langle \eta'(\mathbf{u}^\varepsilon), \Delta \mathbf{u}^\varepsilon \rangle \equiv \Delta \eta(\mathbf{u}^\varepsilon) - \langle \nabla \mathbf{u}^\varepsilon, \eta''(\mathbf{u}^\varepsilon) \nabla \mathbf{u}^\varepsilon \rangle \leq \Delta \eta(\mathbf{u}^\varepsilon)
\]

- Vanishing viscosity limits \(\mathbf{u} = \lim \mathbf{u}^\varepsilon\) are entropic: \(\eta(\mathbf{u})_t + \nabla \cdot \mathbf{f}(\mathbf{u}) \leq 0\)

Entropic solutions for nonlinear conservation laws

- Krushkov (1970), Lax (1971): entropy pair \((\eta'(\mathbf{u}))^\top A_j(\mathbf{u}) = (q_j'(\mathbf{u}))^\top\)

\(\mathbf{u}\) is an entropic solution of \(\mathbf{u}_t + \nabla \cdot \mathbf{f}(\mathbf{u}) = 0\)

- for all convex \(\eta\)'s: \(\eta(\mathbf{u})_t + \nabla \cdot \mathbf{q}(\mathbf{u}) \leq 0\)

- Linear \(\eta\)'s: \(\eta(\mathbf{u}) = \pm u_k \mapsto q_j(\mathbf{u}) = \pm f^j_k : \quad \mapsto \mathbf{u}_t + \nabla \cdot \mathbf{f}(\mathbf{u}) = 0\)

- Nontrivial extension for strictly convex \(\eta\)'s \(\mapsto\) decrease of entropy

- Primary example: from Naiver-Stokes to Euler equations

specific entropy \(s = s(\rho, p) = \ln(p \rho^{-\gamma})\)

\[
s_t + \mathbf{v} \cdot \nabla s = 0 - \kappa \Delta \ln T = \kappa \frac{|
abla T|^2}{T^2} + \frac{2\mu \text{tr}(D^2(\mathbf{v})) + \lambda (\nabla \cdot \mathbf{v})^2}{T} \geq 0
\]

- Convex entropy functions\(^1,^2\): \((\eta, \mathbf{q}) = (-\rho h(s), -\rho \mathbf{v} h(s)), \quad \dot{h} - \gamma \ddot{h} > 0\)

- Physical entropy is concave: \(S(\mathbf{u}) = \rho s\) increases \(\mapsto h(s) = -s\)

\[
S = \rho s : \quad S_t + \nabla \cdot (\mathbf{v} S) \geq 0
\]

Key questions

- Entropic systems of conservation laws

\[
\begin{align*}
\begin{cases}
    u_t + \nabla \cdot f(u) &= 0, \\
    \eta(u)_t + \nabla \cdot q(u) &\leq 0, \quad \forall \text{ convex entropy pairs } (\eta, q)
\end{cases}
\end{align*}
\]

- The notion of entropy — imposing the compatibility requirement

\[
(\eta, q) = (\eta(u), q(u)) \text{ is an entropy pair:}
\]

\[
(q_j'(u))^\top = (\eta'(u))^\top A_j(u) \quad A_j(u) = \frac{\partial f_j(u)}{\partial u}
\]

- Questions:

  What does (\star) really mean?

  How rich is the class of systems endowed with such entropy pairs?

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Entropy and symmetry

- One-dimensional systems: \( u_t + f(u)_x = 0, \quad f = (f_1, \ldots, f_m) \)

\[
(\eta, q) = (\eta(u), q(u)) \text{ is an entropy pair:}
\]

\[
(q'(u))^\top = (\eta'(u))^\top A(u) \quad A_{kj}(u) = \frac{\partial f_k(u)}{\partial u_j}
\]

\[
\frac{\partial q}{\partial u_j} = \sum_k \frac{\partial \eta}{\partial u_k} \frac{\partial f_k}{\partial u_j}
\]

- An entropy Hessian \( \eta'' \) symmetrizes \( A \): \( q'' = \eta'' A + T \), \( T_{ij} = T_{ji} \)

- Conversely: if \( \eta'' A \) is symmetric: \( \exists q \) such that

\[
\sum_k \frac{\partial \eta}{\partial u_k} \frac{\partial f_k}{\partial u_j} = \frac{\partial q}{\partial u_j}
\]

Friedrichs & Lax (1971): Entropy symmetrizer \( H(u) := \eta''(u) \)

\[
u_t + \nabla_x \cdot f(u) = 0 \quad \Rightarrow \quad H u_t + \sum_j H A_j u_{x_j} = 0, \quad H A_j = \text{symmetric}
\]
Entropy and symmetry \( \bullet \) \( u_t + \nabla_x \cdot f(u) = 0 \)

- Sever-Mock (1980)
- If \( \eta(u) \) is a convex entropy of \( \bullet \):
  - Then \( \bullet \) is symmetric w.r.t. the entropy variables \( w := \eta'(u) \)

\[ u_t + \nabla_x \cdot f(u) = 0 \iff \text{symmetric} \ u(w)_t + \nabla_x \cdot f(u(w)) = 0 \]

\[ u(w)_t + \nabla_x \cdot f(u(w)) = \frac{\partial u}{\partial w} w_t + \sum_j \frac{\partial f_j}{\partial u} \frac{\partial u}{\partial w} w_{x_j} = 0 \]

Recall the left symmetrizer \( H = \eta''(u) = \frac{\partial w}{\partial u} : \ HA_j \) are symmetric

"Symmetrization on the right": \( \frac{\partial u}{\partial w} = H^{-1} > 0 \) and \( A_j H^{-1} \) symmetric

- Godunov (1961)
  - If \( \bullet \) is symmetrizable under change of variables:
    - Then the system \( u_t + \nabla_x \cdot f(u) = 0 \) has an entropy

Example (ET 1986). Euler eqs: \( w = -((p \rho)^{-\gamma + 1} \times (E, -\rho v, \rho) \top) \)

So how rich are entropic systems?

- Scalar equations \( (m = 1) \): all convex \( \eta \)'s are entropy functions
- 2 \( \times \) 2 systems \( (m = 2) \): \( \eta''(u_1, u_2) \frac{\partial f(u_1, u_2)}{\partial (u_1, u_2)} \) is symmetric\(^1\):

\[ \eta \in \mathcal{F}_\eta : \ \eta_{u_1} \frac{\partial f_1}{\partial u_2} + \eta_{u_2} \frac{\partial f_2}{\partial u_2} = \eta_{u_1} \frac{\partial f_1}{\partial u_1} + \eta_{u_2} \frac{\partial f_2}{\partial u_1} \]

- There are \( m \times m \) entropic "rich systems"; Temple class\(^2\)
- But general \( m \geq 3 \)-systems are non-entropic:
  - \( m(m - 1)/2 \) symmetry constraints of \( \eta''(u) \frac{\partial f}{\partial u} \) is over-determined

Example: \( u_t + \begin{bmatrix} u_2 \\ u_3 \\ u_1 \end{bmatrix} u_x = 0 \) has no conservative extension

- Many "physically relevant systems" are endowed with entropy functions:
  - Euler eqs, MHD eqs, elasticity eqs, shallow-water eqs, ...

\(^1\)Lax, DiPerna, G.-Q. Chen,... \(^2\)Serre, Temple,...
Entropy and stability

- Kruzkov (1970) — $L^1$-stability theory for scalar eqs ($m = 1$)
  \[ \eta(u; c) = |u - c| : \int_{\mathbb{R}^d} \eta(u_2(x, t); u_1(x, t)) \, dx \leq \int_{\mathbb{R}^d} \eta(u_20(x); u_{10}(x)) \, dx \]

- Tartar (1979), DiPerna (1983): $m = 2$ with $\eta \in \mathcal{F}$
  \[ \eta(u^\epsilon_t) + q(u^\epsilon_x) \hookrightarrow H^{-1}_{x,t} \rightsquigarrow \{u^\epsilon\} \text{ } L^2\text{-compact} \]

- Dafermos (1979), DiPerna (1979) — relative entropies
  \[ \eta(u; c) = \eta(u) - \eta(c) - \eta'(c) \cdot (u - c) \sim |u - c|^2 \]

Entropy stable difference schemes: ET (1987)

- \[ \frac{d}{dt} u_\nu(t) + \frac{1}{\Delta x} \left( f_{\nu+\frac{1}{2}} - f_{\nu-\frac{1}{2}} \right) = \Delta x D_{\Delta x} \left( Q_{\nu-\frac{1}{2}} D_{-\Delta x} w_{\nu} \right) : Q \geq Q^* \]

Key questions - revisited

- Entropic systems of conservation laws
  \[
  \begin{cases}
    u_t + \nabla_x \cdot f(u) = 0, \\
    \eta(u)_t + \nabla_x \cdot q(u) \leq 0, \quad \forall \text{ convex entropies } \eta \text{ (and compatible flux } q) \n  \end{cases}
  \]

- The notion of entropy — imposing the symmetry requirement
  \[ \eta = \eta(u) \text{ is an entropy:} \]
  \[ \eta''(u) A_j(u) = (\eta''(u) A_j(u))^\top, \quad A_j(u) = \frac{\partial f_j(u)}{\partial u} \]

- Questions:
  Why imposing the symmetry requirement ($\star$)?
  Where do entropy symmetrizers come from?
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"Physically relevant systems" — kinetic description

Hydrodynamic description: $\rho, v$

\[
\begin{align*}
\rho_t + \nabla_x \cdot (\rho v) &= 0 \\
v_t + v \cdot \nabla_x v &= -\frac{1}{\rho} \nabla_x P
\end{align*}
\]

$\uparrow$ Human scale: $\varepsilon \to 0$, $(t' = \varepsilon t, x' = \varepsilon x)$

Kinetic description: $f(x, \xi, t)$

\[
f_t + \xi \cdot \nabla_x f = \frac{1}{\varepsilon} Q(f, f)
\]

$\uparrow$ $N \to \infty$ (including meso-scale phenomena)

Particle description: $\{x_i, \xi_i\}_{1 \leq i \leq N}$

\[
\begin{align*}
\dot{x}_i &= \xi_i \\
\dot{\xi}_i &= F_i, \quad F_i \mapsto F(x_i, \xi_i)(x, \xi)
\end{align*}
\]
Entropy at the kinetic level: the $H$-theorem

- Boltzmann equation (1872): \( f_t + \xi \cdot \nabla_x f = Q(f, f) \)

- Conservation of \[
\begin{align*}
\begin{cases}
\text{mass}\ldots\ldots \begin{bmatrix} \rho \\ \rho v \end{bmatrix} & = \int_{\mathbb{R}^3} \begin{bmatrix} 1 \\ \xi \\ 1/2|\xi|^2 \end{bmatrix} f(x, \xi, t) d\xi \\
\text{momentum} & \\
\text{energy}\ldots..& 
\end{cases}
\end{align*}
\]

- Entropy \( S = k \log W \sim -\frac{1}{N} \log W = -\sum f_i \log f_i \):
  \[ W = \# \text{ of microscopic states compatible with the observed macroscopic states} \]

- Maxwell (1872):
  \[
  f \approx M_{\rho, v, T}(x, \xi, t) = \frac{\rho}{(2\pi T)^{d/2}} e^{-\frac{|v-\xi|^2}{2T}}
  \]

  \[
  S = -\int M \log M \, d\xi = \rho \ln(\rho T^{-d/2}) : \quad S_t + \nabla_x \cdot (vS) \geq 0
  \]
Maximum Entropy Principle (MEP)

- Clausius (1865), Gibbs (1875,...,1902):
  Entropy is maximized for a given expectation of energy

- E.T. Jaynes (1957)
  Out of the many possibilities —
  choose the one w/little structure as possible (MaxEnt)

- Shannon (1948): entropy as a measure of uncertainty:
  The only such measure which is increasing and additive:
  \[ S(p_1, \ldots, p_N) = - \sum_i p_i \log p_i \]

maximize entropy subject to observables with unknown probabilities

\[
\max_{\{p_i\}} - \sum p_i \log p_i : \sum p_i E_\mu(\xi_i) \quad \sum p_i = 1
\]

\[ p(\xi) = \frac{e^{-\sum E_\mu(\xi)/T_\mu}}{Z(T)} = \frac{e^{-E(\xi)/k_B T}}{Z(T)} \]

w/ partition \( Z(T) = \sum_{\xi} e^{-\sum_\mu E_\mu/T_\mu} \)

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Variational formulation of entropy solutions for nonlinear conservation laws

Key questions — summary

- Entropic systems of conservation laws
  1. Conservation law: \( u_t + \nabla_x \cdot f(u) = 0 \), symmetry requirement
  2. Entropy condition: \( \eta(u)_t + \nabla_x \cdot q(u) \leq 0 \),
  \( \eta'' A_j = (\eta'' A_j)^\top \)

- Minimum Entropy Principle (for a convex entropy \( \eta \))
  3. MEP: \( \# u(\cdot, t_1) \mapsto u(\cdot, t) \min \int_{\tau=t_1}^{t_2} \int_{\Omega} \eta(u(x, \tau)) \, dx \, d\tau \)

  Maximum Entropy principle \( \leftrightarrow \) concave \( \eta \)

- Imposing the symmetry requirement:
  
  “Now, in many branches of physics .. symmetries play a fundamental role, but all these symmetries—as it seems to me—are assumed and not derived.
  I now wonder whether or not .. symmetries can also be derived.”

  K. O. Friedrichs, 1979 John von Neumann Lecture
A new variational formulation of $(\star)$ $u_t + \nabla_x \cdot f(u) = 0$

- $\eta \in C^1_{0, \text{convex}}, u \in BV(\mathbb{R}^d_x \times \mathbb{R}^+_t \mapsto \mathbb{R}^m)$ is a variational solution of $(\star)$:
  \[
  \text{A critical point of } J(\eta, u) := \int_{t_1}^{t_2} \int_{\Omega} \langle \eta'(u), u_t + \nabla_x \cdot f(u) \rangle \phi \, dx \, dt
  \]

- Original $(m \times m)$-system is embedded in $m + 1$ dimensional phase space
- BV framework for non-conservative products\(^{1,2}\) $\langle \cdot, \cdot \rangle_\phi$
  \[
  u \in BV, a(u) \in L^\infty : \langle a(u), u_x \rangle_\phi := \text{Borel measure } \mu \text{ such that}
  \mu(B) = \int_B \langle a(u), u_x \rangle dx, \quad u \in C(B)
  \]
  \[
  \langle \mu(x_0) \rangle_\phi := \int_{s=0}^{1} \langle a(u(s)), \dot{u}(s) \rangle ds, \quad u(s) = u(\phi(s)) : [0, 1] \mapsto [u_0-, u_0+]
  \]
  "Integration by parts" — Gauss-Green holds
  \[
  \langle A(u)u_x \rangle_\phi \text{ independent of path } \phi \text{ iff } A(u) = \frac{\partial f}{\partial u} : \langle A(u)u_x \rangle_\phi = f_x
  \]

\(^{1}\text{Volpert (1967)}\) \(^{2}\text{DelMaso, LeFloch, Murat (1995)}\)
A new variational formulation of (⋆) \( u_t + \nabla_x \cdot f(u) = 0 \)

- \( \eta \in C^1_{0,\text{convex}}, u \in BV(\mathbb{R}^d_x \times \mathbb{R}^+_t \mapsto \mathbb{R}^m) \) is a variational solution of (⋆):

\[
A \text{ critical point of } J(\eta, u) := \int_{t_1}^{t_2} \int_{\Omega} \langle \eta'(u), u_t + \nabla_x \cdot f(u) \rangle \phi \, dx \, dt
\]

- Original \((m \times m)\)-system is embedded in \(m + 1\) dimensional phase space variation in \(u\)-variable: 

\[
\delta u J(\eta, u) = \langle \delta(\eta(u))', u_t + \nabla_x \cdot f(u) \rangle_\phi = 0
\]

\[
\Rightarrow \quad \text{for all admissible } \delta\eta(\cdot): \langle \delta(\eta(u))', u_t + \nabla_x \cdot f(u) \rangle_\phi = 0
\]

- THEOREM. A variational solution is a weak solution of (⋆)

- Renormalized solution (a la DiPerna-Lions (1989))

- Different paths \(\phi\) “encode” different vanishing viscosity limits

Original \((m \times m)\)-system is embedded in \(m + 1\) dimensional phase space variation in \(\eta\)-variable:

\[
\delta \eta J(\eta, u) = \langle \eta'(u), u_t + \nabla_x \cdot f(u) \rangle_\phi = 0
\]
A new variational formulation of \((\star)\) \(\mathbf{u}_t + \nabla_x \cdot \mathbf{f}(\mathbf{u}) = 0\)

- \(\eta \in C^1_{0, \text{convex}}, \mathbf{u} \in BV(\mathbb{R}^d_x \times \mathbb{R}^+_t \mapsto \mathbb{R}^m)\) is a variational solution of \((\star)\):

A critical point of 
\[
\mathcal{J}(\eta, \mathbf{u}) := \int_{t_1}^{t_2} \int_{\Omega} \langle \eta'(\mathbf{u}), \mathbf{u}_t + \nabla_x \cdot \mathbf{f}(\mathbf{u}) \rangle \phi \, dx \, dt
\]

- Original \((m \times m)\)-system is embedded in \(m + 1\) dimensional phase space variation in \(\mathbf{u}: \delta \mathbf{u} \mathcal{J}(\eta, \mathbf{u}) = \langle \eta'(\mathbf{u}), (\delta \mathbf{u})_t + \sum \partial_{x_j}(A_j(\mathbf{u})\delta \mathbf{u}) \rangle \phi = 0\)

\[
\langle \eta'' \sum_j A_j u_{x_j} - \sum_j A_j^\top(\mathbf{u}) (\eta'(\mathbf{u}))_x \delta \mathbf{u} \rangle \phi = 0
\]

Derivation of an entropy \(\eta:\)

- \(\eta''\) symmetrize \(A_j: \eta'' A_j = A_j^\top \eta'' \leftarrow \eta'(\mathbf{u})A_j(\mathbf{u}) = q_j^\top(\mathbf{u})\)

- Existence of critical point(s) \(\iff\) existence of entropy function(s)!
  Non entropic systems: no extremum is attained.

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Variational formulation of entropy solutions for nonlinear conservation laws 37

A new variational formulation of \((\star)\) \(\mathbf{u}_t + \nabla_x \cdot \mathbf{f}(\mathbf{u}) = 0\)

- \(\eta \in C^1_{0, \text{convex}}, \mathbf{u} \in BV(\mathbb{R}^d_x \times \mathbb{R}^+_t \mapsto \mathbb{R}^m)\) is a variational solution of \((\star)\):

A critical point of 
\[
\mathcal{J}(\eta, \mathbf{u}) := \int_{t_1}^{t_2} \int_{\Omega} \langle \eta'(\mathbf{u}), \mathbf{u}_t + \nabla_x \cdot \mathbf{f}(\mathbf{u}) \rangle \phi \, dx \, dt
\]

- Original \((m \times m)\)-system is embedded in \(m + 1\) dimensional phase space since a critical point \(\eta\) is an entropy: \(\langle \eta', \nabla_x \cdot \mathbf{f} \rangle \phi = \nabla_x \cdot \mathbf{q}\)

Derivation of a minimum entropy principle:

\[
\arg \min \mathcal{J} \quad \iff \quad \arg \min \int_{\Omega \times [t_1, t_2]} \eta(\mathbf{u})_t + \nabla_x \cdot \mathbf{q}(\mathbf{u}) \, dx \, dt
\]

\[
= \arg \min \int_{\Omega} \eta(\mathbf{u}(\cdot, t)) \, dx \bigg|_{t_1}^{t_2} + \int_{t_1}^{t_2} \int_{\partial \Omega} \mathbf{q}(\mathbf{u}) \cdot \mathbf{n} \, d\omega
\]
A variational solution of \((\star)\) \(u_t + \nabla_x \cdot f(u) = 0\)

- **DEFINITION.** \((\eta, u) \in (\text{Con}, XBV)\) is a variational solution of \((\star)\):

\[
\text{A critical point of } J(\eta, u) := \int_{t_1}^{t_2} \int_{\Omega} \langle \eta'(u), u_t + \nabla_x \cdot f(u) \rangle \phi \, dx \, dt
\]

- A variational solution:
  1. Satisfies the conservation law \((\star)\) is a “renormalized sense”;
  2. Admits an entropy function (or approximate one);
  3. Satisfies a minimum entropy principle.

- Justify formal steps
- Topological methods – Palais-Smale condition:
  
  Is the class of “approximate entropy solutions” compact

\[
\eta(u^\epsilon)_t + \nabla_x \cdot q(u^\epsilon) \rightarrow \mu \leq 0
\]