Mathematical aspects of collective dynamics...

A basic paradigm for collective dynamics — environmental averaging

Examples of mathematical models for collective dynamics

- Krause-Hegselmann model for opinion dynamics
- Vicsek model for flocking; phase transition
- Cucker-Smale model for flocking — near and far from equilibrium
Outline

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A basic paradigm in collective dynamics

- A general class of $N \gg 1$ agents identified w/ “traits” $\{p_i(t)\}_{i=1}^N$:

- Environmental Averaging:
  \[ p_i(t + \Delta t) = \sum_j a_{ij} p_j(t) \quad \sum_j a_{ij} = 1 \]

  Alignment w/ frequency $\alpha \sim \frac{1}{\Delta t}$
  \[ \frac{p_i(t + \Delta t) - p_i(t)}{\Delta t} = \alpha \left( \sum_{j \neq i} a_{ij} (p_j - p_i) \right) \]

- Nonlinear influence function $\phi$:
  \[ a_{ij} = \frac{1}{\text{deg}_i} \phi(p_i, p_j) \geq 0 \]
  
  \[ \text{deg}_i := \sum_j \phi(p_i, p_j) \]

  — the degree of influence on agent $i$

- Observations: considerably different models in different contexts

- Despite the variety – similar fundamental features in collective dynamics; notably ...

- A large number of agents, $N \gg 1 \leadsto$ emergence of large scale coherent structures: swarms, colonies, parties, clusters, consensus, flocks, ...
Examples of collective dynamics

- Examples of “living agents” — averaging of orientations, velocities, ...
  - Flocks of birds; schools of fish, colonies of ants, locust, bacteria, ...

- Examples of “thinking agents” (social dynamics) — averaging of opinions and other traits, ...
  - Human crowd; traffic jam; “opinion dynamics”, neural networks

- Examples of “non-living agents” — averaging of “positions”
  - Robots – the rendezvous problem, UAVs, peridynamics, nematic fluids, ...
Outline

A basic paradigm for collective dynamics — environmental averaging

1 Examples of mathematical models for collective dynamics
   - Krause-Hegselmann model for opinion dynamics
   - Vicsek model for flocking; phase transition
   - Cucker-Smale model for flocking — near and far from equilibrium
Example #1: Krause model\(^1\) for opinion dynamics

- State space — vectors of "opinions" \(\{p_i(t)\}_{i=1}^N \leadsto \{x_i(t)\}_{i=1}^N\)
- Krause-Hegselmann model (1997) — interaction through **local** averaging:
  \[
  x_i(t + \Delta t) = \frac{1}{N_i} \sum_{|x_i - x_j| \leq R} x_j(t), \quad N_i := \#\{x_j : |x_j - x_i| < R\}
  \]

- "Environmental averaging":
  \[
  x_i(t + \Delta t) = \sum_j a_{ij} x_j(t) \quad \sum_j a_{ij} = 1
  \]

- Act on difference of opinions:
  \[
  a_{ij} = \frac{\phi(|x_i - x_j|)}{\deg_i} \quad \phi(r) = 1_{[0,R]}(r)
  \]

- \(\deg_i = \sum_k \phi(|x_i - x_k|) \leadsto N_i\) — the degree of influence on agent \(i\)
- A **local** model: agent \(i\) influenced by \(N_i\) "nearest" (?) neighbors

Environmental averaging: \( \mathbf{x}_i(t + \Delta t) = \sum_j a_{ij} \mathbf{x}_j(t) \)

- **Alignment** — models environmental averaging (\( \sum_j a_{ij} = 1 \)):

\[
\frac{\mathbf{x}_i(t + \Delta t) - \mathbf{x}_i(t)}{\Delta t} \frac{d}{dt} \mathbf{x}_i(t) = \alpha \left( \sum_j a_{ij} \mathbf{x}_j(t) - \mathbf{x}_i(t) \right)
\]

Frequency \( \alpha \sim \frac{1}{\Delta t} \)

\( a_{ij}(\mathbf{x}(t)) = \frac{1}{\text{deg}_i} \phi(|\mathbf{x}_i - \mathbf{x}_j|), \quad \text{deg}_i = \text{degree of influence on agent } i \)

- **Local models** involve nearby neighbors: \( \text{deg}_i := \sum_j \phi(|\mathbf{x}_i - \mathbf{x}_j|) \sim N_i \)

  involve ‘neighboring’ agents in finite \( \text{Supp}\{\phi\} \)

- **Global models** involve all agents: \( \text{deg}_i = \sum_j \phi(|\mathbf{x}_i - \mathbf{x}_j|) \sim N \)

  All agents interact within a possibly infinite \( \text{Supp}\{\phi\} \)

\( \ast \ \text{wlog } \phi \leq 1 \) — set \( \phi_{ii} := N - \sum_{j \neq i} \phi_{ij} \in [0, N] \) s.t. \( \sum_j a_{ij} = \frac{1}{N} \sum_j \phi_{ij} = 1 \)

\( \sim a_{ij} \in [0,1] \) but may be “far from equilibrium”: \( \{a_{ij} \} \not\in U[0,1] \)

\( \text{Compared with usual Laplacian — } N_i \equiv 4 \)
Large-time opinion dynamics: parties and consensus

- $d = 1$ — 100 uniformly distributed opinions on $[0, 10]$

- $d = 2$ — 100 opinion lead to the formation of $K = 17$ parties ...

- How to measure difference of opinions — $|x_i - x_j|$ in $d \geq 2$?
Example #2: Sensor-based networks

- State vector — vectors of “positions” \( \{p_i(t)\}_{i=1}^N \mapsto \{x_i(t)\}_{i=1}^N \)

\[
\frac{d}{dt}x_i(t) = \frac{\alpha}{\text{deg}_i} \sum_j \phi(|x_i - x_j|) (x_j(t) - x_i(t)) \equiv -\frac{\alpha}{\text{deg}_i} \nabla_x K(x_i(t))
\]

- Gradient descent \( K(x) = \frac{1}{2} \sum_{\alpha,\beta} k(|x_\alpha - x_\beta|), \quad k(r) := \int_0^r s\phi(s)ds \)

\[
\nabla_x K(x)|_{x=x_i} = \sum_j k'(|x_i - x_j|) \frac{x_i - x_j}{|x_i - x_j|} = -\phi(|x_i - x_j|)(x_j - x_i)
\]

\[
\frac{d}{dt}K(x(t)) = \sum_i \langle \dot{x}_i, \nabla_x K(x_i(t)) \rangle = -\frac{1}{\alpha} \sum_i \langle \dot{x}_i, \text{deg}_i \dot{x}_i \rangle \leq 0
\]

- Robotic agents — the “rendezvous problem”: \( x_i(t) - x_j(t) \xrightarrow{t \to \infty} 0 \)
Example #3: Vicsek model — alignment of orientations

- Fix a speed $s$. Averaging of orientations $\{p_i(t)\}_{i=1}^N \sim \{\omega_i(t)\}_{i=1}^N \in S^{d-1}$

\[
\omega_i(t + \Delta t) = (s \sum_j a_{ij} \omega_j(t) + \text{noise}) \times \frac{1}{|s \sum_j a_{ij} \omega_j(t) + \text{noise}|}
\]

- 2D additive Noise = uniform in angle in $[-\tau, \tau]$ 

\[
a_{ij} = \frac{\phi(|x_i - x_j|)}{\text{deg}_i} \sim \mathbf{v}_i(t) := \frac{s}{\text{deg}_i} \sum_{j : |x_i - x_j| < R} \phi(|x_i - x_j|) \omega_j(t)^{2b}
\]

- A second-order model $a_{ij}(x(t))$: $x_i(t + \Delta t) = x_i(t) + \Delta t \cdot \mathbf{v}_i(t)$

Vicsek model and alignment — Degond & Motsch (2008)

\[
\omega_i(t + \Delta t) - \omega_i(t) d\omega_i = \Delta t P_i \left( \frac{\mathbf{v}_i(t)}{|\mathbf{v}_i(t)|} - \omega_i(t) \right) \alpha P_i \left( \frac{\mathbf{v}_i}{|\mathbf{v}_i|} \right) dt + \Delta t P_i^n \text{noise}
\]

- Projection

\[
P_i = P^{n+1} n + \frac{1}{2} n (\omega_i) := I - \omega_i \omega_i^\perp (\omega_i) = 0 \sim \omega_i(t) + \Delta t) \in S^{d-1}
\]

$^3$T. Vicsek, A. Czirók, E. Ben-Jacob, I. Cohen, O. Shochet (PRL 1995). $^{3b} \phi = 1_{(0, R)}$
Vicsek model: $\mathbf{v}_i(t) = \frac{s}{\text{deg}_i} \sum_j \phi(|\mathbf{x}_i - \mathbf{x}_j|) \omega_j(t) + \text{noise}[−\tau,\tau]$

• Phase transition$^{4,4b}$ in order parameter $\varphi := \frac{1}{sN} \sum_i |\mathbf{v}_i|$

  From global alignment to disorder: $\tau \approx 0 \sim \varphi \approx |\mathbf{v}_\infty| \quad \tau \gg 1 \sim \varphi \approx 0$

• Transitions observed$^{4c}$ in meso-scopic scales of 10s–100s

• Act local think global $^{4d,4e,4f}$ (near critical point):
  ★ Exploration of resources; defense mechanism; improved decision process

$^4$* Vicsek & Zaferis, Collective motion (2012); $^{4b}$Chaté et. al. (2004,2008)
$^{4c}$Cavagna et. al., The starflag project (2008) $^{4d}$Bialek et. al. (2013);
$^{4e}$E. O Wilson (ants) Sociobiology (1975) $^{4f}$I. Couzin (locust, fish)
Example #4: Cucker-Smale model\(^5\)—alignment of velocities

- State space of velocities \( \{v_i(t)\}_{i=1}^N \in \mathbb{R}^d \)

\[
\frac{v_i(t + \Delta t) - v_i(t)}{\Delta t} \frac{d}{dt}v_i(t) = \alpha \left( \sum_j a_{ij}v_j(t) - v_i(t) \right) = \alpha \sum_j a_{ij}(v_j(t) - v_i(t))
\]

- A second-order model:

\[
a_{ij}(x(t)) = \frac{1}{\text{deg}_i} \phi(|x_i(t) - x_j(t)|), \quad \frac{x_i(t + \Delta t) - x_i(t)}{\Delta t} \frac{d}{dt}x_i(t) = v_i(t)
\]

- Global vs. local models, depending on \( \text{deg}_i \):
  
  - **Global models**: interaction involve all agents \( \text{deg}_i \sim N \)
    
    Example of Cucker-Smale: \( \phi(r) = \frac{1}{1 + r^{2\beta}} \), \( \beta > 0 \)
  
  - **Local models**: involve nearby neighbors \( \text{deg}_i \sim N_i \)
    
    "Environmental averaging": \( [\text{Supp}\{\phi\}] \ll [x(t)] := \max_{ij} |x_i - x_j| \)

- More on C-S dynamics: Carrillo, Fornasier, S.-Y. Ha, Illner, Karper, J.-G. Liu,...

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Collective Dynamics 13
Limitations C-S model: state space is not homogeneous

- Global C-S: $\text{deg}_i = N$ — a total of $N$ agents in group cluster $G$

\[
\frac{dv_i(t)}{dt} = \frac{\alpha}{N} \sum_{j \in G} \phi_{ij} (v_j(t) - v_i(t))
\]

$$\phi_{ij} := \phi(|x_i - x_j|) \quad \phi(\cdot) \downarrow$$

- What can go wrong? say $G = G_1 \cup G_2$:

- C-S alignment: $N_1 = \#G_1$, $N_2 = \#G_2$, $G_2 = \{j: |x_j - x_i| \gg R\}$:

\[
\frac{dv_i(t)}{dt} = \frac{\alpha}{N_1 + N_2} \left[ \sum_{1 \leq j \leq N_1} \phi_{ij} (v_j(t) - v_i(t)) + \sum_{1 \leq j \leq N_2} \phi_{ij} (v_j(t) - v_i(t)) \right]
\]

- If $N_2 \gg N_1$, then the far-away flock $G_2$ causes $G_1$ to stop
Example #5: Far from equilibrium

• (with S. Motsch)\(^6\):

\[
a_{ij}(x(t)) = \frac{1}{\text{deg}_i} \phi_{ij}, \quad \phi_{ij} := \phi(|x_i(t) - x_j(t)|)
\]

\[
\frac{v_i(t + \Delta t) - v_i(t)}{\Delta t} \frac{dv_i(t)}{dt} = \frac{\alpha}{\text{deg}_i} \sum_j \phi_{ij} (v_j(t) - v_i(t)), \quad \text{deg}_i = \sum_k \phi_{ik}
\]

\[
\star \quad \text{deg}_i (i \in G_1) \approx N_1 \phi_0 \quad \Rightarrow \quad \frac{dv_i(t)}{dt} = \frac{\alpha}{N_1 \phi_0} \sum_{G_1} \phi_{ij} (v_j(t) - v_i(t))
\]

• \(a_{ij}\) is not symmetric

\(^6\)Motsch & ET, A new model for self-organized dynamics and flocking behavior (2011)
Rules of engagement for self-propelled collective dynamics

- Craig Reynolds (1987)

- Three zone (AAAs) models:
  - Avoidance (Repulsion)
  - Attraction (Cohesion)
  - Alignment: $a_{ij} \geq 0$

... 1998 Academy Scientific and Technical Award in recognition of “pioneering contributions ... development of 3D computer animation for motion picture production.”

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7 Flocks, herds and schools: A distributed behavioral model (1987);
**Example #6: Collective synchronization**

- Kuramoto model\(^8\) \{p_i\} \sim phases \{\theta_i\} or frequencies \{\omega_i = \dot{\theta}_i\}

\[
\frac{d}{dt} \theta_i(t) = \Omega_i + \frac{\alpha}{N} \sum_j \sin(\theta_j - \theta_i) \sim \Omega_i + \frac{\alpha}{\text{deg}_i} \sum_j a_{ij}(\theta_j - \theta_i), \quad a_{ij} = \frac{\sin(\theta_j - \theta_i)}{\theta_j - \theta_i}
\]

\[
\frac{d}{dt} \theta_i(t) = \Omega_i + \alpha r \sin(\psi - \theta_i) \sim \Omega_i - \alpha r \sin(\theta_i), \quad re^{i\psi} := \frac{1}{N} \sum_j e^{i\theta_j}, \quad \langle \Omega \rangle = 0
\]

- \(|\Omega_i| < \alpha r\): steady states; more oscillators recruited into synchronized clusters as \(\alpha > \alpha_c\) increases
- \(|\Omega_i| < \alpha r\): no synchronization is possible for \(\alpha < \alpha_c\)

- As a second-order model:

  Averaging of frequencies\(^8b\) \[
  \frac{d}{dt} \omega_i(t) = \frac{\alpha}{N} \sum_j a_{ij}(\omega_j - \omega_i), \quad a_{ij} := \cos(\theta_j - \theta_i)
  \]

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\(^b\)Ha et. al (2010 –)
Collective Dynamics: consensus, emergence of patterns and social hydrodynamics

Leçons Jacques-Louis Lions 2016
Lecture #2. \( t \to \infty \): alignment and self-organization; consensus, flocking, ...

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Focus on two important limits

Lecture #2. Large time behavior – what happens when $t \to \infty$?

- Alignment self-organizes into $K$ clusters, and in particular, into a flock, consensus, ... ($K = 1$)
- A distinction between global and local models:
  - Global models with unconditional consensus;
  - Local models — clusters, connectivity and heterophilious dynamics\footnote{Motsch & ET., SIAM Review 2014}

Lecture #3. Large number of ”agents” – what happens when $N \gg 1$?

- Social hydrodynamics (opinions, flocking, ...)
- Critical thresholds in social hydrodynamics
Outline

$t \to \infty$: alignment and self-organization — consensus, flocking, ...

- Global models — unconditional consensus/flocking
- Local models — clusters, connectivity and heterophilious dynamics
From local interactions to emerging consensus?

• How group pressure in small scales may lead to a consensus on larger scales
Large-time opinion dynamics: parties and consensus

\[
\frac{x_i(t + \Delta t) - x_i(t)}{\Delta t} = \alpha \left( \sum_j a_{ij} x_j(t) - x_i(t) \right)
\]

\[
a_{ij} = \frac{\phi(|x_i(t) - x_j(t)|)}{\text{deg}_i(x(t))}
\]

- $\alpha > 0$ is a scaling factor s.t. $\phi(\cdot) \lesssim 1$
- Clustering – formation of $K$ “parties”:
- $d = 1$ — 100 uniformly distributed opinions on $[0, 10]$

- $K = 1$ — When does “consensus” emerge: $x_i(t) \to x^\infty$ as $t \to \infty$?
Large-time social dynamics: flocking

\[
\frac{\mathbf{v}_i(t + \Delta t) - \mathbf{v}_i(t)}{\Delta t} = \alpha \left( \sum_j a_{ij} \mathbf{v}_j(t) - \mathbf{v}_i(t) \right)
\]

\[
a_{ij}(\mathbf{x}(t)) = \frac{1}{\text{deg}_i} \phi(|\mathbf{x}_i(t) - \mathbf{x}_j(t)|), \quad \frac{\mathbf{x}_i(t + \Delta t) - \mathbf{x}_i(t)}{\Delta t} = \mathbf{v}_i(t)
\]

- Again — \( \alpha > 0 \) is a scaling factor s.t. \( \phi(\cdot) \lesssim 1 \)
- Clustering — formation of \( K \) “flocks”:
- \( K = 1 \) — When does a “flock” emerge:

\[
\begin{cases}
\mathbf{v}_i(t) \rightarrow \mathbf{v}^\infty \\
|\mathbf{x}_i(t) - \mathbf{x}_j(t)| \leq D^\infty
\end{cases}
\]
as \( t \rightarrow \infty \)?
Emergence as \( t \to \infty \)

\[
\frac{p_i(t + \Delta t) - p_i(t)}{\Delta t} = \alpha \left( \sum_{j \neq i} a_{ij} p_j - p_i \right), \quad \alpha \mapsto 1 \text{ (rescale } \Delta t) \quad A \text{ is stochastic: } \sum_j a_{ij} \equiv 1
\]

Nonlinear Alignment

\[
p(t + \Delta t) = (1 - \Delta t)p(t) + \Delta t A p(t) \quad \text{ } A = A(p(t))
\]

- Seek contractive diameter \([\ldots]\) such that \([A p] \leq (1 - \eta)[p]\)

if \([A p] \leq (1 - \eta)[p] , \quad \eta = \eta(t) > 0 \)

then \([p(t + \Delta t)] \leq (1 - \Delta t)[p(t)] + \Delta t[A p(t)] \leq (1 - \eta \Delta t)[p(t)]\):

\[
\int_{-\infty}^{\infty} \eta(s) ds = \infty \quad \sim \quad \frac{d}{dt} [p(t)] \leq -\eta(t)[p(t)] < 0 \quad \ldots \quad p_i(t) \xrightarrow{t \to \infty} p^\infty(t)
\]

- but \( \sum a_{ii} = 1 \) or \( A \mathbf{1} = \mathbf{1} \) \( \sim \) contract states “separated” from \( p^\infty \equiv 1 \)
Fix any \( i \) and \( j \); set \( \eta_k := \min\{a_{ik}, a_{jk}\} \) so that \( a_{ik} - \eta_k \geq 0 \), \( a_{jk} - \eta_k \geq 0 \). Then, for arbitrary \( w \in \mathbb{R}^d \) and \( \eta = \min_{ij} \sum_k \min\{a_{ik}, a_{jk}\} \)

\[
\langle (Ap)_i - (Ap)_j, w \rangle = \sum_k a_{ik} \langle p_k, w \rangle - \sum_k a_{jk} \langle p_k, w \rangle
\]

\[
= \sum_k (a_{ik} - \eta_k) \langle p_k, w \rangle - \sum_k (a_{jk} - \eta_k) \langle p_k, w \rangle
\]

A is raw stochastic \( \leq \sum_k (a_{ik} - \eta_k) \max_k \langle p_k, w \rangle - \sum_k (a_{jk} - \eta_k) \min_k \langle p_k, w \rangle \)

\[
\eta_k = \eta_k(i,j) \ldots = (1 - \sum_k \eta_k) \left( \max_k \langle p_k, w \rangle - \min_k \langle p_k, w \rangle \right)
\]

\[
\eta := \min_{ij} \sum \eta_k(i,j) \ldots \leq (1 - \eta) \max_{k,\ell} \langle p_k - p_\ell, w \rangle \leq (1 - \eta) \max_{k,\ell} \|p_k - p_\ell\|_w^*
\]


\[
[Ap] = |(Ap)_i - (Ap)_j| = \sup_{w \neq 0} \frac{\langle (Ap)_i - (Ap)_j, w \rangle}{|w|_w^*} \leq (1 - \eta) \max_{k,\ell} [p]
\]
Global models: \( a_{ij}(x) = \frac{1}{\text{deg}_i} \phi(|x_i - x_j|) \geq \eta > 0 \)

- \( \ell_\infty \)-diameter — \( [p] := \max_{i,j} |p_i - p_j| \ \leadsto \ [Ap] \leq (1 - \eta) [p] , \ p \perp 1 \)

1 – \( \eta \) is the coefficient of ergodicity\(^3\) \( \eta = \eta_E := \min_{ij} \sum \min(a_{ik}, a_{jk}) \)

- Opinion dynamics: \( \eta_E = \min \sum \min(a_{ik}, a_{jk}) \geq \frac{N^k}{\text{deg}_i} \min_{r \leq [x_0]} \phi(r) \)

- If \( \{x_0\} \subset \text{Supp}\{\phi\} \) then:

\[
\frac{d}{dt} [x(t)] \leq -\eta [x(t)] \ \leadsto \ [x(t)] \leq e^{-\eta t} [x_0] , \ \eta = \min_{r \leq [x_0]} \phi(r) > 0
\]

- Global connectivity \( \leadsto \) unconditional consensus: \( x(t) \rightarrow x^\infty \in \text{Conv}\{x_0\} \)

Symmetric case (\( \text{deg}_i = N \)): \( x^\infty = \frac{1}{N} \sum_i x_i(t = 0) \)

Non-symmetric case (\( \text{deg}_i = \sum_k \phi_{ik} \)) Q. #1: characterize \( x^\infty \)

- Q. #2: Conditional consensus — dependence on initial configuration

\(^3\) Ipsen & Selee (2011): \( 1 - \eta_E = \frac{1}{2} \max_{ij} \sum_k |a_{ik} - a_{jk}| \); Krause (2014): scrambling
Finite time collision — the rendezvous problem

• Assume the influence function \( \phi(\cdot) \) is decreasing ...

• If \( \{x(0)\} \subset \text{Supp}\{\phi\} \) then:

\[
\frac{d}{dt}[x(t)] \leq -\phi([x(t)])[x(t)] = k'(\[x(t)\]), \quad k(r) = \int_0^r s\phi(s)ds
\]

• Osgood condition:

\[
\int_0^1 \frac{dr}{r\phi(r)} < \infty \quad \implies \quad \text{Finite-time collision} \quad [x(t_c)] = 0;
\]

\[
\int_0^1 \frac{dr}{r\phi(r)} = \infty \quad \implies \quad \text{Global regularity} \quad [x(t)] > 0
\]

\[\text{Example.} \quad \phi(r) \sim r^{-\beta}, \quad \beta \leq 1 \quad \implies \quad \text{unconditional rendezvous}^{4c}\]

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\[\text{Bertozzi,Carrillo,Laurent (2009): symmetric } \dot{R}(t) \lesssim k'(R(t)), \quad R := \max_i |x_i(t) - x^\infty_i|\]

\[\text{Unconditional rendezvous with bearing-only } \phi(r) = 1/r\]
Global models: \( a_{ij}(x) = \frac{1}{\deg_i} \phi(|x_i - x_j|) \geq \eta > 0 \)

Flocking dynamics with a decreasing influence function \( \phi \):

Diameter \([x(t)]\) may grow \( \frac{d}{dt} [x(t)] \leq [v(t)] \) — but remains bounded:

\[
\frac{d}{dt} [v(t)] \leq -\phi([x(t)]) [v(t)] \sim \mathcal{E}(t) := [v(t)] + \int_{[x_0]}^{[x(t)]} \phi(s) ds \text{ decreases}^
\]

\[
\int_{[x_0]}^{[x(t)]} \phi(s) ds \leq [v_0] \text{ if } [v_0] < \int_{[x_0]}^{\infty} \phi(s) ds \text{ then } \exists D_\infty: [v_0] = \int_{[x_0]}^{D_\infty} \phi(s) ds
\]

- Unconditional flocking: \([x(t)] \leq D_\infty \sim [v(t)] \leq e^{-\alpha \phi(D_\infty) t} [v_0] \)

\[
\int_{[x_0]}^{\infty} \phi(s) ds = \infty \text{ — global communication implies unconditional flocking}
\]

Example. \( \phi(r) = \frac{1}{1 + r^{2\beta}}, \quad \beta < \frac{1}{2} \) \( \sim \) flocking velocity \( v(t) \to v^\infty \)

- Q. #1. What is \( v^\infty \) in the non-symmetric case?

\(^5\) S. Y. Ha & J.-G. Liu (2009); non-symmetric case in Motsch & ET (2014)
From $\ell_\infty$ to $\ell_2$ contractivity \ldots \quad \frac{d}{dt} p_i = \sum_j a_{ij} p_j - p_i$

• $\max_p \frac{\langle Ap \rangle}{\langle p \rangle} = 1 - \eta$ — set the $\ell_2$ diameter $\|p\|^2 := \sum |p_i - p_j|^2$

• $A1 = 1$; If $p \perp 1$ then (i) $\|p\|^2 = 2N|p|^2$ and (ii) $Ap \perp 1$ ($A$ is symmetric)

$$\leadsto \max_p \frac{\langle Ap \rangle}{\langle p \rangle} = \max_{p \perp 1} \frac{\langle Ap \rangle}{\langle p \rangle} = \max_{p \perp 1} \frac{|Ap|}{\|p\|} = \max_{p \perp 1} \frac{\langle Ap, p \rangle}{\|p\|^2}$$

• Set the graph Laplacian\footnote{Laplacian $L_\Phi = D - \Phi$ and its normalized $L_A = I - D^{-1}\Phi$} $L_A := I - A$ (note the positivity)

$$a_{ij} = \frac{\phi_{ij}}{\text{deg}_i}$$ is symmetrizable with eigenvalues $0 \leq \mu_N \leq \ldots \mu_2 \leq \mu_1 = 1$

$L_A = I - A$ w/corresponding real eigenvalues $1 \geq \lambda_n \geq \ldots \lambda_2 \geq \lambda_1 = 0$

$$\max_{p \perp 1} \frac{\langle Ap, p \rangle}{\|p\|^2} = 1 - \min_{p \perp 1} \left( 1 - \frac{\langle Ap, p \rangle}{\|p\|^2} \right) = 1 - \min_{p \perp 1} \frac{\langle L_A p, p \rangle}{\|p\|^2} = 1 - \lambda_2(L_A)$$

• $\frac{d}{dt} p = (A - I)p = -L_A p$ contractivity: $\frac{d}{dt} \|p\| \leq -\eta \|p\|, \quad \eta = \lambda_2(L_A)$

• $\ell_2$-contractivity ("shortcut"): $\frac{d}{dt} \|p\|^2 = -2 \langle L_A p, p \rangle \leq -2 \lambda_2(L_A) \|p\|^2$
A graph $G = (V, E)$; vertex set $V = \{p_i\} \subset \mathbb{R}^N$ and edge set $E = \{e_{ij}\} \subset \mathbb{R}^N \times \mathbb{R}^N$

The gradient $\nabla \phi : V \mapsto E$ 

$$\nabla \phi(p)_{ij} := \sqrt{\phi_{ij}}(p_i - p_j)$$

The divergence $\text{div}_\phi : E \mapsto V$ 

$$(\text{div}_\phi(e))_i := \sum_j \sqrt{\phi_{ij}}(e_{ij} - e_{ji})$$

with the usual duality $\langle \nabla p, u \rangle = \langle p, \text{div} u \rangle$ ...

and the Laplacian:

$$\Delta \phi := -\frac{1}{2} \text{div} \circ \nabla \phi : V \mapsto V$$

$$\Delta \phi(p)_i = \sum_j \phi_{ij}(p_i - p_j)$$

Alignment process:

$$\frac{d}{dt} p(t) = -\frac{1}{\text{deg}(p(t))} \Delta \phi(p(t)) = -L_A p(t)$$

Laplacians:

$$L_A = I - D^{-1} \Phi, \quad L_\Phi := D - \Phi, \quad L_{\text{sym}} = I - D^{-\frac{1}{2}} \Phi D^{-\frac{1}{2}}$$

$$\frac{d}{dt} p = -L_A p \quad \Rightarrow \quad \frac{1}{2} \frac{d}{dt} |p|^2 = -\langle L_A p, p \rangle$$

but lacks $\ell_2$-coercivity $\langle L_A p, p \rangle \not\gg |p|^2$

$L^p$ Sobolev inequalities, Poincare inequality


$^7c$Badr & Russ (2009, 2012)

$^7d$Coulhon & Koskela (2004)
revisiting the coercivity of graph Laplacians

• Symmetric Self-alignment \((\text{deg}_i = N)\):

\[
\frac{d}{dt} p = -\frac{1}{2N} \text{div} \nabla p \quad \Rightarrow \quad \frac{1}{2N} \frac{d}{dt} |p|^2 = -\frac{1}{2N} |\nabla p|^2 = -\frac{1}{2N} \sum_{ij} \phi_{ij} |p_i - p_j|^2
\]

• And the corresponding non-symmetric case \(\frac{1}{2N} \frac{d}{dt} |p|^2 = \langle L_A p, p \rangle\):

\[
\langle L_\Phi p, p \rangle = \sum_i \text{deg}_i |p_i|^2 - \sum_{ij} \phi_{ij} \langle p_i, p_j \rangle = \frac{1}{2} \sum_{ij} \phi_{ij} |p_i - p_j|^2 \geq |p|^2
\]

• Symmetry\(^8\) — \(\sum p_i(t) = \sum p_i(0)\)

\[
\frac{1}{2N} \frac{d}{dt} \sum_{ij} |p_i - p_j|^2 = -\frac{1}{N} \sum_{ij} \phi_{ij} |p_i - p_j|^2 \leq -\eta \sum_{ij} |p_i - p_j|^2
\]

• Coercivity with \(\eta = \frac{1}{N} \min_{ij} \phi_{ij}; \ldots; \eta = \lambda_2(L_\Phi)\)

---

\(^8\)Raw stochastic and column stochastic matrices
Fiedler number and spectral graph theory

• $\lambda_2(L\Phi) > 0$ – Fiedler number$^9$ dictates algebraic connectivity of $\{\phi_{ij}\} \succeq 0$

• A (weighted) graph $G = (V, E)$: vertex set of positions: $V = \{p_i\}$ and edge set of connectors: $E = \{\phi_{ij}\}$, $\phi_{ij} = \phi(|p_i - p_j|)$

• Connectivity of graph $G_\Phi(p, \Phi(p))$: $\forall p_i, p_j \exists$ a (shortest) path $\Gamma_{ij}$ s.t.

$$\min_{k_\ell \in \Gamma_{ij}} \phi_{k_\ell, k_{\ell+1}} \geq \mu > 0, \quad \Gamma_{ij} = \{i = k_0 < k_2 < \ldots < k_r = j\}, \text{length}(\Gamma_{ij}) = r_{ij}$$

$$\sum_{ij} \mu |p_i - p_j|^2 \leq r_{ij} \sum_{k_\ell \in \Gamma_{ij}} \mu |p_{k_{\ell+1}} - p_{k_\ell}|^2 \quad p_i - p_j = \sum_{\ell=0}^{r_{ij}} (p_{k_{\ell+1}} - p_{k_\ell})$$

$$\leq r_{ij} \sum_{k_\ell \in \Gamma_{ij}} \phi_{k_{\ell+1}, k_\ell} |p_{k_{\ell+1}} - p_{k_\ell}|^2 \leq \max_{ij}(r_{ij}) N^2 \sum_{\alpha \beta} \phi_{\alpha \beta} |p_\alpha - p_\beta|^2$$

$$\leadsto \lambda_2(L\Phi) = \min_{p \perp 1} \frac{\langle L\Phi p, p \rangle}{|p|^2} = \min_{p \perp 1} \frac{1}{2} \sum_{\alpha \beta} \phi_{\alpha \beta} |p_\alpha - p_\beta|^2 \geq \frac{\mu}{\max_{ij}(r_{ij}) N}$$

• Algebraic connectivity$^{9b}$: $\lambda_2(L\Phi) \geq \frac{\mu}{d(G)N}, \geq \frac{\mu}{N^2}, \quad d(G) := \max_{ij}(r_{ij}) \leq N$

$^9$M. Fiedler, Algebraic connectivity of graphs (1973);$^{9b}$Mohar (1991) $\lambda_2(L\Phi) \geq \frac{4\mu}{N^2}$
On the geometric aspects of Fiedler eigenpair

• The multiplicity of $\lambda_1(L_\Phi) = 0 \leadsto$ the number of connected components

\[
\dot{x} = -\frac{1}{\deg(x)} \Delta \phi(x), \quad \phi(r) = 1_{[0,1]} \text{ vs. } \mu_i(A) = 1 - \lambda_i(L_A)
\]

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{opinions}
\caption{Evolution in time of the opinions $x_i$}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{eigenvalues}
\caption{Eigenvalues $|\lambda_i|$}
\end{figure}

• $L_A = I - D^{-1} \Phi$ similar to $L_{sym} = I - D^{-\frac{1}{2}} \Phi D^{-\frac{1}{2}}$ congruent to $L_\Phi = D - \Phi$

• Fiedler (1973): $\lambda_2(L_A)$ increases as $A$ becomes “more” connected:

If $(V, E)$ is a sub-graph of $(V, F \supset E)$ then $\lambda_2(L_E) \leq \lambda_2(L_F)$

• Spectral bisection\textsuperscript{10} (1975): The Fiedler vector $v_2(L_{sym})$ induces connected decomposition $V = V_- \cup V_+$, $V_\pm = \{i : \pm v_2(i) > 0\}$

\textsuperscript{10}Pothen,Simon,Liou (1990); Newmann (2003): The structure ... of complex networks
Propagation of connectivity

- Consensus/flocking: time-dependence on the underlying graph $G_{A(p(t))}$
- Loss of connectivity (in local models):

<table>
<thead>
<tr>
<th>emergence clusters ($K=5$)</th>
<th>vs. consensus ($K=1$)</th>
</tr>
</thead>
</table>

1. Determine the number of “parties” $K^{11}$

2. In particular — when $K = 1$ (consensus/flocking) depending on ...
   - Propagation of connectivity: $\lambda_2(L_{A(p(0))}) > 0 \Rightarrow \lambda_2(L_{A(p(t))}) > 0$?
   - “Intensity” of connectivity $- \{A^N(p(t))\}_{ij} \geq \mu^N(t) : \int_{\infty}^{\infty} \mu(t) dt = \infty$

---

Alignment yields finite # of parties, Jabin & Motsch (2014), Motsch & ET (2014)
How “rules of engagement” influence the emergence of consensus?

- 100 uniformly distributed opinions: 
  \[ \phi(r) = a \mathbf{1}_{\{r \leq \frac{1}{\sqrt{2}}\}} + b \mathbf{1}_{\{\frac{1}{\sqrt{2}} < r < 1\}} \]

\[ a = b = 1 : \phi(r) = 1_{\{0 < r < 1\}} \]

\[ (a, b) = (0.1, 1) \]

- Homophilious dynamics: align with those that think alike \((a \gg b)\) vs.
- Heterophilious dynamics: ”bonding with the different“ \((a \ll b)\)
Collective Dynamics:
consensus, emergence of patterns and social hydrodynamics

Leçons Jacques-Louis Lions 2016
Lecture #3. $N \to \infty$: from agent-based models to social hydrodynamics

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Laboratoire Jacques-Louis Lions
Universite Pierre et Marie Curie, June 13–15 2016

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Mathematical aspects of collective dynamics...

1 Examples of mathematical models for collective dynamics
   - Krause-Hegselmann model for opinion dynamics
   - Vicsek model for flocking; phase transition
   - Cucker-Smale model for flocking — near and far from equilibrium
   - Questions that arise in different fields

2 $t \to \infty$: alignment and self-organization — consensus, flocking, ...
   - Global models — unconditional consensus/flocking
   - Local models — clusters, connectivity and heterophilious dynamics
   - A new paradigm — tendency to move ahead; emergence of leaders

3 $N \to \infty$: from agent-based models to social hydrodynamics
   - Liouville equation
   - Kinetic description
   - From kinetic to hydrodynamic description of flocking
   - Hydrodynamic alignment — smooth solutions must flock
   - Critical thresholds in flocking hydrodynamics
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Multi-scale phenomena - different models at different scales

**Macroscopic models**: $\rho, u$

\[
\begin{align*}
\rho_t + \nabla_x \cdot (\rho u) &= 0 \\
u_t + u \cdot \nabla_x u + \nabla_x P &= 0
\end{align*}
\]

$\uparrow$ Human scale: $\epsilon \to 0$, $(t' = \epsilon t, x' = \epsilon x)$

**Kinetic models**: $f(t, x, v)$

\[
f_t + v \cdot \nabla_x f = Q(f, f)
\]

$\uparrow$ $N \to \infty$ (including meso-scale phenomena)

**Agent-based models**: $\{x_i, v_i\}_{1 \leq i \leq N}$

\[
\begin{align*}
\dot{x}_i &= v_i \\
\dot{v}_i &= F_i, \quad F_i \mapsto F(x_i, v_i)(x, v)
\end{align*}
\]
Empirical distribution

- Phase space: \( \{(x_i, v_i)\} \subset \mathbb{R}^{Nd} \times \mathbb{R}^{Nd} \quad i = 1, 2, \ldots, N, \quad N \gg 1 \)

Agent based models — in terms of Empirical distribution

\[
\begin{align*}
\frac{dx_i}{dt} &= v_i, \\
\frac{dv_i(t)}{dt} &= F[f^N](x_i, v_i), \\
F^N(t, x, v) &= \frac{1}{N} \sum_{j=1}^{N} \delta_{x_j(t)} \otimes \delta_{v_j(t)}
\end{align*}
\]

- \( F(f^N) \) dictated by interaction kernel — \( a(x, y) = \frac{\phi(|x - y|)}{\text{deg}(x)} \)

\[
\begin{align*}
F(f)(x, v) &= \int \int_{x', v'} a(x, y)(v' - v)f(t, x', v')dx'dv'|_{f=f^N} \\
\Rightarrow F(f^N)(x, v) &= \frac{1}{N \cdot \text{deg}(x)} \sum_j \phi(|x_j - x|)(v_j - v)
\end{align*}
\]
The passage to limit:  
\[ f^N = \frac{1}{N} \sum_j \delta_{x_j(t)} \otimes \delta_{v_j(t)} \rightarrow f \]

- Agents are indistinguishable:  
  \[ f^N(x_{\sigma(1)}, v_{\sigma(1)}, \ldots, x_{\sigma(N)}, v_{\sigma(N)}) \]

- Liouville equation for empirical function \( f^N \):  
  \[ \dot{x}_i = v_i, \quad \dot{v}_i = F(x_i, v_i) \]

\[
\begin{align*}
\frac{\partial}{\partial t} f^N + \sum_{i=1}^N v_i \cdot \nabla x_i f^N + \frac{\alpha}{N} \sum_{i=1}^N \frac{1}{\text{deg}_i} \nabla v_i \cdot \left( \sum_{j=1}^N \phi(|x_i - x_j|)(v_j - v_i)f^N \right) &= 0
\end{align*}
\]

- Marginal distribution: we "probe" it by any of its pairs — take \( (x_1, v_1) \):  
  \[ f^N(x_1, v_1) := \int f^N(x_1, v_1, x_-, v_-) dx_- dv_-, \quad (x_-, v_-) := (x_2, v_2, \ldots, x_N, v_N) \]

\[
\int_{\mathbb{R}^{d(N-1)}} T(f^N(x, v)) dx_- dv_- = v_1 \cdot \nabla x_1 f^N(x_1, v_1)
\]

\[
\int I(f^N) dx_- dv_- = \frac{1}{\text{deg}_1} \int \sum_{j=2}^N \nabla v_1 \cdot \left( \phi(|x_j - x_1|)(v_j - v_1)f^N \right) dx_- dv_-
\]
The passage to a limit - formalities cont’d

• But agents are indistinguishable \((x_j, v_j) \leftrightarrow (x_2, v_2)\):

\[
\int I(f^N)dx_- dv_- = \frac{\alpha}{N \text{deg}_1} \int \sum_{j=2}^{N} \nabla v_1 \cdot \left( \phi(|x_j - x_1|)(v_j - v_1)f^N \right) dx_- dv_-
\]

\[
\leadsto \frac{N - 1}{N} \frac{1}{\text{deg}_1} \int \nabla v_1 \cdot \left( \phi(|x_2 - x_1|)(v_2 - v_1)g^N \right) dx_2 dv_2
\]

where \(g^N(x_1, v_2, x_2, v_2) := \int_{\mathbb{R}^{2d(N-2)}} f^N(x, v)dx_3 dv_3 \ldots dx_N dv_N\)

• Propagation of chaos\(^1,^{1b}\): \(f^N \rightarrow f(x_1, v_1), \quad g^N \rightarrow f(x_1, v_1)f(x_2, v_2)\)

\[
\exists f \text{ satisfies the Vlasov equation: } f_t + v \cdot \nabla_x f + \nabla_v \cdot (F(f)f) = 0
\]

• From deterministic to a statistical description\(^1c\):

\(f(t, x, v)\) — probability of finding agents in \(dx dv\) at time \(t\)

---

\(^1\)Bogoliubov–Born–Green–Kirkwood–Yvon (BBGKY) hierarchy

\(^{1b}\)Convergence of expectation (Sznitman)

\(^{1c}\)F. Golse Lecture notes; Gallagher, Saint-Raymond & Texier
The passage to the limit \( N \to \infty \)

- \( f^N = \frac{1}{N} \sum_j \delta_{x - x_j(t)} \otimes \delta_{v - v_j(t)} \to f(t, x, v) \)

\[ f_t + v \cdot \nabla_x f + \alpha(x) \nabla_v \cdot Q(f, f) = 0 \tau \Delta \omega f \]

- \( Q(f, f) \) assembles binary interactions: alignment, repulsion, noise, ...

\[ Q(f, f) = \int_{\mathbb{R}^{2d}} \frac{\phi(|x - y|)}{\text{deg}(t, x)} (w - v)f(t, x, v)f(t, y, w)dydw = (\overline{v}_{\text{mean}} - v)f \]

- Align w/mean

\[ \overline{v}_{\text{mean}} \coloneqq \frac{v \phi \ast f}{\phi \ast f}, \quad \overline{g} \coloneqq \int_w g(w)dw \quad \alpha(x) = \frac{\phi \ast f}{\text{deg}(t, x)} \]

\[ \text{deg}(t, x) \begin{cases} 1 & \text{Cucker-Smale} \quad \sim \quad \alpha(x) = \phi \ast f \\ \phi \ast f & \text{Motsch-ET} \quad \sim \quad \alpha(x) \equiv 1 \end{cases} \]

\(^2\) Ha & ET (2008); Canizo, Carrillo, Rosado (2009)
Emergence of Dirac masses in velocity space

- **Kinetic description:**  \( f_t + \mathbf{v} \cdot \nabla_x f + \alpha \nabla_v \cdot Q(f, f) = 0 \)

- **Flocking**\(^3\) \((K = 1)\)  \( f \sim \rho(t, x) \delta(\mathbf{v} - \mathbf{u}(t, x)) \)

\[ 0.5 \quad 1 \quad 1.5 \quad 2 \quad 0 \quad 5 \quad 10 \]

Shifted position \( \mathbf{x} - \mathbf{u} t \)  Velocity \( \mathbf{v} - \mathbf{u} \)

- **Recover**  
  \[
  \begin{bmatrix}
  \text{density} \\
  \text{momentum}
  \end{bmatrix} = \int \begin{bmatrix}
  1 \\
  \mathbf{v}
  \end{bmatrix} f(t, \mathbf{x}, \mathbf{v}) d\mathbf{v}
  \]

Requires closure in terms of \( P_{ij} := \int (\mathbf{v}_i - \mathbf{u}_i)(\mathbf{v}_j - \mathbf{u}_j)f(t, \mathbf{x}, \mathbf{v})d\mathbf{v} \)

\(^3\)Ha & ET (2008); Carrillo-Fornasier-Rosado-Toscani (2009), Motsch-ET (2014): 
Unconditional flocking:  \( \sup_{\text{supp} f(t)} |\mathbf{x} - \mathbf{y}| < \infty \);  \( \sup_{\text{supp} f(t)} |\mathbf{v} - \mathbf{v}'| \xrightarrow{t \to \infty} 0 \)
Flocking hydrodynamics

mass: \[ \rho_t + \nabla_x \cdot (\rho u) = 0 \]

momentum: \[ (\rho u)_t + \nabla_x \cdot (\rho u \otimes u + P) = \mathcal{R}, \quad \mathcal{R} = \mathcal{R}(u) \]

energy: \[ (\rho E)_t + \nabla_x \cdot (\rho E u + Pu + q) = S, \quad S = S(u, u) \]

- non-local action \[ \mathcal{R}(u) = \alpha \int_{\mathbb{R}^d} a(x, y)(u(t, y) - u(t, x))\rho(t, x)\rho(t, y)dy \]

stress tensor \[ P = (p_{ij}): \quad p_{ij} = \int_{\mathbb{R}^d} (v_i - u_i)(v_j - u_j)f(t, x, v)dv \]

- Derivation of \( P \) — empirical; closure of moments:
- Closure — in terms of local Maxwellian: \( P = P(f) \mapsto P(M_f) \)
- Mono-phase CS\(^4\): \( M_{\{\rho, u\}}(v) = \rho(t, x)\delta(v - u(t, x)) \sim P_{ij} \equiv 0 \)
- Other closures...\(^4b\): \( M_{\{\rho, u\}}(v) = \rho(t, x)\frac{1}{(2\pi)^{d/2}}e^{-\frac{|v - u(x)|^2}{2}} \sim P_{ij} \equiv \rho I \)

Brownian effect: \[ f_t + v \cdot \nabla_x f + \nabla_v \cdot Q(f, f) = \frac{1}{\epsilon} \nabla_v \cdot ((v - u)f) + \frac{1}{\epsilon} \nabla_v f \]
Flocking hydrodynamics. Alignment with non-local means

Mono-phase model \( P \equiv 0^{4b} \):
\[
\begin{align*}
\rho_t + \nabla_x \cdot (\rho u) &= 0 \\
u_t + (u \cdot \nabla_x) u &= \alpha(x)(\bar{u}(t,x) - u(t,x))
\end{align*}
\]

- Tendency to align w/mean
  \[ \bar{u}_{\text{mean}}(t,x) = \frac{\rho \bar{u}}{\rho}, \quad \bar{w} := \int_y \phi(|x - y|)w(y)dy \]

Cucker-Smale: \( \alpha(x) \equiv \phi(|\cdot|) \ast \rho; \quad \text{RHS} = \phi \ast (\rho u) - u \phi \ast \rho; \)

Motsch-ET: \( \alpha(x) \equiv 1; \quad \text{RHS} = \phi \ast (\rho u) - u \)

- Projected mean
  \[ \bar{u}_{\text{mean}}(t,x) := \frac{\rho \chi_{x,y} u}{\rho}, \quad \chi_{x,y} := \frac{\langle u(x), u(y) \rangle}{|u(y)|^2} \]

- \( \phi(|x|) \mapsto \epsilon^{-(d+2)} \phi \left( \frac{|x|}{\epsilon} \right) \quad \epsilon \to 0 \quad (\rho u)_t + \text{div}(\rho u \otimes u) = C \text{div}(\rho \nabla u) \)

- Singular influence: fractional Laplacian \( \phi(|x|) \sim |x|^{-(d+2)} \)

- Non-local means: smooth \( \phi \in C^1 \) — local vs. global

\(^5\) Ha & ET(2008); Carrillo et. al.(2012); \(^5\) Caffarelli-Vasseur, Kiselev-Nazarov, Constantin-Vicol, … \(^5\) Mellet, Vasseur, C. Yu
Agent-base model vs. hydrodynamic description

Vicsek model: agent-base model vs. hydrodynamic description

Particles at $t = 0.00$

Density and velocity at $t = 0.00$
Classical solutions must flock (with C. Tan)

\[
\rho_t + \nabla x \cdot (\rho u) = 0 \quad \text{subject to compactly supported } \rho_0
\]

\[
u_t + (u \cdot \nabla x) u = \alpha(x)(\bar{u} - u), \quad \bar{u}(t, x) := \int_y a(x, y)(u(t, y)\rho(t, y) \, dy
\]

Theorem\(^6\). Set diameter \([u(t)] := \sup_{x,y \in \text{Supp } \rho(t)} |u(t, x) - u(t, y)|

\[
\text{If } u \in C^1 \quad \implies \quad \frac{d}{dt}[u(t)] \leq -\alpha \min_{x,x'} \eta(x, x') [u(t)]:
\]

\[
\eta(x, x') \text{ is coefficient of ergodicity} := \int_{\text{Supp } \rho(t)} \min \{a(x, y), a(x', y)\} \, dy
\]

\[
\eta(x, x') \geq \phi([x(t)]) \quad \implies \quad \frac{d}{dt}[u(t)] \leq -\alpha \phi([x(t)]) [u(t)], \quad \frac{d}{dt}[x(t)] \leq [u(t)]
\]

Then — Unconditional flocking\(^6b\): \[\int_\infty \phi(s)ds = \infty \text{ implies}

(i) \([\text{Supp}(\rho)](t, \cdot) \leq S_\infty < \infty; \quad \text{(ii) } [u(t)]_{\text{Supp}(\rho)} \xrightarrow{t \to \infty} 0
\]

Proof — Lagrangian...

• Does \(u(t, \cdot) \in C^1\)? “expects” conditional regularity \(\mapsto\) critical thresholds

Critical threshold - 1D alignment

- 1D alignment
  \[ \rho_t + (\rho u)_x = 0 \]
  \[ u_t + uu_x = \int \phi(|x - y|)(u(t, y) - u(t, x))\rho(t, y)dy \]
  \[ \rho' := (\partial_t + u\partial_x)\rho = -\rho d, \quad \{\cdot\}' := \partial_t + \partial_x \quad \text{and} \quad d := u_x \]
  \[ d' + d^2 = - (\phi \ast \rho)' - d(\phi \ast \rho) \]

- Differentiate along the particle path: set \{\cdot\}' := \partial_t + \partial_x \quad \text{and} \quad d := u_x

  \[ \rho = -\rho d \rho' = -\rho d \]
  \[ - (\phi \ast \rho)_{tt} \]
  \[ d' + d^2 = \phi \ast (\rho u)_x - u(\phi \ast \rho)_x - u_x(\phi \ast \rho) = -(\partial_t + u\partial_x)\phi \ast \rho - d(\phi \ast \rho) \]

- Riccati balanced by alignment: \( (d + \phi \ast \rho)' = -d(d + \phi \ast \rho) \)

- Sub-critical data\(^7\) — Global smooth solution iff \( u_0'(x) + \phi \ast \rho_0(x) \geq 0 \)

\(^7\)Y.-P. Choi, J. Carrillo, ET, C. Tan (2015)
Alignment and Poisson forcing

\[ u_t + uu_x = \int \phi(|x - y|) (u(t, y) - u(t, x)) \rho(t, y) dy - \kappa \psi_x, \quad -\psi_{xx} = \rho \]

\[ \rho_t + (\rho u)_x = 0 \]

- Critical threshold:

  Global smooth solution if \( u'_0(x) > -\phi \ast \rho_0(x) + \sigma_+(x) \)

  Finite time breakdown if \( \exists x \) s.t. \( u'_0(x) > -\phi \ast \rho_0(x) - \sqrt{2\kappa \rho_0(x)} \)
Systems: spectral dynamics

- \[ u_t + u \cdot \nabla_x u = \nabla_x \psi[u, \nabla_x u, \rho \ldots], \quad \rho_t + \nabla_x \cdot (\rho u) = 0 \]

- Key issue: control of the matrix \( D := \left( \frac{\partial u_i}{\partial x_j} \right), \quad i, j = 1, 2, \ldots, d \)

satisfies a Riccati equation

\[
D_t + u \cdot \nabla_x D + D^2 = \partial_x^2 \psi, \quad \partial_x^2 \psi = \left( \frac{\partial^2 \psi}{\partial x_i \partial x_j} \right)_{i,j=1,\ldots,d}
\]

- Spectral dynamics: \( \lambda(D) \) an eigenvalue w/eigenpair \( \langle \ell, r \rangle = 1 \)

\[
\partial_t \lambda_i + u \cdot \nabla_x \lambda_i + \lambda_i^2 = \langle \ell_i, \partial_x^2 \psi r_i \rangle \quad i = 1, 2, \ldots, d
\]

— Difficult interaction of eigenstructure–forcing \( \cdots \langle \ell, \partial_x^2 \psi r \rangle \)

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Liu & ET, Spectral dynamics of the velocity gradient field in restricted flows (2002)
Critical thresholds in 2D Eulerian dynamics

- **2D rotation** \( u_t + u \cdot \nabla_x u = \frac{1}{\alpha} u^\top \): 

  Global smooth solution iff 
  \[ 2\alpha \omega_0 + \alpha^2 \eta_0^2 < 1, \quad \eta := \lambda_2 \left( \frac{\partial u_i}{\partial x_j} \right) - \lambda_1 \left( \frac{\partial u_i}{\partial x_j} \right) \]

- Rotation prevents finite-time breakdown...shallow water equations

- **Restricted dynamics:** \( R_i R_j(\ast) \mapsto \frac{1}{d}(\ast)I_d \)

\[
D_t + u \cdot \nabla_x D + D^2 = \partial_x^2 \Delta_x^{-1} \text{trace}(D^2) = R_i R_j(\text{trace}(D^2)) \sim \frac{1}{d} \text{trace}(D^2)I_d
\]

- **Euler-Poisson:** 

\[
D_t + u \cdot \nabla_x D + D^2 = \kappa \partial_x^2 \Delta_x^{-1}(\rho) = \kappa R_i R_j(\rho) \sim \frac{1}{d} \rho I_d
\]

The solution of 2D REP remains smooth for all time iff...

Dependence on divergence \( d := \lambda_1 + \lambda_2 \) and the spectral gap \( \eta := \lambda_2 - \lambda_1 \)

\[
\begin{cases}
\eta_0(x) \leq 0 \quad \text{and} \quad \begin{cases}
d_0(x) \geq 0, & \text{if } \rho_0(x) = 0 \\
d_0(x) \text{ arbitrary}, & \text{if } \rho_0(x) > 0
\end{cases} \\
\text{OR} \\
\rho_0(x) > 0, \eta_0(x) > 0 \quad \text{and} \quad d_0(x) > g(\rho_0(x), \eta_0(x))
\end{cases}
\quad \forall x \in \mathbb{R}^2
\]

- **Critical surface:** 

\[
g(\rho, \eta) := \text{sgn}(\eta^2 - 2\kappa \rho) \sqrt{\eta^2 - 2\kappa \rho + 2\kappa \rho \ln \left( \frac{2\kappa}{\eta^2} \right)}
\]

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Critical thresholds in 2D flocking hydrodynamics

- What about the 2D case? Critical threshold in 2D self-alignment

\[
\begin{align*}
\rho_t + \nabla_x \cdot (\rho u) &= 0 \text{ subject to compactly supported } \rho_0 \\
\mathbf{u}_t + (\mathbf{u} \cdot \nabla_x)\mathbf{u} &= \alpha(\mathbf{x}) (\mathbf{u}(t, x) - \mathbf{u}(t, x))
\end{align*}
\]

- Ricatti for \( D := \{\partial_i u_j\} \) is studied in terms of its spectral dynamics\(^\text{10}\) ...

\[{\partial_i u_j}(t, \cdot) \text{ remain bounded (and hence flock) if}
\]
(i) the initial divergence \( \text{div}_x \mathbf{u}(0) \) is not too negative;
(ii) the spectral gap \( \eta := \lambda_2(D) - \lambda_1(D) \) is not too large:

Critical threshold: \( \text{div}_x \mathbf{u}_0 > \sigma_+(\eta_0) \implies \left. \nabla_x \mathbf{u}(t, x) \right|_{\text{Supp}(\rho)} < \infty \)

- Flocking hydrodynamics with tendency ...

- Why the spectral gap?

\(^{10}\) ET & C. Tan, Critical thresholds in flocking hydrodynamics... (2014)
THANK YOU