WAVE INTERACTIONS AND NUMERICAL APPROXIMATION FOR TWO–DIMENSIONAL SCALAR CONSERVATION LAWS

MATANIA BEN–ARTZI

This talk is concerned with the approximation of solutions to multidimensional scalar conservation laws by high resolution numerical schemes. Our approach here is based on the comparison of the numerical approximation with analytic solutions. Such solutions are obtained in two cases, to be discussed in detail: The Burgers equation and the Guckenheimer equation. In both cases, the "Riemann–type" problems to be studied are genuinely two–dimensional, leading to non–trivial wave interactions, which nonetheless can be obtained analytically. As already pointed out by Lindquist , " These solutions to two–dimensional Riemann problems also supply a set of problems for testing of finite difference schemes. The richness of structure of these solutions lends itself to this purpose". We note right away that a nonlinear one–dimensional conservation law can be "rotated" in the (x, y) plane, thus forming a "two–dimensional" problem. This, however, cannot lead to the kind of wave interactions discussed here, and will not be considered (even though the consideration of such problems is important in testing the basic features of a numerical scheme).

There are three ingredients in the present talk: (a) Analytic solutions involving wave interactions due to the two–dimensional geometry. (b) A high–resolution scheme for one–dimensional conservation laws. (c) A "spatial splitting" technique which enables us to convert the one–dimensional scheme into a two–dimensional one. The point here is to try and study the "mutual interaction" of these ingredients. In particular, while (a)–(b) seem to be well studied, the interaction between (b) and (c) is not yet fully understood . This interaction is influenced by (at least) two factors, namely, the adaptivity of the particular one–dimensional scheme to "spatial splitting" and the geometric complexity of the problem. The latter includes also the interplay between a Cartesian grid and strong curvilinear waves .

We consider the initial value problem (IVP) for the equation,

$$u_t + f(u)_x + g(u)_y = 0, \tag{1}$$

$$u(x, y, 0) = \phi(x, y), \quad (x, y) \in \mathbb{R}^2, \tag{2}$$

where $u(x, y, t)$ is a real (scalar) function and $f(u)$, $g(u)$ are real smooth flux functions.

A "Riemann type" problem for (1) is the IVP where $\phi(x, y)$ is finitely valued and homogeneous of order zero,

$$\phi(x, y) = u_0(\theta), \quad \theta = \arg(x, y)(= \arctan \frac{y}{x}), \tag{3}$$

and $u_0(\theta)$ is piecewise constant in $[0, 2\pi]$ with finitely many jumps.

Recall that, for any initial function $\phi \in L^\infty(\mathbb{R}^2)$, there exists a unique (weak) solution $u(x, y, t)$ to (1)–(2). The entropy condition (which includes already the fact that u is indeed a weak solution) can be described as follows.

Let $U(s)$ be a real convex function and $F(s)$ and $G(s)$ functions such that

$$F'(s) = U'(s)f'(s), \quad G'(s) = U'(s)g'(s). \tag{4}$$

Then, in the sense of distributions,

$$U(u)_t + F(u)_x + G(u)_y \leq 0. \tag{5}$$

The initial value (2) is attained in the sense that

$$u(x, y, t) \to \phi(x, y) \quad \text{in } L^1_{loc}(\mathbb{R}^2), \quad \text{as } t \to +0. \tag{6}$$

When the initial data is given by (3), the uniqueness implies that the solution is "self–similar", namely,

$$u(x, y, t) = u(x/t, y/t, 1), \quad t > 0. \tag{7}$$

The solutions to the Riemann–type problem (1)–(3) display a rich variety of wave patterns, some of which are far from being "evident". Our intention in this talk is to show that this variety can serve as a basis for the investigation of "fine points and subtleties" pertinent to high resolution schemes. We first discuss the detailed structure of the solutions. For the Burgers equation we have $f(u) = g(u) = \frac{1}{2}u^2$. Even in this rather elementary case, we demonstrate various possibilities of wave interactions. Next we describe the solution for the "Guckenheimer equation", where $f(u) = \frac{1}{2}u^2$ and $g(u) = \frac{1}{3}u^3$.

This equation was first studied by Guckenheimer (1975) . Here we take the initial data

$$u_0(\theta) = \begin{cases} 0, & 0 < \theta < \frac{3\pi}{4}, \\ 1, & \frac{3\pi}{4} < \theta < \frac{3\pi}{2}, \\ -1, & \frac{3\pi}{2} < \pi < 2\pi. \end{cases} \tag{8}$$

The structure of the soluition can be described in the $(\xi = x/t, \eta = y/t)$ plane as follows. Outside of a large disk we obtain three shocks:

(a) A shock emanating from $y = 0$ and moving at speed $\frac{1}{3}$ in the positive y direction (note that u^3 is concave on $[-1, 0]$). In the (ξ, η) plane it is given by $\eta = \frac{1}{3}$.

(b) A standing shock along $\xi = 0$ $(\eta < 0)$.

(c) A shock emanating from the line $x + y = 0$. In the (ξ, η) plane it is given by $\xi + \eta = \frac{5}{6}$.

The interaction of these three shocks in a disk around $(0, 0)$ form a very complex wave pattern, which can be described as follows (see attached Figure).

At a certain point $(0, b)$, $0 < b < \frac{1}{3}$, the shock (b) bifurcates into a centered rarefaction wave (CRW) whose leading characteristic is a sonic shock, across which the solution v jumps from -1 to a (still unknown) value $\tilde{v}$. Then v increases across the rarefaction from $\tilde{v}$ to 1. The rarefaction wave modifies shock (c) .

Institute of Mathematics, The Hebrew University of Jerusalem, Jerusalem, 91904, Israel