Viscosity Solutions of Nonlinear Hyperbolic Systems

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Abstract: Since the classical work of Glimm [4], weak solutions of the Cauchy problem for a hyperbolic systems of conservation laws

$$u_t + f(u)_x = 0\,, \tag{1}$$

$$u(0,x) = \bar{u}(x)\,, \tag{2}$$

have been obtained by a compactness argument. In one space dimension, the total variation of small BV solutions can indeed be controlled by means of an interaction potential. Wave-interaction functionals also play a key role in [3], estimating the $\mathbf{L}^1$ distance between any two weak solutions. This provides the well-posedness of the Cauchy problem within a space of BV functions.

A further development of these ideas was recently achieved in [1], where new Lyapunov functionals were constructed, which control the total amount of oscillations in viscous approximations to hyperbolic systems. The results apply more generally to vanishing viscosity approximations of nonlinear strictly hyperbolic systems, not necessarily in conservation form:

$$u_t + A(u)u_x = \varepsilon\, u_{xx}\,. \tag{3}$$

As the viscosity coefficient $\varepsilon \to 0+$, the solutions of the viscous system (3) converge to a unique limit, continuously depending on the initial data (2). In the conservative case $A = Df$, this limit provides a weak solution to the system of conservation laws (1). Under the additional assumption of genuine nonlinearity or linear degeneracy of each characteristic field, this solution coincides with the one obtained by Glimm.

The talk will review the main ideas involved in the construction of these new interaction functionals and the uniform control of oscillations in viscous approximations. The possibility of uniform BV bounds and stability estimates for other types of approximations will be also briefly discussed.

References

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