Burgers-type equations with vanishing hyper-viscosity

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OVERVIEW

- **Burgers’-type equations**

  \[ u_t + f(u)x = -\frac{1}{\epsilon} P(\epsilon \partial_x)u, \quad |f''(u)| \geq \text{Const.} \]

  ⊗ Main question: the behavior of \( u = u^\epsilon \) as \( \epsilon \downarrow 0 \)

- **Prototype example**

  \[ u_t + \left( \frac{1}{2} u^2 \right)_x = -\frac{1}{\epsilon} F^{-1} \left[ \frac{\epsilon^2 \xi^2}{1 + \epsilon^2 \xi^2} \hat{u}(t, \xi) \right] = \epsilon F^{-1} \left[ \frac{1}{1 + \epsilon^2 \xi^2} \hat{u}(t, \xi) \right]_{xx} \]

  From viscosity at low \( \xi \)'s, \( \sim \epsilon u_{xx} \) to relaxation at large \( \xi \)'s, \( \sim -\frac{u}{\epsilon} \)

  ⊗ 'Chapman-Enskog' asymptotic expansion: \( \epsilon \) as mean-free path

    \[ u_t + \left( \frac{1}{2} u^2 \right)_x = \epsilon u_{xx} + \epsilon^3 u_{xxxx} + \ldots \]

  ⊗ Instability of Burnett and super-Burnett equations
Regularized Chapman-Enskog

\[ u_t + \left( \frac{1}{2} u^2 \right)_x = -\frac{1}{\epsilon} \mathcal{F}^{-1} \left[ \frac{\epsilon^2 \xi^2}{1 + \epsilon^2 \xi^2} \hat{u}(t, \xi) \right] = \frac{1}{\epsilon} [Q \epsilon \ast u - u], \quad Q = \frac{1}{2} e^{-|x|} \]

- Radiating gas, convolution model...: \( \phi := \epsilon \left( 1 - \epsilon^2 \partial_x^2 \right)^{-1} u_x = \epsilon Q \epsilon \ast u_x \)

Rosenau, Schochet-ET, Kawashima-Nishibata, Serre, Lattanzio-Marcati, Liu-ET, ...

\[ u_t + \left( \frac{1}{2} u^2 \right)_x = \phi_x \]

\[ \epsilon^2 \phi_{xx} = \phi - \epsilon u_x \]

- Finite time breakdown and critical threshold phenomena
- Sharper shock profile (Schochet-ET)
Krushkov and entropic convergence

\[ u_t + f(u)_x = \frac{1}{\epsilon} [Q_\epsilon * u - u], \quad 0 \leq Q, \quad \int Q(y) dy = 1. \]

- The viscous case: Krushkov theory

\[ \frac{\partial}{\partial t} \left| u_2(\cdot, t) - u_1(\cdot, t) \right| + \frac{\partial}{\partial x} F(u_2(\cdot, t), u_1(\cdot, t)) \leq 0 \]

- The convolution model: \( L^1 \) contraction but no Krushkov pairs;

- Instead — one-sided Lip condition: \( u_{x}^\epsilon(\cdot, t) \leq \text{Const} \) (ET et. al.)

\[ \| u(\cdot, t) - u^\epsilon(\cdot, t) \|_{W^{-s, p}} \leq \text{Const} \epsilon^{\frac{sp+1}{2p}}, \quad 0 \leq s \leq 1 \]

- Hyperbolic scaling \( (t, x) \rightarrow \left( \frac{t}{\epsilon}, \frac{x}{\epsilon} \right) \)
Burgers Poisson equation - dispersive effects

- Euler-Poisson: Fellner & Schmeiser ...

\[ u_t + \left( \frac{1}{2}u^2 \right)_x = \phi_x \]

\[ \epsilon^2 \phi_{xx} = \phi \pm u \]

- Chapman-Enskog: ‘dispersive’ expansion: \( \epsilon^2 u_{xxx} + \ldots \)

- Re: Camassa-Holm

\[ u_t + \left( \frac{1}{2}u^2 \right)_x = \phi_x \]

\[ \phi_{xx} = \phi + 2\kappa u + u^2 + \frac{1}{2}u_x^2 \]

- Vanishing diffusion-dispersion problem (Schonbek...)

\[ u_t + f(u)_x = \epsilon u_{xx} + \delta \epsilon u_{xxx}, \quad \delta \epsilon \sim \epsilon^2 \]

- Diffusion dominates if \( \delta \epsilon \ll \epsilon^2 \),
Reversing time – hyper-viscosity

\[ u_t - f(u)_x = + \frac{1}{\epsilon} \mathcal{F}^{-1} \left[ \frac{\epsilon^2 \xi^2}{1 + \epsilon^2 \xi^2} \hat{u}(t, \xi) \right] = - \epsilon u_{xx} - \epsilon^3 u_{xxxx} + \ldots \]

- Vanishing Kuramoto-Sivashinsky

- Hyper viscosity: \( u_t + f(u)_x = - \epsilon^3 u_{xxxx} \)

- Lack of monotonicity

- Nonlinear extensions: lubrication, thin film
  Bertozzi et. al., Otto-Westdickenberg, ...

\[ u_t + \epsilon^3 \left( \sigma(u) u_{xxx} \right)_x = 0 \]
Burgers-type equation w/vanishing hyper-viscosity

\[ \frac{\partial u^\epsilon}{\partial t} + \frac{\partial}{\partial x} f(u^\epsilon(x, t)) = (-1)^s + 1 \epsilon^{2s-1} \frac{\partial^{2s}}{\partial x^{2s}} u^\epsilon(x, t), \quad f'' > 0. \]

⊙ Krushkov BV theory for monotone viscous case \( s = 1 \)

⊙ Hyper-viscosity case, \( s > 1 \), lacks monotonicity

⊙ Instead, use compensated compactness theory

⊙ Hyper-viscosity with \( s > 1 \): weaker entropy dissipation bound than in the viscosity dominated case \( s = 1 \).
Hyper-viscosity and hyper-dissipation

\[
\frac{\partial u_N}{\partial t} + \frac{\partial}{\partial x} f(u_N(x, t)) = \frac{(-1)^{s+1}}{N^{2s-1}} \frac{\partial^{2s}}{\partial x^{2s}} u_N(x, t) =: \mathcal{V}(u_N), \quad \epsilon \sim \frac{1}{N}.
\]

- \( u_N \) has a smallest scale of order \( 1/N \).

- Quadratic entropy dissipation + production:

\[
\frac{1}{2} \frac{\partial}{\partial t} u_N^2 + \frac{\partial}{\partial x} \int_{u_N}^{u_N} \xi f'(\xi) d\xi = \frac{(-1)^{s+1}}{N^{2s-1}} u_N \frac{\partial^{2s}}{\partial x^{2s}} u_N =: \mathcal{E}(u_N)
\]

- Convergence: \( H^{-1} \) compactness of \( \mathcal{V}(u_N) \) and \( \mathcal{E}(u_N) \)

- “Differentiation by parts” (periodic BCs, say):

\[
\mathcal{E}(u_N) \equiv \frac{1}{N^{2s-1}} \sum_{p+q = 2s-1} (-1)^{s+p+1} \frac{\partial}{\partial x} \left[ \frac{\partial^p u_N}{\partial x^p} \frac{\partial^q u_N}{\partial x^q} \right] \quad - \quad \frac{\mathcal{E}_2(u_N)}{N^{2s-1} \left( \frac{\partial^s u_N}{\partial x^s} \right)^2}
\]
Decomposition into low and high modes

• Entropy dissipation estimate:

\[ \| u_N(\cdot, T) \|_{L^2}^2 + \frac{1}{N^{2s-1}} \| \partial_x^s u_N \|_{L^2(x,t)}^2 \leq \| u_N(\cdot, 0) \|_{L^2(x)}^2 \leq K_0^2. \]

\[ u_N(x, t) = \sum_{|k| \leq N} \hat{u}_N(k, t)e^{ikx} + \sum_{|k| > N} \hat{u}_N(k, t)e^{ikx} =: u^I_N(x, t) + u^H_N(x, t). \]

• Entropy dissipation of \( u^I_N \) becomes weaker for \( s > 1 \):

\[ \frac{1}{N^{2s-1}} \| \partial_x^s u_N \|_{L^2(x,t)}^2 = N \sum_{|k| \leq N} \left( \frac{|k|}{N} \right)^{2s} |\hat{u}(k, t)|_{L^2[0,T]}^2 \leq K_0^2 \]

\[ \| \partial_x^s u_N \|_{L^2} \sim N^{s-1/2}; \text{ interpolate 'gain' on } L^2\text{-growth of } \partial_x^q u_N: \]

\[ \left\| \frac{\partial^q u_N}{\partial x^q} \right\|_{L^2(x,t)} \leq \text{Const.} \left\| \partial_x^s u_N \right\|_{L^2}^{\frac{q}{s}} \times \left\| u_N \right\|_{L^2}^{1-\frac{q}{s}} \leq \text{Const.} N^{q-\delta}, \quad 1 \leq q \leq s \]
Estimates

⊙ Interpolate for \( s \leq q \leq 2s \):

\[
\left\| \frac{\partial^q u_N}{\partial x^q}(x, t) \right\|_{L^2(x,t)} \leq C_\infty \cdot N^{q-\delta}, \quad \delta_s = \frac{1}{2s}
\]

⊙ Small scale \( \sim \frac{1}{N} \): for \( 1 \leq p < s \), (clear for \( u^I_N \); entropy dissipation \( \Rightarrow \))

\[
\left\| \frac{\partial^p u_N}{\partial x^p}(x, t) \right\|_{L^2_t(L^\infty_x)} \leq K_\infty \cdot N^p, \quad K_\infty := \| u_N \|_{L^2_t(L^\infty_x)}
\]

\[
\left\| \left[ \frac{\partial^p u_N \partial^q u_N}{\partial x^p \partial x^q} \right] \right\|_{L^1_t(L^2_x)} \leq \| \frac{\partial^p u_N}{\partial x^p} \|_{L^2_t(L^\infty_x)} \times \| \frac{\partial^q u_N}{\partial x^q} \|_{L^2(x,t)} \leq \]

\[
\leq C_{pq} \cdot N^{p+q-\delta}, \quad p < s \leq q < 2s
\]

\[
\left\| \mathcal{E}_1(u_N) \right\|_{L^1_t(H^{-1}_x)} \leq \frac{1}{N^{2s-1}} \sum_{\begin{array}{c} p+q = 2s-1 \\ q \geq s \end{array}} \left\| \frac{\partial}{\partial x} \left[ \frac{\partial^p u_N \partial^q u_N}{\partial x^p \partial x^q} \right] \right\|_{L^1_t(H^{-1}_x)} \leq \]

\[
\leq \frac{1}{N^{2s-1}} \sum_{\begin{array}{c} p+q = 2s-1 \\ q \geq s \end{array}} C_{pq} \cdot N^{p+q-\delta} \leq \frac{C_s}{N^\delta} \to 0, \quad C_s \sim s^s
\]
THEOREM. Consider the hyper-viscosity solution subject to $L^2$-bounded initial data, $\|u^\epsilon(\cdot, 0)\|_{L^2} \leq K_0$ so that

$$N^{-\left(s - \frac{1}{4s}\right)} \left\| \frac{\partial^s}{\partial x^s} u_{N}(\cdot, 0) \right\|_{L^2} \leq Const.$$  

Assume $u^\epsilon(\cdot, t)$ remains uniformly bounded, $\|u^\epsilon(\cdot, t)\|_{L^\infty} < K_\infty$. Then $u^\epsilon$ converges to the unique entropy solution of the convex conservation law.

REMARK. The issue of an $L^\infty$ bound for vanishing hyper-viscosity of order $s > 1$ remains an open question.

NOTE. Entropic convergence and consistency with quadratic entropy: Panov, De Lellis et. al. ($\eta(w) = w^2, f(u) \in L^2$)
Spectral- and hyper spectral-viscosity

\[
\frac{\partial u_N}{\partial t} + \frac{\partial}{\partial x}[P_N f(u_N)] = -N \sum_{|k| \leq N} \sigma \left( \frac{|k|}{N} \right) \hat{u}_k(t)e^{ikx}.
\]

\(\sigma(\xi)\) is a symmetric low pass filter satisfying \(\sigma(\xi) \geq \left( |\xi|^{2s} - \frac{1}{N} \right)^+\)

- Spectral viscosity: \(s = 1\) viscous free modes \(|k| \leq \sqrt{N}\)
- Hyper-spectral viscosity: \(s > 1\); requires \(C_s N^{-\delta} \rightarrow 0\):
- Spectral viscosity at modes \(\left\{ k \mid \theta \frac{N}{(\log N)^{-\mu/2}} \leq |k| \leq N \right\}\)

\[
\frac{\partial u_N}{\partial t} + \frac{\partial}{\partial x}[P_N f(u_N)] = -N \sum_{m_N \leq |k| \leq N} \left( \frac{|k|}{N} \right)^s \hat{u}_k(t)e^{ikx}, \quad m_N \sim \frac{N}{(\log N)^{-\mu/2}}.
\]

⊙ No monotonicity: spurious and Gibbs oscillations are stable and contain high order information
Hyper spectral viscosity: H.-Y. Li, H.-P. Ma, ET

HSV solution based on $s = 1$ and (a) $2N = 64$ modes, (b) $2N = 128$ modes

HSV solution after post-processing
HSV solution based on $s = 2$ and (a) $2N = 64$ modes, (b) $2N = 128$ modes

HSV solution after post-processing
HSV solution based on $s = 3, 2N = 64$ modes before and after post-processing

Log-error plot of the post-processed HSV solutions for

(left) $2N = 64, s = 1, 3, 2$, (right) $2N = 128, s = 1, 2$ (dotted) and $2N = 32, s = 2$ (circled)
Vanishing Kuramoto-Sivashinsky dissipation

\[
\frac{\partial u_N}{\partial t} + \frac{\partial}{\partial x} f(u_N(x,t)) = -\frac{1}{N} \frac{\partial^2 u_N}{\partial x^2} - \frac{1}{N^3} \frac{\partial^4 u_N}{\partial x^4} := \mathcal{V}(u_N).
\]

- Rescaling such that $u_N$ has a smallest scale of order $1/N$,

\[
\left\| \frac{\partial^p u_N}{\partial x^p} (x,t) \right\|_{L^2_t(L^\infty_x)} \leq \text{Const.} N^p \cdot \| u_N(x,t) \|_{L^2_t(L^\infty_x)}, \quad p < 2
\]

- Entropy dissipation:

\[
\frac{1}{2} \frac{\partial}{\partial t} u_N^2 + \frac{\partial}{\partial x} \int_{u_N}^{u_N} \xi f'(\xi) d\xi = -\frac{1}{N} u_N \frac{\partial^2 u_N}{\partial x^2} - \frac{1}{N^3} u_N \frac{\partial^4 u_N}{\partial x^4} := \mathcal{E}(u_N)
\]

\[
\| u_N(\cdot,t) \|_{L^2}^2 + \frac{1}{N^3} \left\| \frac{\partial^2 u_N}{\partial x^2} \right\|_{L^2_t(L^2)}^2 \leq \| u_N(\cdot,0) \|_{L^2}^2 + \frac{1}{N} \left\| \frac{\partial u_N}{\partial x} \right\|_{L^2}^2
\]

- Giacomelli & Otto: in fact $\| u_N(\cdot,t) \|_{L^2} \downarrow$ as $t \uparrow \infty$
Extensions

- The multidimensional problem:
  2D compensated compactness

- The discrete framework:
  entropic discretizations of nonlinear convection

- Nonlinear hyper-viscosity
THANK YOU