Central schemes for the modified Buckley–Leverett equation

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In this paper, we extend the second and third order classical central schemes for the hyperbolic conservation laws to solve the modified Buckley–Leverett (MBL) equation which is of pseudo-parabolic type. The MBL equation describes two-phase flow in porous media, and it differs from the classical Buckley–Leverett (BL) equation by including a balanced diffusive–dispersive combination. The classical BL equation gives a monotone water saturation profile for any Riemann problem; on the contrast, when the dispersive parameter is large enough, the MBL equation delivers non-monotone water saturation profiles for certain Riemann problems as suggested by the experimental observations. Numerical results in this paper confirm the existence of non-monotone water saturation profiles consisting of constant states separated by shocks.

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1. Introduction

The classical Buckley–Leverett (BL) equation [1] is a simple model for two-phase fluid flow in a porous medium. One application is secondary recovery by water-drive in oil reservoir simulation. In one space dimension the equation has the standard conservation form

\[

t + f(u)_x = 0 \quad \text{in} \quad Q = (x, t) : x > 0, t > 0 \\
\]

\[
u(x, 0) = 0 \quad x \in (0, \infty) \\
u(0, t) = u_0 \quad t \in [0, \infty)
\]

with the flux function \( f(u) \) being defined as

\[
f(u) = \begin{cases} 
0 & u < 0, \\
u^2 & 0 \leq u \leq 1, \\
u^2 + M(1-u)^2 & u > 1.
\end{cases}
\]

In this context, \( u : \overline{Q} \to [0, 1] \) denotes the water saturation (e.g. \( u = 1 \) means pure water, and \( u = 0 \) means pure oil), \( u_0 \) is a constant which indicates water saturation at \( x = 0 \), and \( M > 0 \) is the water/oil viscosity ratio. The classical BL Eq. (1.1) is a prototype for conservation laws with convex–concave flux functions. The graph of \( f(u) \) and \( f'(u) \) with \( M = 2 \) is given in Fig. 1.

The classical BL Eq. (1.1) has been well studied (see [10] for an introduction). Let \( \alpha \) be the solution of \( f'(u) = \frac{f(u)}{u} \), i.e.,

\[
\alpha = \sqrt{\frac{M}{M + 1}}.
\]

The entropy solution of the classical BL equation can be classified into two categories:
1 If \(0 < u_B \leq \alpha\), the entropy solution has a single shock at \(\xi = \frac{u_B}{\alpha}\).

2 If \(\alpha < u_B < 1\), the entropy solution contains a rarefaction between \(u_B\) and \(\alpha\) for \(f'(u_B) < \frac{\alpha}{\xi} < f'(\alpha)\) and a shock at \(\xi = \frac{u_B}{\alpha}\).

These two types of solutions are shown in Fig. 2 for \(M = 2\). In either case, the entropy solution of the classical BL Eq. (1.1) is a non-increasing function of \(x\) at any given time \(t > 0\). However, the experiments of two-phase flow in porous medium reveal complex infiltration profiles, which may involve overshoot, i.e., profiles may not be monotone [4]. This suggests the need of modification to the classical BL Eq. (1.1).

Hassanizadeh and Gray [5,6] have included a third order mixed derivatives dispersive term, which models the dynamic effects in the capillary pressure difference between the two phases. Following the linearization and rescaling in [14–16], the modified Buckley–Leverett equation (MBL) is derived as

\[
\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = \epsilon \frac{\partial^2 u}{\partial x^2} + \epsilon^2 \tau \frac{\partial^3 u}{\partial x^3 \partial t},
\]

where \(\epsilon\) is the diffusion coefficient. van Duijn et al. [15] showed how \(\epsilon\) and \(\tau\) determine the type of the solution profile. In particular, for certain Riemann problems, the solution profile of (1.4) is not monotone when \(\tau\) is larger than the threshold value \(\tau_\ast\), where \(\tau_\ast\) was numerically determined to be 0.61 [15]. The non-monotonicity of the solution profile is consistent with the experimental observations [4].

The classical BL Eq. (1.1) is hyperbolic, and the numerical schemes for hyperbolic equations have been well developed (e.g. [10,11,2,3,13,8]). The MBL Eq. (1.4), however, is pseudo-parabolic. van Duijn et al. [15] have developed a first order finite difference scheme to solve the MBL Eq. (1.4). In this paper, we will illustrate how to extend the second and third order central schemes [13,8,9] to solve (1.4) numerically. The local discontinuous Galerkin method has been applied to solve equations involving mixed derivatives \(u_{ext}\) [18,19]. To the best knowledge of the authors, the central schemes have not been applied to solve equations of this kind. The main advantage of the central schemes is the simplicity. The “direction of the wind” is not required to be identified, and hence the field-by-field decomposition can be avoided.

Unlike the finite domain of dependence for the classical BL Eq. (1.1), the domain of dependence for the MBL Eq. (1.4) is infinite. This naturally raises the question for the choice of computational domain. To answer this question, Wang et al. [17] studied the MBL equation equipped with two types of domains, one is the half line domain \(x \in [0, + \infty)\), and the other one is finite interval domain \(x \in [0, L]\). Wang et al. [17] have shown that the difference between the solutions of these two types of problems decays exponentially with respect to the length of the interval \(L\) for practically interesting initial profiles. This provides a theoretical justification for the choice of the computational domain. Therefore, the numerical results in this paper are sought on the finite interval domain \(x \in [0, L]\) with sufficiently large \(L\).

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**Fig. 1.** \(f(u)\) and \(f(u)\) with \(M = 2\).

**Fig. 2.** The entropy solution of the classical BL equation \((M = 2, \alpha = \sqrt{\frac{\alpha}{\xi}} \approx 0.8165)\). (a) \(0 < u_B = 0.7 \leq \alpha\), the solution consists of one shock at \(\xi = \frac{u_B}{\alpha}\); (b) \(\alpha < u_B = 0.98 < 1\), the solution consists of a rarefaction between \(u_B\) and \(\alpha\) for \(f'(u_B) < \frac{\alpha}{\xi} < f'(\alpha)\) and a shock at \(\xi = \frac{u_B}{\alpha}\).
The organization of this paper is as follows. In Section 2, the second and third order central schemes will be developed for MBL equation in the finite interval domain. We provide a detailed derivation on how to extend the central schemes [13,8] for conservation laws to solve the MBL Eq. (1.4). The idea of adopting numerical schemes originally designed for hyperbolic equations to pseudo-parabolic equations is not restricted to central type schemes only [18,19]. The numerical results in Section 3 show that the water saturation profile strongly depends on the dispersive parameter \( \tau \) as studied in [15]. For \( \tau > \tau ^* \), the MBL Eq. (1.4) gives non-monotone water saturation profiles for certain Riemann problems as suggested by experimental observations [4]. Section 4 gives the conclusion of the paper and the possible future directions.

2. Numerical schemes

In this section, we show how to apply the central schemes [13,8] originally designed for hyperbolic conservation laws to numerically solve the MBL Eq. (1.4), which is of pseudo-parabolic type. Specifically, we solve the following finite domain initial boundary value problem

\[
\begin{align*}
\frac{\partial u}{\partial t} + f(u) \frac{\partial u}{\partial x} &= \epsilon u_{xx} + \epsilon^2 u_{xx} \quad x \in (0, L), \ t > 0, \\
u(0, t) &= \epsilon g_0(x) \\
u(0, t) &= \epsilon g_t(t), \ u(L, t) = 0.
\end{align*}
\]

(2.1)

We first collect all the terms with time derivative and rewrite MBL Eq. (2.1) as

\[
\frac{\partial u}{\partial t} + f(u) \frac{\partial u}{\partial x} = \epsilon u_{xx}.
\]

(2.2)

For the sake of convenience, we let

\[
\begin{align*}
w_j &= u(x, t) - \epsilon u_{xx}, \\
\lambda &= \max_{j \in S_j} \left| \frac{\partial f(u(x, t))}{\partial u} \right|.
\end{align*}
\]

(2.3)

and the CFL condition

\[
\lambda \cdot \max_{j \in S_j} \left| \frac{\partial f(u(x, t))}{\partial u} \right| \leq \frac{1}{2}, \quad \text{where} \quad \lambda = \frac{\Delta t}{\Delta x}.
\]

(2.4)

is met. For MBL Eq. (2.4), we have that at \( t > 0 \),

\[
\begin{align*}
\epsilon u_{xx} &= w, \\
u(0) &= \epsilon g_0(t), \quad u(L, t) = \epsilon g_L(t).
\end{align*}
\]

(2.5)

To solve the boundary value problem (2.12), we let \( v(x) = \frac{w(x)}{1 + \epsilon^2 \tau}, \) then

\[
\begin{align*}
u(x) &= (1 - \epsilon^2 \tau) \frac{w(x)}{1 + \epsilon^2 u_{xx}(x)}, \\
u(x) &= (1 - \epsilon^2 \tau) \frac{w(x)}{1 + \epsilon^2 \tau}. \quad x \in (0, L).
\end{align*}
\]

(2.6)

We denote the staggered piecewise-constant functions \( \frac{\partial u}{\partial t} \) as

\[
\begin{align*}
\lambda^*_k &= \max_{j \leq x \leq j+1} \left| \frac{\partial f(u(x, t))}{\partial u} \right| \cdot \lambda_k
\end{align*}
\]

(2.7)

and the CFL condition

\[
\lambda^* \lambda \cdot \max_{j \leq x \leq j+1} \left| \frac{\partial f(u(x, t))}{\partial u} \right| \leq \frac{1}{2}, \quad \text{where} \quad \lambda^* = \frac{\Delta t}{\Delta x}.
\]

(2.8)

Therefore, the CFL condition is

\[
\frac{\Delta t}{\Delta x} \leq \frac{1}{2}.
\]

(2.9)

Evolve the piecewise linear interpolant (2.6) by integrating (2.4) over \([x_j, x_{j+1}] \times [t, t + \Delta t]\)

\[
\begin{align*}
\frac{\partial w_j}{\partial t} &= \frac{w_{j+1}}{2} - \frac{w_j}{2} + \frac{\int_{x_j}^{x_{j+1}} f(u(x, t)) \, dx}{\Delta x} - \frac{\int_{x_j}^{x_{j+1}} f(u(x, t)) \, dx}{\Delta x}
\end{align*}
\]

(2.10)

We calculate each term on the right hand side of (2.9) below. For \( \frac{\partial w_j}{\partial t} \), applying the definition of \( L_j(x, t) \) and \( L_{j+1}(x, t) \) given in (2.6) to (2.8), we have that

\[
\begin{align*}
\frac{\partial w_j}{\partial t} &= \frac{1}{\Delta x} \int_{x_j}^{x_{j+1}} L_j(x, t) \, dx + \frac{1}{\Delta x} \int_{x_j}^{x_{j+1}} L_{j+1}(x, t) \, dx
\end{align*}
\]

(2.11)

The middle two integrands can be approximated by the midpoint rule

\[
\begin{align*}
\int_{x_j}^{x_{j+1}} L_j(x, t) \, dx &= f(u(x_j, t + \Delta t/2)) \Delta t + O(\Delta t^3)
\end{align*}
\]

(2.12)

We denote the staggered piecewise-constant functions \( \frac{\partial u}{\partial t} \) as

\[
\begin{align*}
\lambda^*_k &= \max_{j \leq x \leq j+1} \left| \frac{\partial f(u(x, t))}{\partial u} \right| \cdot \lambda_k
\end{align*}
\]

(2.13)

and the CFL condition

\[
\lambda^* \lambda \cdot \max_{j \leq x \leq j+1} \left| \frac{\partial f(u(x, t))}{\partial u} \right| \leq \frac{1}{2}, \quad \text{where} \quad \lambda^* = \frac{\Delta t}{\Delta x}.
\]

(2.14)

Therefore, the CFL condition is

\[
\frac{\Delta t}{\Delta x} \leq \frac{1}{2}.
\]

(2.15)
In the numerical computations in Section 3, we chose $\frac{\Delta t}{\Delta x} = 0.1$. In (2.11), to estimate $u \left( \cdot, t + \frac{\Delta t}{2} \right)$’s, we use Taylor expansion and the conservation law (2.4):

$$w(x_j, t + \frac{\Delta t}{2}) = w_j(t) + \frac{\partial w}{\partial t} \frac{\Delta t}{2} + O(\Delta t^2)$$

$$= w_j(t) + \left( \varepsilon \frac{\partial^2 u}{\partial x^2} - \frac{\partial f}{\partial x} \right) \frac{\Delta t}{2} + O(\Delta t^2)$$

$$= w_j(t) + (\varepsilon \Delta x D^2 u_j - f_j)^\lambda \frac{\Delta t}{2},$$

where $D$ is the discrete central difference operator

$$D^2 u_j = \frac{u_{j-1} - 2u_j + u_{j+1}}{\Delta x^2},$$

and the second-order accuracy is met if

$$\frac{f_j'}{\Delta x} = \frac{\partial f(u(x_j), t)}{\partial x} + O(\Delta x).$$

(2.14)

The choices for $\{w_j'\}$ in (2.7) and $\{f_j'\}$ in (2.14) can be found in [13], and we chose

$$w_j' = MM(\Delta w_j + \Delta w_{j-1}), \quad f_j' = MM(\Delta f_j + \Delta f_{j-1})$$

(2.15)

where $MM(x, y) = \min\{(x), y\}$, $\min\{(x), (y)\}$ and $\Delta w_j = w_{j+1} - w_j$. Notice that (2.15) determines $w_j'$ and $f_j'$ values ultimately based on one-sided difference, which makes the proposed schemes not purely central. However, this choice ensures the proposed schemes to be non-oscillatory.

Combining (2.9)–(2.11), we obtain

$$\bar{w}_{j+\frac{1}{2}}(t + \Delta t) = \bar{w}_{j+\frac{1}{2}}(t)$$

$$- \lambda \left[ f \left( u_{j+1} \left( t + \frac{\Delta t}{2} \right) \right) - f \left( u_j \left( t + \frac{\Delta t}{2} \right) \right) \right]$$

$$+ \frac{\varepsilon}{\Delta x} \int_t^{t+\Delta t} \int_{x_j}^{x_{j+1}} \frac{\partial^2 u(x, s)}{\partial x^2} \, dx \, ds.$$  

(2.16)

Next, we will re-write (2.16) in terms of $u$. $\bar{w}_{j+\frac{1}{2}}(t + \Delta t)$ is approximated as

$$\frac{1}{\Delta x} \int_{x_j}^{x_{j+1}} u_{x\bar{w}} \, dx = \frac{1}{\Delta x} \left( u(x_{j+1}, t) - u_j(t) \right),$$

and using the cell averages, it becomes

$$\bar{u}_{x\bar{w}}_{j+\frac{1}{2}} = \frac{1}{\Delta x} \left( \frac{\bar{u}_{j+\frac{3}{2}} - \bar{u}_{j+\frac{1}{2}}}{\Delta x} - \frac{\bar{u}_{j+\frac{1}{2}} - \bar{u}_{j-\frac{1}{2}}}{\Delta x} \right)$$

$$= \frac{\bar{u}_{j+\frac{3}{2}} - 2\bar{u}_{j+\frac{1}{2}} + \bar{u}_{j-\frac{1}{2}}}{(\Delta x)^2}$$

$$= D^2 \bar{u}_{x\bar{w}}_{j+\frac{1}{2}}.$$  

(2.17)

Notice that the linear interpolation (similar to (2.6))

$$\int_t^{t+\Delta t} \int_{x_j}^{x_{j+1}} \frac{\partial^2 u(x, s)}{\partial x^2} \, dx \, ds$$

$$= \Delta x \int_t^{t+\Delta t} \int_{x_j}^{x_{j+1}} \frac{\partial^2 u(x, s)}{\partial x^2} \, dx \, ds$$

$$= \frac{\Delta x \Delta t}{2} \left( \bar{w}_{x\bar{w}}_{j+\frac{1}{2}}(t + \Delta t) \right) + O(\Delta t^3)$$

(2.18)

with $O(\Delta t^3)$ error. Combining with (2.17) and (2.19), we can get the trapezoid scheme

$$\left( I - \left( \varepsilon^2 \tau + \frac{\varepsilon \Delta t}{2} \right) D^2 \right) u_{j+\frac{1}{2}}(t + \Delta t)$$

$$= \left( I - \left( \varepsilon^2 \tau - \frac{\varepsilon \Delta t}{2} \right) D^2 \right) \bar{u}_{j+\frac{1}{2}}(t)$$

$$- \lambda \left[ f \left( u_{j+1} \left( t + \frac{\Delta t}{2} \right) \right) - f \left( u_j \left( t + \frac{\Delta t}{2} \right) \right) \right].$$

(2.19)

The flow chart of the trapezoid scheme is given in (2.22)
2.12. Midpoint scheme

In this scheme, we use the notion (2.8) and the midpoint rule to calculate the integral numerically as follows:

\[
\int_t^{t + \Delta t} \int_{x_j}^{x_{j+1}} \frac{\partial^2 u(x, s)}{\partial x^2} \, dx \, ds = \Delta x \int_t^{t + \Delta t} \left( \bar{u}_{j+\frac{1}{2}}(s) \right) \, ds \\
= \Delta x \Delta t \mathbb{E}_{j+\frac{1}{2}} \left( t + \frac{\Delta t}{2} \right).
\]

Combining with (2.17) and (2.19), we can get the midpoint scheme

\[
(I - \varepsilon^2 t^2) u_j(t + \Delta t) = \bar{w}_{j+\frac{1}{2}}(t) - \lambda \left[ f \left( u_{j+1} \left( t + \frac{\Delta t}{2} \right) \right) - f \left( u_j \left( t + \frac{\Delta t}{2} \right) \right) \right] \\
+ \varepsilon \Delta t^2 \Pi_{j+\frac{1}{2}} \left( t + \frac{\Delta t}{2} \right).
\]

The flow chart of the midpoint scheme is given in (2.24)

\[
\begin{align*}
\bar{w}_{j+\frac{1}{2}}(t) & \xrightarrow{(2.10)} u_j(t) \\
\bar{w}_{j+\frac{1}{2}}(t) & \xleftarrow{(2.23)} u_j(t)
\end{align*}
\]

\[
\begin{align*}
\bar{w}_{j+\frac{1}{2}}(t) & \xrightarrow{(2.18)} \bar{w}_{j+\frac{1}{2}}(t + \frac{\Delta t}{2}) & \bar{w}_{j+\frac{1}{2}}(t + \frac{\Delta t}{2}) & \xrightarrow{(2.18)} \bar{w}_{j+\frac{1}{2}}(t + \frac{\Delta t}{2}) \\
\bar{w}_{j+\frac{1}{2}}(t + \Delta t) & \xrightarrow{(2.23)} u_j(t + \Delta t)
\end{align*}
\]

2.2. A third order semi-discrete scheme

Similarly, we can extend the third order scheme to solve MBL Eq. (1.4), however, it is more involved. But the third order semi-discrete central scheme proposed in [8] can be extended to solve the MBL equation in a straightforward manner. In order to make the paper self-contained, we include the formulation below.

\[
\frac{d\bar{w}_j}{dt} = -H_{j+1/2}(t) - H_{j-1/2}(t) + \varepsilon Q_j(t)
\]

where \(\bar{w}(x, t)\) denotes the cell average of \(w\)

\[
\bar{w}_j(t) = \frac{1}{Ax} \int_{x_{j-1/2}}^{x_{j+1/2}} w(x, t) \, dx.
\]

\(H_{j+1/2}(t)\) is the numerical convection flux and \(Q_j(t)\) is a high-order approximation to the diffusion term \(u_{xx}\)

\[
H_{j+1/2}(t) = \frac{f(u_{j+1/2}^+(t)) + f(u_{j+1/2}^-(t))}{2} - \frac{a_{j+1/2}(t)}{2} \left[ w_{j+1/2}^+(t) - w_{j+1/2}^-(t) \right]
\]

where \(u_{j+1/2}^+(t), u_{j+1/2}^-(t)\) denote the left and right intermediate values of \(u(x, t)\) at \(x_{j+1/2}\), and their values are converted from the \(w_{j+1/2}^+(t), w_{j+1/2}^-(t)\) using (2.3). The way to calculate \(w_{j+1/2}^+(t), w_{j+1/2}^-(t)\) and \(a_{j+1/2}(t)\) is

\[
\begin{align*}
w_{j+1/2}^+(t) &= A_{j+1} + \frac{Ax}{2} B_{j+1} + \frac{\Delta x^2}{8} C_{j+1}, \\
w_{j+1/2}^-(t) &= A_j + \frac{Ax}{2} B_j + \frac{\Delta x^2}{8} C_j, \\
a_{j+1/2}(t) &= \max \left\{ \frac{\partial f}{\partial u} (u_{j+1/2}^-(t)), \frac{\partial f}{\partial u} (u_{j+1/2}^+(t)) \right\},
\end{align*}
\]

where

\[
\begin{align*}
A_j &= \bar{w}_j^0 - \frac{C_j L}{12} (\bar{w}_{j+1} - 2\bar{w}_j^0 + \bar{w}_{j-1}^0), \\
B_j &= \frac{1}{\Delta x} \left[ C_{j+1} (\bar{w}_{j+1}^0 - \bar{w}_j^0) + C_j (\bar{w}_j^0 - \bar{w}_{j-1}^0) + C_{j-1} (\bar{w}_{j-1}^0 - \bar{w}_{j-2}^0) \right], \\
C_j &= 2C_j L (\bar{w}_{j+1}^0 - 2\bar{w}_j^0 + \bar{w}_{j-1}^0), \\
C_i &= \frac{\partial^2 f}{\partial u^2} (u_j) = \frac{c_i}{\varepsilon a^m_j (c_0 + IS_j)^p}, \\
c_i &= c = 1/4, \\
I &= 1/2, \\
\epsilon_0 &= 10^{-6}, \\
p &= 2.
\end{align*}
\]

The diffusion \(u_{xx}\) is approximated using the following fourth-order central differencing form

\[
Q_j(t) = -\Pi_{j-2} + 16 \Pi_{j-1} - 30 \Pi_j + 16 \Pi_{j+1} - \Pi_{j+2}.
\]

(2.25)

The boundary conditions (2.1) are extended to the ghost points at the boundaries. The scheme is semi-discrete in the sense that the discretization is done in space first, and then the time evolution equation can be solved as a system of ordinary differential equations using any ODE solver of third order or higher. In this paper, we simply use the standard fourth order Runge–Kutta methods. Notice that to achieve the third order accuracy, the linear solver that converts \(u\) from \(w\) using (2.3) need also to be high order, and (2.25) is used to discretize \(u_{xx}\) in our convention.

3. Computational results

In this section, we show the numerical solutions to the MBL equation

\[
\begin{align*}
(\partial_t + f(u) \partial_x) u &= \varepsilon u_{xx} + \varepsilon^2 u_{xx} \\
u_t(x, 0) &= u_0(x), \\
u_x(x, 0) &= 0.
\end{align*}
\]

(3.1)

To validate the order analysis given in Section 2 for various schemes proposed, we first test the order of our schemes numerically with a smooth initial condition

\[
u_0(x) = u_0(x - 5, 5),
\]

where

\[
H(x, \xi) = \begin{cases} 
1 & \text{if } x < -\xi \\
1 - \frac{1}{2} \left( 1 + \frac{x}{\xi} + \frac{1}{\xi} \sin \left( \frac{\pi x}{\xi} \right) \right) & \text{if } -\xi \leq x \leq \xi \\
0 & \text{if } x > \xi
\end{cases}
\]
The final time $T = 1$ was employed, so that there was no shock created, $\epsilon$ in the MBL Eq. (3.1) is taken to be 1, $M$ is taken to be 2, and the computational interval is $[-10, 20]$. The $L_1$, $L_2$, $L_\infty$ order tests of the trapezoid scheme and the third order semi-discrete scheme with different parameter $\tau$ values and the initial condition $u_B$ are given in Tables 1 and 2. Table 1 shows that the trapezoid rule achieved second order accuracy for all the tested cases in $L_1$, $L_2$, $L_\infty$ sense. Table 2 shows that the semi-discrete scheme has the order of accuracy greater than 2.3 for all the cases, and exceeds 3 for some cases. This confirms the accuracy study given in Sections 2.1.1 and 2.2 respectively.

We will now use examples to study the solutions to MBL Eq. (3.1) using the numerical schemes proposed in Section 2. We first notice that if we scale $\tau$ and $x$ as follows

$$\tilde{\tau} = \frac{\tau}{\epsilon}, \quad \tilde{x} = \frac{x}{\epsilon},$$

then MBL (3.1) equation can be written in terms of $\tilde{\tau}$ and $\tilde{x}$ as follows

$$u_{\tilde{\tau}} + (f(u))_{\tilde{x}} = u_{\tilde{\tau} \tilde{x}} + \tau u_{\tilde{xx}}$$

(3.2)

The scaled Eq. (3.2) shows that it is the magnitude of $\frac{\tau}{\epsilon}$ and $\frac{x}{\epsilon}$ that determine the asymptotic behavior, not $\tau$, $x$, neither $\epsilon$ alone [15]. In addition, (3.2) also shows that the dispersive parameter $\tau$ denotes the relative importance of the dispersive term $u_{\tilde{xx}}$. The bigger $\tau$ is, the more dispersive effect Eq. (3.1) has. This can be seen from the computational results to be shown later in this section.

van Duijn et al. [15] numerically provided a bifurcation diagram (Fig. 3) of MBL (3.1) equation as the dispersive parameter $\tau$ and the post-shock value $u_B$ of the initial condition vary. The solution of (3.1) has been proven to display qualitatively different profiles for parameter values $(\tau, u_B)$ falling in different regimes of the bifurcation diagram. In particular, for every fixed $\tau$ value, there are two

![bifurcation diagram]

Fig. 3. The bifurcation diagram of the MBL Eq. (1.4) with the bifurcation parameters $(\tau, u_B)$.
critical $u_B$ values, namely, $\bar{u}$ and $u$. From the bifurcation diagram (Fig. 3), it is clear that, when $\tau > \tau_c$, $\bar{u} = u = \alpha$. For a fixed $\tau$ value, the solution has three different profiles.

(a) If $u_B \in [\bar{u}, 1]$, the solution contains a plateau value $u_B$ for $0 \leq \frac{x}{t} \leq \frac{df}{du}(u_B)$, a rarefaction wave connecting $u_B$ to $\bar{u}$ for $\frac{df}{du}(\bar{u}) \leq \frac{df}{du}(u_B) \leq \frac{df}{du}(\bar{u})$, another plateau value $\bar{u}$ for $\frac{df}{du}(\bar{u}) < \frac{df}{du}(u_B)$, and a shock from $\bar{u}$ down to 0 at $\frac{x}{t} = \frac{\bar{u}}{u_B}$ (see Fig. 4(a)).

(b) If $u_B \in (0, \bar{u})$, the solution contains a plateau value $u_B$ for $0 \leq \frac{x}{t} \leq \frac{\bar{u}}{u_B}$, a shock from $u_B$ up to $\bar{u}$ at $\frac{x}{t} = \frac{\bar{u}}{u_B}$, another plateau value $\bar{u}$ for $\frac{\bar{u}}{u_B} < \frac{x}{t} < \frac{\bar{u}}{u_B}$, and a shock from $\bar{u}$ down to 0 at $\frac{x}{t} = \frac{\bar{u}}{u_B}$ (see Fig. 4(b)). The solution may exhibit a damped oscillation near $u = u_B$.

(c) If $u_B \in (0, \bar{u})$, the solution consists of a single shock connecting $u_B$ and 0 at $\frac{x}{t} = \frac{f(u_B)}{u_B}$ (see Fig. 4(c)). It may exhibit oscillatory behavior near $u = u_B$.

**Fig. 4.** Given a fixed $\tau$, the three qualitatively different solution profiles due to different values of $u_B$. In particular, when $\tau > \tau_c$ and $\bar{u} < u_B < \bar{u}$, the solution profile (b) displays non-monotonicity, which is consistent with the experimental observations [4]. (a)–(c) are demonstrative figures.
Table 3

<table>
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<th>Example 4</th>
<th>Example 5</th>
<th>Example 6</th>
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<td>(5.0, 0.75)</td>
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Notice that when τ > τ∗ and u < uB < Π, the solution profiles (Fig. 4(b)) displays non-monotonicity, which is consistent with the experimental observations [4].

In the numerical computation we show below, we will therefore test the accuracy and capability of central schemes for different parameter values (τ and uB) that fall into various regimes of the bifurcation diagram, and therefore display qualitatively different solution profiles. The numerical experiments were carried out for M = 2, e = 0.001 and T = 4000 × e, i.e. T = 4000 to get the asymptotic solution profiles, and Δx was chosen to be Δx = 0.01 and Δt was chosen to be 0.01. The scheme used in the computation is the second order Trapezoid scheme as shown in Section 2.1.1. The Midpoint scheme delivers similar computational results, hence is omitted here. The solution profiles at x = 0.25 (blue), x = 0.5 (green), x = 0.75 (magenta) and T (black) are chosen to demonstrate the time evolution of the solutions. The red dashed lines are used to denote the theoretical shock locations and plateau values for comparison purpose.

We start with τ > 1. Based on the bifurcation diagram (Fig. 3), we choose three representative uB values, i.e. uB = 0.9 < α, uB = α = \sqrt{\frac{M}{2+e}} = \sqrt{\frac{2}{\tau}} (for M = 2) and uB = 0.75 < α. For each fixed uB, we choose three representative τ values, i.e. τ = 0.2 < τ∗ < 0.61, τ = 1 < τ∗ with uB = 0.75 < Π, the solution is consistent with the experimental observations. Notice that as τ increases from 1 to 5, the dispersive effect increases, the inter-shock interval length increases at every fixed time (compare Fig. 5(e) with Fig. 5(f)). In addition, for a fixed τ = 1 (τ∗ = 5 respectively), as time progresses, the inter-shock interval length increases in the linear fashion (see Fig. 5(e) and (f) respectively).

Example 3. \( (τ, uB) = (0.2, 0.75), (τ, uB) = (1.0, 0.75), (τ, uB) = (5.0, 0.75) \).

When uB = 0.75 < α is fixed, we increase τ from 0.2 to 1 to 5 (Fig. 5(g)–(i)), the start to dominate the solution profile in the similar fashion as \( uB = 0.9 \) and \( uB > α \). Notice that when τ = 1, since uB = 0.75 is very close to uB−1, the solution displays oscillation at \( \frac{uB}{uB^−1} = 0.75 \). If we increase τ further to τ = 5, the dispersive effect is strong enough to create a plateau value at \( Π ≈ 0.98 \) (see Fig. 5(i)).

Example 4. \( (τ, uB) = (0.2, 0.9), (τ, uB) = (1.0, α), (τ, uB) = (5.0, 0.75) \).

Now, we fix τ = 0.2. decrease uB from 0.9 to 0.75 to 0.5 (Fig. 5(a), (d) and (g)). If uB > α the solution consists a rarefaction wave connecting uB down to α, then a shock from α to 0, otherwise, the solution consists a single shock from uB down to 0. In all cases, since \( τ < τ∗ < α \), regardless of the uB value, the solution will not display non-monotone behavior, due to the lack of dispersive effect.

Example 5. \( (τ, uB) = (1.0, 0.9), (τ, uB) = (1.0, α), (τ, uB) = (5.0, 0.75) \).

Now, we fix τ = 1, decrease uB from 0.9 to α to 0.75 (Fig. 5(b), (e) and (h)). If uB = 0.9 > Π, the solution consists a rarefaction wave connecting uB and Π, and a shock connecting Π down to 0 (Fig. 5(b)).

Example 6. \( (τ, uB) = (5.0, 0.9), (τ, uB) = (5.0, α), (τ, uB) = (5.0, 0.75) \).

Now, we fix τ = 5, decrease uB from 0.9 to α to 0.75 (Fig. 5(c), (f) and (i)). For all three uB, they are between uB−5 and Π−5, hence all increase to the plateau value Π−5 ≈ 0.98 before dropping to 0. Notice that as uB decreases, the inter-shock interval length decreases at every fixed time (compare Fig. 5(c), (f) and (i)). This shows that when the dispersive effect is strong (τ > τ∗), the bigger uB is, the bigger region the solution stays at the plateau value.

Example 7. \( (τ, uB) = (0.0, 0.9), (τ, uB) = (0.0, α), (τ, uB) = (0.0, 0.75) \).

We now show the solution profiles for the extreme τ value, i.e. τ = 0 in Fig. 6(a) (uB = 0.9), (b) (uB = α) and (c) (uB = 0.75). Notice that these are cases of classical BL equation with small diffusion \( \varepsilon \). We compare Fig. 6(a)–(c) with the solution of the classical BL equation given in Fig. 2(a) and (b), it is clear that they show qualitatively same solution profiles. The difference is that due to the diffusion term in the MBL equation, as shown in Fig. 6, the solutions do not have sharp edges right at the shock, instead, the solutions smears out a little. Notice that this smearing effect is also partially introduced by the central scheme. It is well known that central scheme is non-oscillatory, i.e. it generates numerical viscosity. If we compare Fig. 6(a)–(c) with Fig. 5(a), (d) and (g), there is no visible difference. This shows that once \( τ < τ∗ \), solution profile will stay the same for a fixed uB value.

Example 8. \( (τ, uB) = (5.0, 0.99), (τ, uB) = (5.0, 0.98), (τ, uB) = (5.0, 0.97) \).

We also study the solution profiles for uB close to Π. For example, when τ = 5, Π ≈ 0.98, we hence choose uB = 0.99, uB = 0.98, uB = 0.97 and solutions are shown in Fig. 7(a)–(c). If uB = 0.99 > Π−5 ≈ 0.98, the solution drops to the plateau value Π, then drops to 0 (see Fig. 7(a)). If uB = 0.98 < Π−5 = 0.98, the solution remains at plateau value Π−5 and then drop down to 0 (see Fig. 7(b)). If uB = 0.97 < Π−5 < Π−5, the solution increases to the plateau value Π−5 ≈ 0.98, then drops to 0 (see Fig. 7(c)). In all cases, the transition from uB to Π−5 ≈ 0.98 takes very small space. In the majority space, the solution keeps to be the plateau value Π−5 ≈ 0.98.
Fig. 5. Numerical solutions to MBL equation with parameter settings fall in different regimes of the bifurcation diagram (Fig. 3). The color coding is for different time: \(\frac{1}{4}T\) (blue), \(\frac{1}{2}T\) (green), \(\frac{3}{4}T\) (magenta) and \(T\) (black). The results are discussed in examples 1–6. In figures (d)–(f), \(\alpha = \sqrt{\frac{M}{N}} = \sqrt{2}\) for \(M = 2\). (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

Fig. 6. The numerical solutions of the MBL equation at \(T = 1\) with \(\tau = 0\) and different \(u_B\) values. The results are discussed in example 7.

Fig. 7. Numerical solutions to MBL equation with \(u_B\) close to \(B_{u_B} \approx 0.98\). The color coding is for different time: \(\frac{1}{4}T\) (blue), \(\frac{1}{2}T\) (green), \(\frac{3}{4}T\) (magenta) and \(T\) (black). The results are discussed in example 8. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)
Fig. 8. Numerical solutions to MBL equation with \( u_0 \) close to \( \eta_{-5} = 0.68 \). The color coding is for different time: \( \frac{1}{4}T \) (blue), \( \frac{2}{4}T \) (green), \( \frac{3}{4}T \) (magenta) and \( T \) (black). The results are discussed in example 9. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

Fig. 9. Numerical solutions to MBL equation with small constant \( u_0 = 0.6 \) and different \( \tau \) values. The figures on the second and third rows are the magnified versions of the first row at \( t = \frac{1}{4}T \) and \( t = T \) respectively. The color coding is for different time: \( \frac{1}{4}T \) (blue), \( \frac{2}{4}T \) (green), \( \frac{3}{4}T \) (magenta) and \( T \) (black). The results are discussed in examples 10. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)
Example 9. \((\tau, u_B) = (5, 0.7), (\tau, u_B) = (5, 0.69), (\tau, u_B) = (5, 0.68), (\tau, u_B) = (5, 0.67), (\tau, u_B) = (5, 0.66)\).

In addition, we study the solution profiles for \(u_B\) close to \(u\). For example, when \(\tau = 5\), \(u_B = 0.68\), we hence choose \(u_B = 0.7, u_B = 0.69, u_B = 0.68, u_B = 0.67, u_B = 0.66\) and solutions are shown in Fig. 8(a)–(e). As \(u_B\) decreases crossing \(\frac{\Upsilon}{\Delta S} \approx 0.68\), the solution gradually stops increasing to the plateau value \(\bar{u}_{\tau = 5}\), and the inter-shock interval length decreases (compare Fig. 8(a)–(c)). The oscillation in Fig. 8(d) and (e) are due to the fact that \(u_B\) values are too close to \(\bar{u}_{\tau = 5}\). This confirms that even with big dispersive effect (say \(\tau = 5\)), if \(u_B\) is too small (e.g. \(u_B < u\)), the solution will not exhibit non-monotone behavior.

Example 10. \((\tau, u_B) = (0.2, 0.6), (\tau, u_B) = (1, 0.6), (\tau, u_B) = (5, 0.6)\).

We fix \(u_B\) to be small, and in this example, we take it to be \(u_B = 0.6\). We vary the \(\tau\) value, from \(\tau = 0.2 < \tau < 1\) barely larger than \(\tau\) to \(\tau > 5 > \tau\). The numerical solutions are given in Fig. 9(a)–(c). As \(\tau\) increases, the post-shock value remains the same, but there will be oscillation generated as \(\tau\) becomes larger than \(\tau\). Fig. 9(d)–(f) show that as \(\tau\) increases, the oscillation amplitude increases and oscillates more rounds. Notice that \(\tau\) is the dispersive parameter, and this means that even for small \(u_B\) value, different dispersive parameter values still give different dispersive effects, although none can bring the solution to the plateau value \(\bar{u}\). Comparing Fig. 9(d)–(f) with Fig. 9(g)–(i), it is clear that the oscillation amplitude remains steady with respect to time.

Example 11. \(\epsilon = 0.001, \epsilon = 0.002, \epsilon = 0.003, \epsilon = 0.004, \epsilon = 0.005\).

In this example, we will compare the solution profiles for different \(\epsilon\) values. Fixing \(T = 0.5, \Delta x = 0.0001, \lambda = \frac{\Delta x}{\Delta \tau} = 0.1\), we show the numerical results in Fig. 10 for \(\epsilon = 0.001\) (blue), \(\epsilon = 0.002\) (yellow), \(\epsilon = 0.003\) (magenta), \(\epsilon = 0.004\) (green), and \(\epsilon = 0.005\) (black). For the purpose of cross reference, we choose the same nine sets of parameter settings as in examples 1–6. To assist the observation, the figures in Fig. 10 are zoomed into the regions where different \(\epsilon\) values introduce different solution profiles. The numerical solutions clearly show that as \(\epsilon\) increases, the numerical solution is smeared out, and the jump location becomes less accurate. Notice that \(\tau\) is responsible for the competition between the diffusion and dispersion, which in turn determines the plateau values. Hence varying \(\epsilon\) value does not affect the plateau location.

4. Conclusion

We extended the second and third order classical central schemes originally designed for the hyperbolic systems to solve the MBL equation, which is of pseudo-parabolic type. The numerical solutions for qualitatively different parameter values \(\tau\) and initial conditions \(u_B\) show that the jump locations are consistent with the theoretical calculation and the plateau heights are consistent with the numerically obtained values given in [15]. In particular, when \(\tau > \tau^*\), for \(u_B \in (\bar{u}, \bar{f})\), the numerical solutions give non-monotone
water saturation profiles, which is consistent with the experimental observations. In addition, the order tests show that the proposed second and third order central schemes achieved the desired accuracies.

In [16,14], the two-dimensional space extension of the modified Buckley–Levett equation has been derived. One of the future directions is to develop high order numerical schemes to solve the two-dimensional MBL equation. Central schemes have been used to solve high dimensional hyperbolic problem and dispersive problem [7,12], which makes it a good candidate for such a task.

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References