Nonlocal aggregation models.

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The model.
Actin filaments.
Mesh of actin filaments.
Cross-linking proteins.

filamin, ABP-50, fibrillin, villin, fascin...
Actin filaments with or without cross-linking proteins.

A control  

B 8-Br-cGMP
Action of Cross-linking proteins.

\[ G'(x) : \text{moment force between two filaments.} \]
We the case where the density of proteins is homogeneous in space.

- \( \rho(t, \cdot) \in M^1(S^1 \text{ or } \mathbb{R}) \). \( \rho(t, x) \) is the density of filaments of orientation \( x \). We normalise it by:

\[
\int_{\mathbb{R}^n} d\rho(t, \cdot) = 1,
\]

- \( G(x) \) is the interaction potential between two filaments. We assume that this potential is symmetric.
The force applied to one filament is:

$$\partial_x (G \ast \rho).$$

We assume that the rotating speed of a particule is proportionnal to the moment applied. Then, $\rho$ evolves as:

$$\begin{cases} 
\rho(0, \cdot) = \rho_0, \\
\partial_t \rho = \partial_x (\rho \partial_x (G \ast_x \rho)).
\end{cases}$$

(1)
simulations
The model. Local stability analysis

simulations
non-unicity of steady-states
non-unicity of steady-states
Local stability analysis
Pseudo-inverse

Let consider a measure $\rho \in M^1(\mathbb{R})$, of total mass 1: $\int_{\mathbb{R}} d\rho = 1$. Then,

$$x \mapsto \int_{-\infty}^{x} d\rho,$$

is a increasing function $\mathbb{R} \to [0, 1]$. One can then define its pseudo-inverse $u: [0, 1] \to \mathbb{R}$ as:

$$u(z) = \inf \left\{ x \in \mathbb{R}; \int_{-\infty}^{x} d\rho \geq z \right\}.$$
The Pseudo-inverse equation

$u$ then satisfies the following equation:

$$\partial_t u(t, z) = \partial_x (G \ast_x \rho(u(t, z))),$$

(2)

or:

$$\partial_t u(t, z) = \int_{0}^{1} G'(u(t, \xi) - u(t, z)) \, d\xi.$$
Proposition

If \( G \) is analytical, \( G \in L^1(\mathbb{R}) \) and \( |\int_\mathbb{R} G| < \infty \), \( \int_\mathbb{R} G \neq 0 \), then every steady solution of equation (1) is a sum of Dirac masses

\[ \bar{\rho} = \sum_{i \in \mathbb{N}} \rho_i \delta_{x_i}, \]

where the sequence \((x_i)_i\) has no accumulation point (this sum is finite if the support of \( \bar{\rho} \) is bounded).

Proof:

\[ \forall z \in [0, 1], \quad 0 = \partial_x (G \ast \bar{\rho}(\bar{u}(z))). \]

Then, if \( u([0, 1]) \) has an accumulation point, \( 0 = \partial_x (G \ast \bar{\rho}) \). Then \( Cte = G \ast \bar{\rho} \), and if we apply a Fourier transform and evaluate it in 0, we get :

\[ Cte \delta_0(0) = \left( \int_\mathbb{R} G \right) \left( \int_\mathbb{R} d\rho \right), \]

which is absurd.
condition to be a steady-state

\[ \partial_t u(t, z) = \int_0^1 G'(u(t, \xi) - u(t, z)) \, d\xi. \]

**Proposition**

\[ \bar{\rho} = \sum_{i=1}^n \bar{\rho}_i \delta \bar{u}_i, \bar{\rho}_i \neq 0 \text{ is steady state of eq. (1) if and only if } (\bar{\rho}_i)_i \in \text{Ker} \left( (G'(\bar{u}_i - \bar{u}_j))_{i,j} \right), \]

that is \( \forall i = 1, \ldots, n, \partial_x (G * \rho)(u_i) = 0. \)

**Proof:**

\[
\begin{align*}
\partial_t \bar{u} & = \int_0^1 G' (\bar{u}(t, \xi) - \bar{u}(t, z)) \, d\xi \\
& = \sum_i \bar{\rho}_j G' (\bar{u}_j - \bar{u}_i)
\end{align*}
\]
Proposition

For a steady solution \( \bar{\rho} = \sum_{i=1}^{n} \rho_i \delta_{x_i}, \, \rho_i \neq 0 \) of eq. (1) to be linearly stable under small dislocations, it is necessary that:

\[ \forall i = 1, \ldots, n, \quad \partial^2_{xx}(G \ast \rho)(u_i) > 0. \]
Necessary conditions for linear stability 2

Proposition

For a steady solution \( \bar{\rho} = \sum_{i=1}^{n} \rho_i \delta u_i, \rho_i \neq 0 \) of eq. (1) to be linearly stable under small perturbations of the \( u_i \), it is necessary that the linear application \( \mathcal{L}_M \) defined by the matrix

\[
M = \left( \rho_i G''(u_i - u_j) \right)_{i,j} - \text{diag} \left( (G''(u_i - u_j))_{i,j} (\rho_j) \right),
\]

has a spectrum included in \( \mathbb{R}^* \times i\mathbb{R} \) when restricted to the hyperspace \( \{(w_i)_{i=1,...,n}; \sum_{i=1}^{n} w_i = 0\} \).
Local stability with support conditions

**Proposition**

A steady-state \( \bar{\rho} = \sum_{i=1}^{n} \bar{\rho}_i \delta_{\bar{\mu}_i} \) is locally stable under a support condition, that is:

\[
\| u(t) - \bar{u} \|_\infty \leq C e^{-\kappa t}, \quad \kappa = \kappa (\| G''' \|_\infty, n) > 0,
\]

as soon as \( \| u(0) - \bar{u} \|_\infty \) is small enough,

if and only if it is satisfies the linear stability conditions 1 and 2.
Proof1

We consider a perturbation \( u = \bar{u} + \nu \) of the steady-state \( \bar{u} \). We first estimate \( \int_{l_i} \nu \):

\[
\partial_t \nu = \int_0^1 G' (u(\xi) - u(z)) \, d\xi \\
= \sum_{j=1}^n G'' (\bar{u}_j - \bar{u}_i) \int_{l_j} \nu(\xi) \, d\xi - \nu(z) \sum_{j=1}^n G'' (\bar{u}_j - \bar{u}_i) \bar{\rho}_j \\
+ O(\|\nu\|_\infty^2).
\]

Then, if we integrate the equation over \( l_i \),

\[
\frac{d}{dt} \int_{l_i} \nu = \left[ (\rho_i G''(u_i - u_j))_{i,j} - \text{diag} \left( (G''(u_i - u_j))_{i,j} (\rho_j)_{j}\right) \right] (\int_{l_i} \nu)_i \\
+ O(\|\nu\|_\infty^2).
\]
We now get estimates on $|v|$:

$$
\partial_t |v| = sgn(v(z)) \int_0^1 G'(u(\xi) - u(z)) \, d\xi \\
= -G'' * \bar{\rho}(x_i)|v| + O \left( \left\| \left( \int_{l_i} v \right)_i \right\|_\infty \right) \\
+ O(\|v\|_\infty^2)
$$
Proof3

So, if we consider the vector \( w := \begin{pmatrix} \| v \|_{\infty} \\ \int_{I_1} v \\ \vdots \\ \int_{I_n} v \end{pmatrix} \), we get thanks to previous estimates :

\[
\frac{d}{dt} w = \begin{pmatrix} -G'' \ast \bar{\rho}(x_{i_0}) \\ 0 \\ (\rho_i G''(u_i - u_j))_{i,j} - \text{diag} \left( (G''(u_i - u_j))_{i,j} (\rho_j)_j \right) \end{pmatrix} + O(1) + O(\| w \|^2),
\]

and a Gronwall lemma shows the proposition.
A Lemma

**Lemma**

If the steady state $\bar{\rho}$ satisfies the linear stability condition 2, there exist $\eta > 0$, such that if $\| (\tilde{\rho}_i)_{i=1,\ldots,n} - (\bar{\rho}_i)_{i=1,\ldots,n} \| < \eta$, then there exist a unique $(\tilde{u}_i)_{i=1,\ldots,n}$ close to $(\bar{u}_i)_{i=1,\ldots,n}$ such that $\tilde{\rho} := \sum_{i=1}^{n} \tilde{\rho}_i \delta \tilde{u}_i$ is a stable steady solution of (1), and such that $\tilde{\rho}$ has the same center of mass as $\bar{\rho}$.

Notice that the steady state $\tilde{\rho}$ obtained this way is also stable.
Proof

We consider the function

$$F((\tilde{u}_i)_i, (\tilde{\rho}_i)_i) := (G'(\tilde{u}_i - \tilde{u}_j))_{i,j}(\tilde{\rho}_i)_i.$$ 

We want to apply the implicit function theorem to find to describe the set \( \{ F((\tilde{u}_i)_i, (\tilde{\rho}_i)_i) = 0 \} \). The derivative to consider then is:

$$D_{(\tilde{u}_i)_i} F((\tilde{u}_i)_i, (\tilde{\rho}_i)_i),$$

which is equal to the matrix of the linear stability assumption 2.
Proposition

For a steady state $\bar{\rho} = \sum_{i=1}^{n} \bar{\rho}_i \delta \bar{x}_i$ to be linearly stable under distant invasions, it is necessary that there exist $(y_i)_{i=1,\ldots,n-1}$ such that:

$$G' * \rho < 0 \text{ on } (-\infty, x_1), \quad G' * \rho > 0 \text{ on } (x_n, +\infty),$$

and $G'' * \rho(y_i) < 0$.

(3)
Local stability without support conditions

Proposition

If a steady-state \( \bar{\rho} = \sum_{i=1}^{n} \rho_i \delta_{x_i} \) satisfy the three stability conditions, then it is locally stable, that is:

If \( \|u(0) - \bar{u}\|_{L^1} \) is small enough, and is not flat on neighbourhoods of the points \((y_i)_i\) defined in (3), then there exist \( \tilde{\rho} = \sum_{i=1}^{n} \tilde{\rho}_i \delta_{\tilde{x}_i} \) close to \( \bar{\rho} \), such that:

\[
\|u(t) - \tilde{u}\|_{L^1} \leq Ce^{-ct},
\]

where \( c > 0 \).
Idea of the proof 1
Idea of the proof 2
Idea of the proof 3

We want to estimate $\rho$ over the green interval $I$ of size $\varepsilon$. At all times, there exist $y(t) \in I$ such that $\partial_x (G \ast_x \rho)(y(t)) = 0$. Then, for $z \in I$,

$$\partial_t u(t, z) = (\partial^2_{xx} (G \ast_x \rho)(y(t)) + O(\varepsilon)) (u(t, z) - u(t, y(t))).$$

Then,

$$u(t, z) = \left[ u(0, z) - \int_0^t \left( y(s) e^{\int_0^s G'' \ast \rho(y(\tau)) + O(\varepsilon) d\tau} \right) ds \right] e^{\int_0^t G'' \ast \rho(y(\tau)) + O(\varepsilon) d\tau},$$

that is $u(t) = c(t) u(0) + d(t)$, where:

$$c(t) = e^{\int_0^t G'' \ast \rho(y(\tau)) + O(\varepsilon) d\tau} \sim e^{ct} \rightarrow \infty.$$

And all the mass escapes from the interval $I$. 

Remarks
refined model.

A more complex model has been introduced by Kang, Perthame, Primi, Stevens and Velazquez:

\[ \partial_t \rho(t, x) = \int T[\rho](y, x) \rho(t, y) \, dy - \int T[\rho](x, y) \rho(x, t) \, dx, \]

where:

\[ T[\rho](x, y) = \int \Gamma_\sigma \left( y - x - G'(z - x) \right) \rho(t, z) \, dz. \]
A refined kinetic model.

They say that for $\sigma$ small, an approximation of their kinetic model is:

$$\partial_t \rho(t, x) = \frac{\sigma^2}{2} \partial_{xx} \rho(t, x) + \partial_x \left( \rho(t, x) \int G'(x - y) \rho(t, y) \, dy \right)$$

The local stability result doesn’t pass to this generalised model: the diffusion can transport mass from one Dirac to another. My guess would be that the result is true under the additional assumption that $\forall i, j$, $G \ast_x \tilde{\rho}(\tilde{u}_i) = G \ast_x \tilde{\rho}(\tilde{u}_j)$...
Attractive singular kernels

If $G$ is repulsive and singular at 0, $\rho$ explodes in finite time:

Theorem

*(Bertozzi, Carrillo, Laurent)* Let $\rho$ be a solution of (1) in $\mathbb{R}^N$, with a nonnegative compactly supported initial data in $L^\infty$. Let $G$ satisfy $\int_0^1 \frac{1}{G'(x)} \, dx < +\infty$ and $\frac{G'(r)}{r}$ monotone decreasing. Then there exists a maximal time $T^\ast st < \infty$ and a unique solution $\rho$ on the interval $[0, T^\ast)$. Moreover,

$$\lim_{t \to T^\ast} \| \rho(\cdot, t) \|_{L^q} = \infty,$$

for $q \in [2, \infty]$ if $N > 2$ and $q \in (2, \infty]$ if $N = 2$.

And if $\int_0^1 \frac{1}{G'(x)} \, dx = +\infty$, the solution is global in time.
If we add a diffusion to the model and if $G = -\frac{1}{2\pi} \log |x|$ (in dimension 2), then (1) becomes the Keller-Segel model:

$$\begin{cases} 
\partial_t n = \Delta n - \xi \nabla \cdot (n \nabla c), \\
-\Delta c = n.
\end{cases}$$
If $G := \delta_0$, equation (1) becomes the porous medium equation:

$$\partial_t \rho = \partial_x (\rho \partial_x \rho).$$

If $G$ is singular and repulsive, it also behaves numerically as a slow diffusion:
repulsive singular kernels

We only have very few results on this up to now:

**Proposition**

If $G$ is Lipschitz continuous and $C^2$ on $\mathbb{R} \setminus \{0\}$, then (1) has a unique solution.

**Proposition**

Let $G_\varepsilon \rightarrow (x \mapsto -|x|)$ a sequence of regular kernels, $\bar{\rho}_\varepsilon$ and $\bar{\rho}$ steady-states of:

$$\partial_t \rho = \partial_x (\rho \partial_x (G \ast_x \rho) + V).$$

Then,

$$\bar{\rho}_\varepsilon \rightarrow \bar{\rho} \text{ in } M^1.$$