Uniform (as opposed to weak) semiclassical limits with the help of infinite order operators: theory and applications.

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Wigner measures have been used successfully in the description of semiclassical wave propagation in various contexts [4, 3, 5]. This work concerns a modification of the Wigner transform (WT), and how it can be used (after the necessary functional analytic framework is worked out) for the homogenization of wave propagation. The main application presented is a new approach for semiclassical problems: a much stronger convergence to the semiclassical limit, allowing the formulation of a hierarchy of high-order models

$$W^\epsilon = W_{\text{approx}}^\epsilon + O\left(\epsilon^a(N)\right), \quad N = 1, 2, 3...$$

with errors measured in appropriate Banach spaces, which are easy to implement in practice (e.g. numerically). This must be contrasted with the traditional Wigner measure approach, where the approximations are weak, and working with anything but the Wigner measure $W_0$ is intractable. This rigidity, i.e. the difficulty of WT based models to incorporate corrections, has already been seen to be crucial in some problems [1] and/or has led to other modifications of the WT [2].

The basic idea is to use a smoothed version of the Wigner transform (the smoothed Wigner transform, SWT) in the place of the WT. This is much better behaved in several respects, especially qualitative and numerical. On the other hand, developing a calculus and trace formula for a smoothed object gives rise to infinite order operators, which cannot be studied in the more common framework of (finite-order) Sobolev spaces and Schwarz distributions. The appropriate function spaces are constructed, and they are found to be closely related to Gelfand-Shilov-type test-functions (and their duals, ultradistributions). Interesting links also arise with modulation spaces (this is very interesting as the original motivation behind modulation spaces is in fact related with the WT and its variations, albeit in a different context).

After the appropriate framework is understood, the precise formulation of a calculus and a trace formula for the SWT is possible. This in particular allows to derive the exact equations that govern the SWT of a wavefield (of the solution of a wave equation), as well as extract observables (such as energy, energy flux etc) from the SWT. It must be stressed that, unlike for the WT, these equations must be considered tractable computationally. This has to do with the absence of the so-called ‘interference terms’, fast oscillations in phase spaces that create overwhelming (and quite redundant) complexity in the respective equations for the WT.

Semiclassical asymptotics of smoothed Wigner equations are carried out as an application; in particular the approximation of the exact (non-local, infinite-order) operators by a finite order differential operator yields now much stronger results than the respective approximation for the WT (of finite-order non-local equations by differential operators), namely uniform approximation. That is, although the operators are ‘more complicated’, the function itself is ‘much better’, leading to (significantly) stronger results altogether.

Another issue studied is the formulation of numerical methods for the implementation of the exact operators. Once more we have to look beyond the more commonly used ideas, since they tend to avoid the study of infinite-
order operators. A particle method is formulated and analyzed in an appropriate Gelfand-Shilov type space; it is seen that infinite-order operators can be implemented numerically. The analysis also highlights the – more or less intuitive and expected – limitations.

This work is a continuation of research started in the author’s PhD thesis (Princeton, 2007).

This presentation contains joint work with Thierry Paul and Norbert J. Mauser. The author would like to acknowledge financial support from an ERCIM “Alain Bensoussan” fellowship, and to thank the Wolfgang Pauli Institut for its hospitality.

References


