The Zero Electron Mass Limit in the Euler Poisson System

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The Euler Poisson system is derived in semiconductor or plasma physics to study the time evolution of charged fluids. These models can be obtained from Boltzmann equation for charged particles, i.e. the electrons and ions (or holes in semiconductor). The system consists of the conservation laws for the electron(ion) density and current density for electron(ion), coupled to the Poisson equation for the electrostatic potential. More precisely, these (scaled) hydrodynamic equations for the electron density n_e with charge $q_e = -1$, the density n_i of the positively charged ions with charge $q_i = +1$, the respective velocities v_e, v_i, and the electrostatic potential ϕ,

$$\begin{aligned} \partial_t n_\alpha + \nabla \cdot (n_\alpha v_\alpha) &= 0, \quad \alpha = e, i, \\ m_\alpha \partial_t (n_\alpha v_\alpha) + m_\alpha \nabla \cdot (n_\alpha v_\alpha \otimes v_\alpha) + \nabla p_\alpha(n_\alpha) &= -q_\alpha n_\alpha \nabla\phi - m_\alpha \frac{n_\alpha v_\alpha}{\tau_\alpha}, \\ -\lambda^2 \Delta\phi &= n_i - n_e - C(x) \quad \text{for } x \in R^d,\ t > 0, \end{aligned} \tag{1}$$

where $d \geq 1$. The initial conditions are given by

$$n_\alpha(\cdot, 0) = n_{I,\alpha}, \quad v_\alpha(\cdot, 0) = v_{I,\alpha} \quad \text{in } R^d,\ \alpha = e, i.$$

In the above equations, p_α are the pressure functions, usually given by $p_\alpha(x) = c_\alpha x^{\gamma_\alpha}$, $x \geq 0$, where $c_\alpha > 0$ and $\gamma_\alpha \geq 1$. In this work, we only assume that p_α is strictly monotone and smooth. The function $C(x)$ models fixed charged background ions (doping profile). The (scaled) physical parameters are the particle mass m_α, the relaxation time τ_α, and the Debye length λ. We assume that the value of the integral $\int_{R^d} \phi dx$ is fixed; for instance, $\int_{R^d} \phi dx = 0$.

There are a lot of mathematical works, both on wellposedness and different kinds of limit problems, for Euler-Poisson system in the literature, for example, by G. Ali, L. Hsiao, A. Jüngel, H-L. Li, S. Wang, H-J. Zhao, K-J. Zhang etc., we omit the references here. Among the limit problems, relaxation time limit ($\tau_{i,e} \to 0$) by P. Marcati, R. Natalini, A. Jüngel, Y-J. Peng, G. Alí, D. Bini, and S. Rionero., and the quasineutral limit ($\lambda \to 0$) by S. Cordier and E. Grenier, I. Gasser and P. Marcati., S. Wang, are well studied. The zero mass limit ($m_e/m_i \to 0$) was left unsolved in the literature.

In this work, we restrict ourselves to a situation in which the ion density is given and the initial data is ill prepared. The parameter m_e is essentially the ratio of the electron mass to the ion mass. We assume that the ion mass is much larger than the electron mass such that the limit $m_e \to 0$ makes sense. The limit has the goal to achieve simpler models containing the essential physical phenomena. We notice that in plasma physics, zero-electron-mass assumptions are widely used.

Before we present the main results, it is convenient to write the main part of the system into symmetric hyperbolic form. Setting $n = n_e$, $v = v_e$, $p(n) = p_e(n_e)$, and $\varepsilon^2 = m_e$ and introducing the *enthalpy* $h = h(n_e)$, defined by $h'(n) = p'(n)/n$ and $h(1) = 0$, clearly for smooth solutions, the system is equivalent to the following system with symmetric hyperbolic structure

$$\begin{aligned} (\partial_t + v\cdot\nabla)h + p'(n)\nabla\cdot v &= 0, \\ \varepsilon^2(\partial_t + v\cdot\nabla)v + \nabla h &= \nabla\phi - \varepsilon^2 v, \\ \Delta\phi &= n(h) - N, \quad x\in R^d,\ t>0, \end{aligned} \tag{2}$$

with initial conditions

$$h(\cdot,0) = h_I^\varepsilon = h(n_I), \quad v(\cdot,0) = v_I^\varepsilon \quad \text{in } R^d. \tag{3}$$

where, we have set $\tau_e = \lambda = 1$ in order to simplify the notation. As we suppose that the pressure function is invertible, so does $h(n)$ and we denote its inverse by $n(h)$.

The limit of vanishing electron mass of this system, i.e. $\varepsilon \to 0$ with ill prepared initial data is discussed. Although it has some relations to the incompressible limit of Euler equation, i.e. the limit velocity satisfies the incompressible Euler equations with damping, things are more complicated due to the linear singular perturbation including the coupling with the Poisson equation. We first prove the uniform existence by a reformulation of the equations in terms of the enthalpy, higher-order energy estimates and a careful use of the Poisson equation. (Actually, by the same idea, one can get the estimates for time derivatives for well prepared initial data which is enough for the limit discussion). Now our focus is on the case of ill prepared initial data, a careful analysis on the structure of the linear perturbation has been done so that we are able to show the convergence away from time $t=0$. Here are the main theorems we have obtained,

Theorem 1 (Local uniform existence) Let $s > d/2+1$ and $N>0$. The initial data $(n_I^\varepsilon, v_I^\varepsilon)$ satisfy

$$\left\| \frac{n_I^\varepsilon - N}{\varepsilon} \right\|_s + \|v_I^\varepsilon\|_s \le M_0,$$

with $M_0 > 0$ is a constant independent of ε. Then there exist constants $T_0 > 0$ and $M_0' > 0$, independent of ε, and $\varepsilon_0(M_0) > 0$ such that for all $0 < \varepsilon < \varepsilon_0(M_0)$, the problem (2)-(3) has a classical solution $(n^\varepsilon, v^\varepsilon, \phi^\varepsilon)$ in $[0, T_0]$ satisfying

$$|||\frac{n^\varepsilon - N}{\varepsilon}|||_{s,T_0} + |||v^\varepsilon|||_{s,T_0} + |||\frac{\nabla\phi^\varepsilon}{\varepsilon}|||_{s,T_0} \le M_0'.$$

Theorem 2 (Zero mass limit) Let the assumptions of Theorem 1 hold and let $(n^\varepsilon, v^\varepsilon, \phi^\varepsilon)$ be a classical solution to (2)-(3) in $[0,T_0]$ with $T_0 > 0$ independent of ε. Then, as $\varepsilon \to 0$,

$$\begin{aligned} n^\varepsilon \to n^0, \quad \nabla\phi^\varepsilon \to 0 &\qquad \text{strongly in } L^\infty(0,T_0; H^s(R^d)), \\ v^\varepsilon \rightharpoonup v^0 &\qquad \text{weakly* in } L^\infty(0,T_0; H^s(R^d)), \\ v^\varepsilon \to v^0 &\qquad \text{strongly in } C^0_{loc}((0,T_0]\times R^d), \end{aligned}$$

where $v^0 \in L^\infty([0,T_0]; H^s(R^d))$ is the unique solution of the following incompressible Euler equations with damping,

$$\begin{gathered} \nabla\cdot v^0 = 0, \quad (\partial_t + v^0\cdot\nabla)v^0 + v^0 = \nabla\pi, \quad x\in R^d,\ t>0, \\ v^0(\cdot,0) = Pv_I \quad \text{in } R^d, \end{gathered} \tag{4}$$

for some $\pi \in L^\infty([0,T_0]; H^s(R^d))$. P is the orthogonal projection of H^s onto the subspace $\{v\in H^s : \nabla\cdot v = 0\}$.