A Space-Time hp-Adaptive Scheme for Evolution Equations

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Many applications require the solution of complex non-linear systems of evolution equations of the general form

$$\partial_t U(t,x) + \nabla \cdot \big(F(U(t,x),t,x) + a(U(t,x),t,x)\nabla U(t,x)\big) = S(U(t,x),t,x)\ . \tag{1}$$

Here $t > 0$ and $x \in \Omega \subset R^d$ denote the time and space variable and the unknown function is $U : R^+ \times \Omega \mapsto R^m$, F is a general flux function, a a diffusion matrix, and S a source term. Examples are the compressible Navier-Stokes equations, the equations of Magnetohydrodynamics, and different models for shallow water flow. Depending on the application governed by this system, both a high degree of accuracy and a high degree of efficiency have to be combined during the design of a numerical scheme. These goals can be achieved on the one hand by a careful reduction of the complexity of the mathematical model and on the other hand by using local grid adaptivity and parallelization strategies. If possible these techniques should be simple to use for the developer and a modification of the mathematical model should not require a major redesign of the code. In the software library [1] both the required flexibility and efficiency is achieved by using generic software design techniques based on static interfaces implemented in C++. The basis of this project is the Discontinuous Galerkin method using implicit/explicit Runge-Kutta methods for the time discretization. DUNE-FEM is based on the grid interface concept developed in the DUNE project. Here general concepts for using locally adapted grids in a parallel environment are developed [2].

The Discontinuous Galerkin (DG) method is a higher order discretization method for evolution equations of the form (1), which can be easily adapted to both the diffusion and the advection dominated case [3]. Due to its small stencil, the method can be easily implemented on non-conform grid structures and thus simplifies the use of local adaptivity and parallelization based on domain decomposition. For the case of an advection-diffusion equation the compact DG method presented for general elliptic equations [4] can be used to reduce the stencil to the optimal case of one neighboring layer. In the case of small or zero viscosity the higher order versions of the DG method become unstable and some limiting of gradients is required. Here unstructured grids prove to be difficult; while on structured Cartesian grids *minmod* type limiting is very effective [3], limiters on unstructured grids either lead to a severe degeneration of the approximation in smooth regions or are unstable in extreme cases.

In [5] we present an a-posteriori error estimate for the higher order Discontinuous Galerkin method for scalar conservation laws

$$\partial_t u(t,x) + \nabla \cdot f(u(t,x)) = 0\ . \tag{2}$$

The result holds for the semi-discrete scheme on arbitrary grid structures and is heuristicly extended to a fully discrete scheme by using explicit Runge-Kutta methods for the time discretization. The error estimate is then used to locally reduce the order of the scheme (p-adaptivity) and simultaneously the local grid resolution is adapted (h-adaptivity) - thus the stability and efficiency of the scheme is achieved by hp-adaptivity in space.

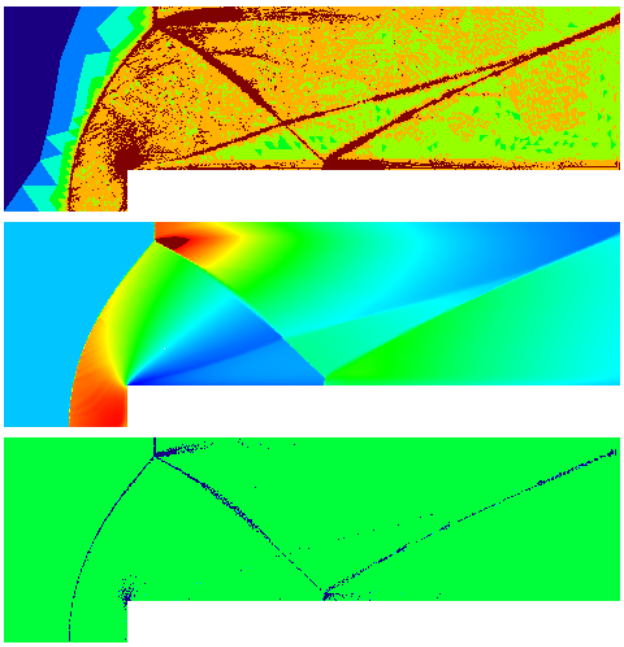

Figure 1: The figure shows results The middle figure shows the density

Figure 1 shows results for the system of Euler equations. Shown are the density, the polynomial degree and the local grid level.

In this talk we present the results from [5] and present a method for extending the hp-adaptive scheme also to the time discretization by using local time stepping based on a class of multistep methods. These schemes are an extension of the SSP methods presented in [6]; we extend these methods to include varying time steps and local time stepping. We also present the DUNE project and the software techniques used for solving evolution equations of the form (1).

References

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