A Convergent Finite Element Method for the Stokes Approximation Equations

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The Navier-Stokes system for a compressible barotropic viscous gas has the form

\[
\begin{align*}
\rho_t + \text{div}(\rho \mathbf{u}) &= 0, \\
\rho (\mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u}) &= \mu \Delta \mathbf{u} + (\lambda + \mu) \nabla \cdot \mathbf{u} - D p(\rho).
\end{align*}
\]

Here, the unknowns are the density \( \rho = \rho(t, \mathbf{x}) \geq 0 \) and velocity \( \mathbf{u} = \mathbf{u}(t, \mathbf{x}) \in \mathbb{R}^N, \mathbf{x} \in \mathbb{R}^N, N = 2, 3, t \geq 0 \). \( p(\rho) \) is the pressure governed by the equation of state given by Boyle’s law, \( p(\rho) = a\rho^\gamma, a > 0 \). Typical values of \( \gamma \) ranges from a maximum of \( \frac{5}{3} \) for monoatomic gases, through \( \frac{7}{5} \) for diatomic gases including air, to lower values close to 1 for polyatomic gases at high temperatures. \( \mu, \lambda \) are viscosity coefficients and are assumed to be constants satisfying \( \mu > 0 \) and \( N\lambda + 2\mu \geq 0 \). \( \text{div} \) and \( \nabla \) are the usual spatial divergence and gradient operators and \( \Delta = \text{div} \nabla \) is the Laplace operator.

Through the years, numerous numerical methods approximating the Navier-Stokes system (1) – (2) have been proposed. In one dimension, convergence results are largely due to David Hoff and collaborators with discretizations performed in lagrangian coordinates. However, in more than one spatial dimension, the convergence properties of any numerical method is still unknown. That is, it is unclear whether any of these numerical solutions in fact converges to an actual (weak) solution as discretization parameters goes to zero.

The major difficulty in analysis of the compressible Navier-Stokes system (1) – (2) are the nonlinearities in both convection and pressure and their interactions. Thus, as a step on the way to establish a convergent numerical method for the compressible system (1) – (2) it seems natural to consider a simplification. The most well-known simplification of this system is the *Stokes approximation*:

\[
\begin{align*}
\tilde{\rho}_t + \text{div}(\tilde{\rho} \mathbf{u}) &= 0, \\
\tilde{\rho} \mathbf{u}_t - \mu \Delta \mathbf{u} - \lambda D \text{ div } \mathbf{u} + a D \tilde{\rho}^\gamma &= 0,
\end{align*}
\]

where \( \tilde{\rho} = \text{const} > 0 \) is the mean density. This is a good approximation for strongly viscous fluids where convection can be neglected.

In this talk we will present, and prove the convergence of, a finite element method approximating the system (3)–(4) on a bounded open domain \( \Omega \subset \mathbb{R}^N, N = 2, 3 \). In addition, we will also discuss approximation of it’s semi–stationary version,

\[
\begin{align*}
\tilde{\rho}_t + \text{div}(\tilde{\rho} \mathbf{u}) &= 0, \\
-\mu \Delta \mathbf{u} - \lambda D \text{ div } \mathbf{u} + a D \tilde{\rho}^\gamma &= 0.
\end{align*}
\]
For both systems, we will at the boundary of $\Omega$, $\partial\Omega$, use the boundary conditions:

\[
\mathbf{u} \cdot \nu = 0, \quad \text{and} \quad \begin{cases}
\text{curl } \mathbf{u} = 0, \quad N = 2, \\
\text{curl } \mathbf{u} \times \nu = 0, \quad N = 3.
\end{cases}
\] (5)

The first condition is the standard non-penetration requirement and the second is a Navier slip condition requiring that the shear stress vanish at the boundary.

With the Navier slip condition (5), the natural weak formulation of the system (3)–(4) is in mixed form with $\mathbf{w} = \text{curl } \mathbf{u}$ as an auxiliary variable. As a consequence, the natural finite element discretization in this weak formulation approximates the velocity $\mathbf{u}$ with div–conforming elements and the auxiliary variable $\mathbf{w}$ with curl–conforming elements. For approximation of the velocity $\mathbf{u}$ and the auxiliary variable $\mathbf{w}$ we will use the first order Nedelec elements of first kind. The density $\rho$ will be approximated using the space of piecewise constants.

With this choice of elements, we have that the approximation spaces satisfies a discrete version of the De Rham sequence. In particular, the velocity can be orthogonally decomposed as $\mathbf{u} = \text{curl } \xi + \mathbf{z}$. Where $\mathbf{z}$ is only connected to the density $\rho$ and $\xi$ is only connected to the vorticity $\mathbf{w}$. This type of discrete Hodge decomposition then easily allow us to obtain an equation for the effective viscous flux, $P_f = p(\rho) - (\lambda + \mu) \text{div } \mathbf{u}$. In a similar fashion to the classical works of Lions and Feireisl we can then prove both higher integrability of the pressure and strong convergence of the density.

However, in order to first prove convergence of the product $\rho \mathbf{u}$ we will need spatial compactness of the velocity in $L^2$. Since $\mathbf{u}$ is now only approximated using div–conforming elements, this is not trivial. In fact, we will present a new result on compactness in $L^2$ for weakly curl free div–conforming approximations.