Some recent results on 2D surface quasi-geostrophic equation

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In this talk, I will review some recent results on the properties of solutions of the 2D surface quasi-geostrophic equation (SQGE). The equation is given by

$$\begin{aligned} \theta_t + u \cdot \nabla\theta + (-\Delta)^\alpha \theta = 0, \\ u = (u_1, u_2) = (-R_2\theta, R_1\theta) \\ \theta(x,0) = \theta_0(x). \end{aligned} \tag{1}$$

Here θ is a scalar function defined on a two-dimensional torus T^2, R_2, R_1 are the usual Riesz transforms and $\alpha \geq 0$. Considering equation on the torus is equivalent to periodic initial data in R^2. The equation (1) arises in geophysical studies of strongly rotating fluid flows [3]. The most interesting cases from the physical point of view are $\alpha = 1/2$ and the conservative case when the viscous term $(-\Delta)^\alpha\theta$ is absent from the equation. This talk will consist of two parts: in the first part, I will review the proof of the global regularity of the equation (1) in the case $\alpha = 1/2$, as well as consider some related applications. In the second part, I will present a result on the growth of high order Sobolev norms of solutions to the conservative SQGE.

One important property of the SQGE is the maximum principle for the L^∞ norm of the solution [7, 13]. This makes the value $\alpha = 1/2$ critical for (1). It was known for a while (see [4, 13]) that for $\alpha > \frac{1}{2}$ (the so-called subcritical case), the initial value problem $\theta(x,0) = \theta_0(x)$ with C^∞-smooth periodic initial data θ_0 has a global C^∞ solution. This result can be proved by classical approach, deriving differential inequalities for the Sobolev norms of solutions using Gagliardo-Nirenberg inequalities to control nonlinearity. The case $\alpha = 1/2$ requires new techniques. The following result has been recently proved in [9].

Theorem 1. *The critical surface quasi-geostrophic equation with periodic smooth initial data $\theta_0(x)$ has a unique global smooth solution. Moreover, the following estimate holds:*

$$\|\nabla\theta\|_\infty \leq C\|\nabla\theta_0\|_\infty \exp\exp\{C\|\theta_0\|_\infty\}. \tag{2}$$

A result similar to Theorem 1 has also been proved in [2] by a completely different method. The main idea in [9] is a new, nonlocal maximum principle which allows to control the size of the gradient of the solution. Namely, one can find a modulus of continuity $\omega(\xi)$ such that it is preserved by evolution: if $|\theta_0(x) - \theta_0(y)| \leq \omega(|x-y|)$ for any x, y, then $|\theta(x,t) - \theta(y,t)| \leq \omega(|x-y|)$ for any x, y and $t > 0$. The modulus of continuity has the behavior $\omega(\xi) \sim \xi$ for small ξ and $\omega(\xi) \sim \log\log\xi$ for large ξ. Moreover, a simple scaling argument shows that if $\omega(\xi)$ is preserved by the evolution, then so is $\omega_B(\xi) \equiv \omega(B\xi)$ for any $B > 0$. Together, these properties allow to prove global regularity. First, for any smooth θ_0 the growth of ω for large ξ allows to find B such that θ_0 has the modulus of continuity ω_B. Then it follows from preservation of ω_B and the behavior of ω for small ξ that $|\nabla\theta(x,t)| \leq B$ for any x, t. This control is sufficient to prove global regularity. Two remarks are in order. First, the proof of the preservation of ω under evolution is genuinely nonlocal. Second, it is also essential in the following sense: numerical simulations show that the gradient of the solution can grow in time, at least initially. This suggests that it may be difficult to find a local quantity which is controlled in a way similar to the modulus of continuity.

The modulus of continuity method introduced in [9] can be applied to a number of other problems involving fractional dissipation. One example is one-dimensional Burgers equation with fractional dissipation and periodic initial data

$$u_t + uu_x + (-\Delta)^\alpha u = 0, \;\; u(x,0) = u_0(x). \tag{3}$$

Let us denote H^s the usual scale of Sobolev spaces, and $\|\cdot\|_s$ the corresponding norm. The following result is proved in [10]

Theorem 2. *Assume that $\alpha \geq 1/2$, and $u_0 \in H^s$, $s \geq 3/2-2\alpha$. Then the dissipative Burgers equation has unique solution $u(x,t)$ which belongs to $C([0,\infty), H^s)$ and is real analytic for any $t > 0$. On the other hand if $\alpha < 1/2$, then there exist smooth initial data such that the solution blows up in finite time in any H^s with $s > 3/2 - 2\alpha$.*

A similar result in a slightly different setting has been proved in [1] (with an exception of $\alpha = 1/2$ case, which is handled in [10] using the modulus of continuity approach). Observe that Theorem 2 describes the sharp transition in the behavior of solutions of Burgers equation depending on α. The proof of the blow up direction is based on a time splitting argument and well-understood free Burgers equation dynamics. The question of blow up or regularity for $\alpha < 1/2$ in the SGQE case remains open. The reason for that is relatively poor understanding of the conservative SQGE dynamics. This subject will be the focus of the second part of my talk.

There has been considerable recent interest in trying to understand singularity formation (or lack of it) for the conservative SQGE. Much of the research focused on proving upper bounds on the growth of solutions in certain dangerous scenario, in particular identified by numerical experiments (see [3, 5, 6]). However there has been little progress on proving lower bounds for the growth of the Sobolev norms of solutions. In fact, even for the classical case of two-dimensional Euler equation, until recently the only results on lower bounds were proved for flows in domains with a boundary, and boundary effects or geometry played an important role [14, 12]. Only very recently Denisov [8] proved a superlinear growth for the L^∞ norm of the vorticity gradient in a 2D Euler equation for a cellular flow. For the SQG equation, the following result is proved in [11].

Theorem 3. *For any $A > 0$, and any $\epsilon > 0$, there exists periodic initial data u_0 which is a polynomial (so that only finite number of Fourier coefficients are nonzero), and for any given s we can satisfy $\|u_0\|_s < \epsilon$. However the corresponding solution satisfies $\limsup_{t\to\infty} \|\theta(x,t)\|_{H^{11}} > A$.*

The description of the initial data u_0 is explicit. The theorem shows that high order Sobolev norms of solutions corresponding to very smooth initial data can become large. The main idea of the proof is to consider small perturbations of the stationary solution $\theta(x) = \cos x$. This solution can be shown to be stable in L^2 using two SQGE conservation laws: $\|\theta\|_{L^2} = const$ and $\sum_k |\hat{\theta}(k)|^2 |k|^{-1} = const$. The flow u corresponding to θ is just a shear flow. One can expect that the stable shear flow will stretch the perturbation, introducing linear in time growth in the gradient of solution. We cannot prove such result at this point, settling for a weaker Theorem 3. The proof revolves around an attempt to find a Lyapunov functional, a monotone quantity built out of the solution. We construct something much weaker - a quadratic form in Fourier coefficients of solutions, defined by

$$J(t) = \sum_{k \in Z_+^2} (k_1 + \frac{1}{2}) \hat{\theta}(k) \hat{\theta}(k + e_1).$$

Here e_1 stands for $(1,0)$ vector in Z^2. The derivative of $J(t)$ is a trilinear form which can be split in two parts: terms containing large coefficients $\hat{\theta}(\pm e_1)$ and the rest. It turns out that the contribution of the first part is always positive, however in general this part does not dominate the error term. Nevertheless, it turns out that the positivity of the first term is sufficient for the proof of Theorem 3.

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