A diminishing functional for nonclassical entropy solutions
selected by kinetic relations

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We consider the initial-value problem associated with a hyperbolic conservation law in one-space variable:

\[ u_t + f(u)_x = 0, \]
\[ u(0, \cdot) = u_0, \]

where \( u_0 \) is a function with bounded variation on \( \mathbb{R} \), and the flux \( f \in C^2(\mathbb{R}) \) is a smooth, concave-convex function. Following LeFloch [1], we impose that solutions satisfy a single entropy inequality associated with a convex entropy and we consider nonclassical entropy solutions which contain (classical) compressive discontinuities satisfying Lax entropy inequalities as well as (nonclassical) undercompressive shock waves connecting left- and right-hand states \( u_\pm \) characterized by a kinetic relation

\[ u_+ = \varphi^b(u_-). \]

In practice, the choice of a kinetic relation is (essentially) equivalent to the choice of a regularization for the conservation law, for instance a regularization by (possibly nonlinear and) vanishing diffusion-dispersion terms.

The purpose of the present work is to provide a new definition of the wave strength \( \sigma(W) \) of a (classical or nonclassical) wave \( W \), which improves upon the one proposed in [1] and is inspired (but different) from the one used in LeFloch and Shearer [2] to handle the perturbation of a single nonclassical shock. Importantly, our new definition is “natural” since it relies solely on the most fundamental properties of the kinetic function, namely:

1. \( \varphi^b : \mathbb{R} \to \mathbb{R} \) is a Lipschitz continuous, one-to-one map;
2. \( d\varphi^b/du(u) < 0 \) for all \( u \neq 0 \); and \( \varphi^b(0) = 0 \);
3. \( \varphi^b \circ \varphi^b \) is a strict contraction, that is, for all \( u \neq 0 \)

\[ |\varphi^b \circ \varphi^b(u)| < |u|. \]

The new notion of wave strength allows us to introduce a Total Variation-like functional and to deal with the existence of solutions to (1) via Dafermos’ front tracking scheme. The total variation of a front-tracking approximation is equivalent to

\[ V(u(t)) := \sum_{\text{wave } W} \sigma(W), \]
and the functional $V$ induces a “norm” on $\text{BV}(\mathbb{R})$. The mapping $\Phi : \text{BV}(\mathbb{R}) \to \text{BV}(\mathbb{R})$ defined by $u \mapsto \varphi^0_0 \circ u$, where $\varphi^0_0$ denotes the zero diffusion kinetic function, is an isometry of $\text{BV}(\mathbb{R})$.

Our main result is that the change in $V(u(t))$ during interactions in front tracking approximations either vanishes, when there are no rarefactions, or is negative and proportional to the strength of the rarefaction that is cancelled by the interaction. This leads to a clarification of previous estimates established in [1] and implies the existence of nonclassical solutions.

We also show, in contrast to the case of classical entropy solutions [3], that there exists no Glimm-type interaction potential $Q(u(t))$ such that for some constant $K > 0$ and sufficiently small initial data $u_0$, the functional $V(u(t)) + KQ(u(t))$ is strictly decreasing. Only a functional allowing for a weaker control of solutions is possible.

**References**

